Netflix problem and realization of (hyper)graphs

slides of talks presented by A. Skopenkov on 20.09.2022 and 3.02.2023

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1 Motivation

'Matrix completion is the task of filling in the missing entries of a partially observed matrix... One example is the movie-ratings matrix, as appears in the Netflix problem (from machine learning): Given a ratings matrix in which each entry (i, j) *represents the rating of movie* j *by customer* i*, if customer* i *has watched movie* j *and is otherwise missing, we would like to predict the remaining entries in order to make good recommendations to customers on what to watch next...'* [Wiki: Matrix completion: Low rank matrix completion]. The remaining entries are predicted so as to minimize the *rank* of the completed matrix. For a brief overview of the history of this and related problems, see the Wikipedia article and [Kogan, arXiv:2104.10668, Remark 4].

Here for simplicity we consider matrices with entries in the set $\mathbb{Z}_2 = \{0, 1\}$ of all residues modulo 2 (with the sum and product operations). This is sufficient for the topological applications, see below. We start with algorithms estimating minimal rank for the particular case of unknown elements *on the diagonal* (Theorem 2.3). Then we study a more general problem, in which instead of knowing specific matrix elements, we know linear relations on such elements. We estimate the minimal rank of matrices with such relations (Theorems 4.2, 5.1, 6.3).

We present applications of these results to embeddings

• of graphs to surfaces (more precisely, embeddings with rotation systems, and embeddings modulo 2), and

• of k-dimensional hypergraphs to $2k$-dimensional surfaces.

2 Weak realizability of graphs in surfaces

A *hieroglyph* on n letters is an unoriented cyclic letter sequence of length $2n$ such that each letter from the sequence appears in the sequence twice.

Take a hieroglyph on n letters. Take a convex polygon with $2n$ sides. Put the letters in the hieroglyph on the sides of the convex polygon in the nonoriented cyclic order. For each letter glue the ends of a ribbon to the pair of sides corresponding to the letter so that the glued ribbons are pairwise disjoint. The ribbons can be either twisted or not twisted. Call the resulting surface a *disk with ribbons* corresponding to the hieroglyph (see Figure 1).

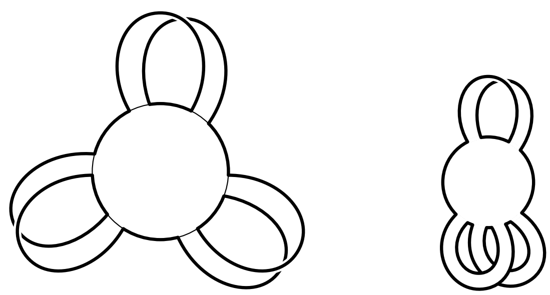

Figure 1: Disk with ribbons corresponding to the hieroglyph *aabbcc* (left) and *aabcbc* (right)

A hieroglyph H is called *weakly realizable* on the Möbius band if some disk with ribbons corresponding to H can be cut out of the Möbius band. Analogously one defines weak realizability on *the disk with k Möbius bands* shown on the left of fig. 1. This is the union of a disk and k pairwise disjoint ribbons having their ends glued to $2k$ pairwise disjoint arcs on the boundary circle of the disk (the ribbons do not have to lie in the plane of the disk) so that

• the orientations of the ends of each ribbon given by an orientation of the boundary circle of the disk have 'the same direction along the ribbon', and

• the ribbons are 'separated', i. e. there are k pairwise disjoint arcs A_i on the boundary circle of the disk such that the ends of the i-th ribbon are glued to two disjoint arcs contained in A_i, $i = 1, 2, \dots, k$.

Theorem 2.1. *(a) (Bikeev, 2020, [Bi20]) There is an algorithm with the complexity of $O(n^2)$ deciding whether an hieroglyph with n ribbons is weakly realizable on the Möbius band.*

(b) (Kogan, 2021, [Ko21]) For any fixed k there is an algorithm with the complexity of $O(n^{k+3})$ deciding whether an hieroglyph with n ribbons is weakly realizable on the disk with k Möbius bands.

This is proved using the realizability criterion in terms of the 'intersection form' of a surface, see Theorem 2.2 below, and a linear algebraic argument, see Theorem 2.3 below.

Two letters a, b in a hieroglyph H *overlap in H* if they interlace in the cyclic sequence of the hieroglyph (i. e., if they appear in the sequence in the order $abab$ but not $aabb$). Define the *overlap matrix* $M(H) \in \mathbb{Z}_2^{n\times n}$ of a hieroglyph H as follows. Put zeros on the main diagonal. Put 1 in the cell (i, j) for $i \neq j$ if the letters i, j overlap in H, and put 0 otherwise.

For a matrix $M \in \mathbb{Z}_2^{n\times n}$ let $R(M)$ be the minimal rank of all the matrices obtained by changing some entries on the main diagonal of M.

Theorem 2.2. *(a) Hieroglyph H is weakly realizable on the Möbius band if and only if $R(M(H)) \leq 1$.*

(b) Hieroglyph H is weakly realizable on the disk with k Möbius bands if and only if $R(M(H)) \leq k$.

Theorem 2.2 is a corollary of [Mo89, Theorem 3.1] (see also [Sk20, §2.8, statement 2.8.8(c)]).

Theorem 2.3 (Kogan, 2021, [Ko21]). *(a) To make a square matrix of rank k out of a square matrix of rank n by changing some diagonal entries, one needs to change at least $|n-k|$ entries.*

(b) For any fixed k there is an algorithm with the complexity of $O(n^{k+3})$ deciding for a matrix $M \in \mathbb{Z}_2^{n\times n}$ whether $R(M) \le k$.

Part (a) is easily implied by the following result.

Lemma 2.4 (subadditivity of rank). *Let P, Q be matrices of the same size with entries in $\mathbb{Z}_2$. Then $\operatorname{rk}(P+Q) \le \operatorname{rk} P + \operatorname{rk} Q$.*

Proof of Theorem 2.3. A matrix is said to be **diagonal** if all its entries outside of the main diagonal are zeros. Any matrix obtained by changing some diagonal elements of a matrix $M \in \mathbb{Z}_2^{n\times n}$ can be uniquely represented as the sum $M+D$, where D is a diagonal matrix.

(a) Take a matrix M of rank n. Take a diagonal matrix Q such that $\operatorname{rk}(M+Q) = k$. Apply Lemma 2.4 for $P = M+Q$ (then $P+Q = M$). We obtain $\operatorname{rk} Q \ge \operatorname{rk} M - \operatorname{rk}(M+Q) = n-k$. Analogously $\operatorname{rk} Q \ge k-n$. Thus Q has at least $|n-k|$ units on the main diagonal.

(b) By (a), for M non-degenerate the inequality $R(M) \le k$ is equivalent to the existence of a diagonal matrix D with at most k zeroes on the main diagonal such that $\operatorname{rk}(M+D) \le k$. The algorithm constructs a non-degenerate matrix $\overline{M}$ from M (using [Ko21]), and then adds to $\overline{M}$ every diagonal matrix with at most k zeroes on the main diagonal. □

See more in [Bi20], [Ko21, Appendix], [Sk20, §2].

3 Modulo 2 embeddings of graphs to surfaces

Denote by S the torus, or sphere with handles, or the Möbius band, or the Klein bottle, or a 2-dimensional surface. Their simple definitions can be found e. g. in §2.1 of [Sk20].

Below graph drawings on S may have self-intersections. An *embedding* (or realization) is a graph drawing without self-intersections.

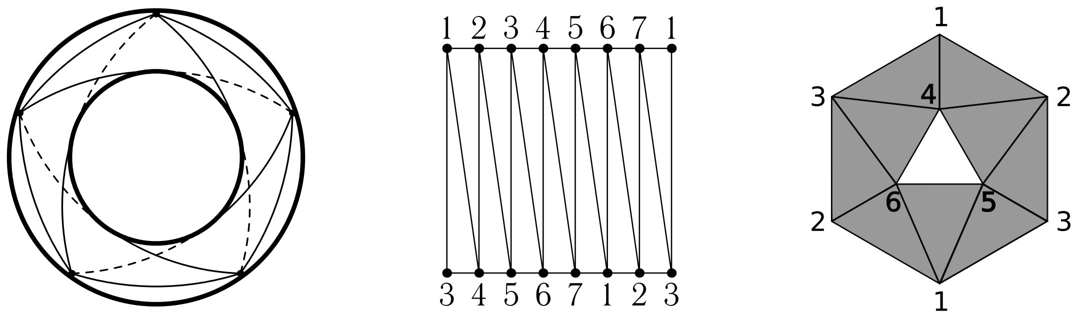

Figure 2: Realization of nonplanar graphs

3.1. *(a) Beautiful realizations of the graphs K_5 and K_7 on the torus are shown in Figure 2, left and middle.*

(b) A beautiful realization of K_6 in the Möbius band is shown in Figure. 2, right.

(c) There is an embedding of K_m in the sphere with some number of handles (depending on m).

Draw the graph K_m in the plane with only double self-intersection points. In a small neighborhood of every double point, attach a handle and lift one of the edges 'bridgelike' over the other edge to the handle, see Figure 3.

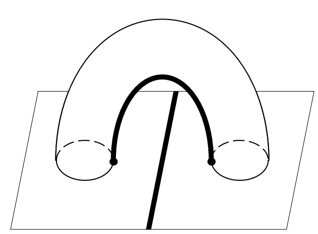

Figure 3: Resolving intersection by adding a handle

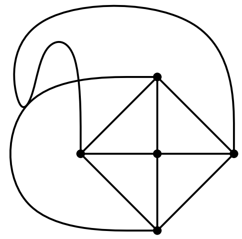

Figure 4: A 'non-general position even drawing' of K_5 in the plane. The drawings (i. e., the images of) every two non-adjacent edges intersect at an even number of points.

A *self-intersection point* of a drawing is a point on the drawing to which corresponds more than one point of the graph itself.

A graph drawing is said to be **general position** if

• to every self-intersection point there corresponds exactly two points of the graph;

• every drawing of a vertex is not a self-intersection point,

• the drawing has finitely many self-intersection points, and

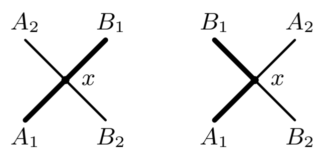

Figure 5: A transverse intersection and a non-transverse intersection

• at every such point the self-intersection is transverse (Figure 5).

A general position graph drawing is a $\mathbb{Z}_2$**-embedding** if the drawings of every two non-adjacent edges intersect at an even number of points. Let S be either the plane or the torus or the Möbius band. If a graph has a $\mathbb{Z}_2$-embedding to S, then the graph embeds to S (Hanani-Tutte; Fulek-Pelsmajer-Schaefer, 2020; Pelsmajer-Schaefer-Stasi, 2009). However, there is a graph having a $\mathbb{Z}_2$-embedding to the sphere with 4 handles but not an embedding in the sphere with 4 handles (Fulek-Kynčl, 2017). See references in [Bi21, Remark 1.3.b,c].

Theorem 3.2 (Fulek-Kynčl, 2019)**.** *If a graph K has a $\mathbb{Z}_2$-embedding to the sphere with g handles, then*

(a) $g \geq (m-5)^2/16$ *for* $K = K_m$.

(b) $g \geq (n-2)^2/4$ *for* $K = K_{n,n}$.

Theorem 3.2 is proved by showing that on a surface to which a large graph has a $\mathbb{Z}_2$-embedding, the intersections of closed curves are sufficiently complicated (in the sense of rank of certain matrix; cf. Assertion 4.1). More precisely,

• the weaker estimation $g \geq (m-4)/3$ for $K = K_m$ [PT19] follows by Theorems 3.3.a and 4.2 together with Assertion 4.1 (all below);

Theorem 3.2.a follows by Theorem 3.2.b (prove!);

• Theorem 3.2.b is proved in [FK19] (see a well-structured exposition in [DS22, §3]).

Analogously, Assertion 4.1 and Theorem 4.2 (together with Theorem 3.3.b) imply the non-$\mathbb{Z}_2$-embeddability of K_7 to the Möbius band. (They also imply the non-embeddability of K_7 to the Klein bottle, which does not follow from the Euler inequality.)

There is an analogous non-embeddabiliy result in higher dimensions (Theorem 6.1).

Denote by $|X|_2 \in \mathbb{Z}_2$ the parity of the number of elements in a finite set X.

Closed curves $\gamma_1, \ldots, \gamma_p$ on S are said to be in **general position** if the graph drawing (of disjoint union of p cycles) formed by this curves is in general position. Their *intersection* $p \times p$*- matrix* G is defined as

$$G_{i,j} := \begin{cases} |\gamma_i \cap \gamma_j|_2, & i \neq j, \\ |\gamma_i \cap \gamma_i'|_2, & i = j, \end{cases}$$

where γ_i' is a curve close to γ_i in general position to γ_i.

Theorem 3.3 (Homology Betti Theorem)**.** *For any closed general position curves* $\gamma_1, \ldots, \gamma_p$ *on*

(a) the sphere with g *handles the rank of their intersection matrix does not exceed* $2g$*.*

(b) the disk with m *Möbius bands the rank of their intersection matrix does not exceed* m*.*

4 Embeddings of complete graphs

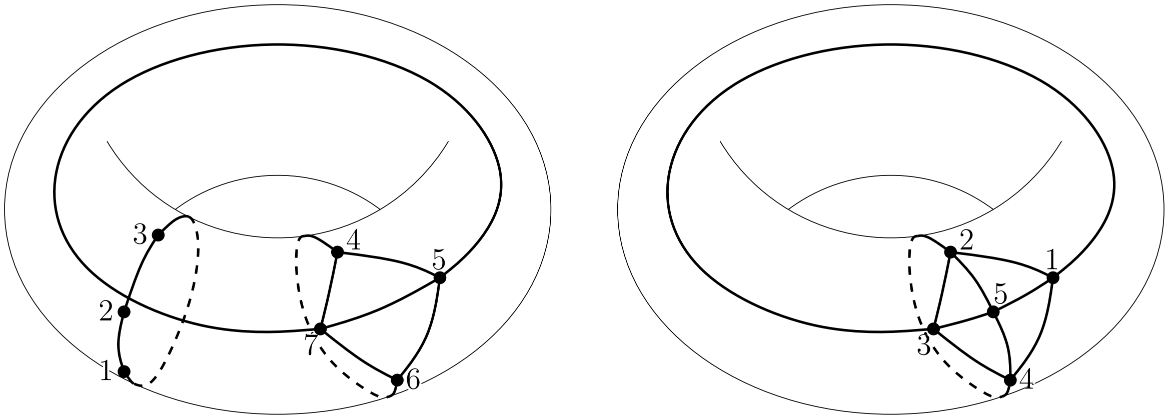

Figure 6: Left: K_3 and K_4 on the torus. Right: K_5 on the torus

4.1. *Take any embedding (or $\mathbb{Z}_2$-embedding) $f\colon K_n \to S$. Take any map $f'\colon K_n \to S$ in general position to f, and close to f. For any pairwise different numbers $i, j, k \in [n]$ denote by $\langle ijk\rangle$ the cycle of length 3 in K_n passing through i, j, k. Let*

$$A_{ijk,pqr} = A_{\{i,j,k\},\{p,q,r\}} = ijk \wedge pqr := \big|f\langle ijk\rangle \cap f'\langle pqr\rangle\big|_2.$$

Then

(4.1.1) $\quad 123 \wedge 456 = 0$, *i.e.,* $A_{123,456} = 0$.

(4.1.2) $\quad 123 \wedge 456 + 123 \wedge 567 + 123 \wedge 467 + 123 \wedge 457 = 0.$

$123 \wedge 345 + 123 \wedge 346 + 123 \wedge 356 + 123 \wedge 456 = 0.$

$123 \wedge 234 + 123 \wedge 235 + 123 \wedge 245 + 123 \wedge 345 = 0.$

$123 \wedge 123 + 123 \wedge 124 + 123 \wedge 134 + 123 \wedge 234 = 0.$

See Figure 6, left. For one formula covering these four formulas see the linear dependence property below. [This follows from $abc \oplus abd \oplus acd \oplus bcd = \emptyset$.]

(4.1.3) $\quad 125 \wedge 345 + 135 \wedge 245 + 145 \wedge 235 = 1.$

See Figure 6, right. This is easily deduced from (B) below.

We denote by $\oplus$ the mod 2 summation (i.e., the symmetric difference) of sets.

Remark. (A) For any pairwise distinct points A_1, A_2, A_3, A_4 in the line there is exactly one 'intertwined' coloring into two colors.

(B) For any pairwise distinct points A_1, A_2, A_3, A_4 on the circle

$$|A_1A_2 \cap A_3A_4| + |A_1A_3 \cap A_2A_4| + |A_1A_4 \cap A_2A_3| = 1.$$

(B') For any 'general position' map $f\colon K_5 \to \mathbb{R}^2$ the number of intersection points in $\mathbb{R}^2$ formed by images of disjoint edges is odd.

A simple deduction of $(A) \Rightarrow (B')$ is presented in [Sk14] (for the linear case; for the PL case the deduction is analogous). Observe that (B') does not follow from Euler formula for planar graphs. Analogously, the non-$\mathbb{Z}_2$-embeddability to surfaces (unlike the non-embeddability) does not follow from the Euler inequality for surfaces [Sk20, §2.4].

We shorten $\{i\}$ to i. An $\binom{[m]}{3}$*-matrix* is a symmetric square matrix with $\mathbb{Z}_2$-entries whose rows and whose columns correspond to all 3-element subsets of $[m]$, and for which the following properties hold:

(triviality) $A_{P,Q} = 0$ if $P \cap Q = \emptyset$;

(linear dependence) for each 4-element and 3-element subsets $F, P \subset [m]$

$$\sum_{i \in F} A_{F-i,P} = 0.$$

(non-triviality) for each $i \in [m]$ and 4-element subset $F \subset [m]-i$ we have $A_{F,i} = 1$, where

$$A_{F,i} := \sum_{\{X,Y\}\ :\ F \cup i = X \cup Y,\ X \cap Y = i,\ |X|=|Y|=3} A_{X,Y}.$$

By Assertion 4.1, an $\binom{[m]}{3}$-matrix is constructed from a $\mathbb{Z}_2$-embedding $f\colon K_m \to S$ to a surface.

Theorem 4.2 (Patak-Tancer, arXiv:1904.02404)**.** *If A is an $\binom{[m]}{3}$-matrix, then* $\operatorname{rk} A \geq \dfrac{m-4}{3}$.

5 Embeddings of complete bipartite graphs

Take any embedding (or $\mathbb{Z}_2$-embedding) $f\colon K_{n,n} \to S$. Take any map $f'\colon K_{n,n} \to S$ in general position to f, and close to f. For 4-cycles $P, Q \subset K_{n,n}$ denote

$$A(f)_{P,Q} := |f(P) \cap f'(Q)|_2 \in \mathbb{Z}_2.$$

The obtained square matrix $A(f)$ is symmetric and has size

$$s = s_{n,1} := \left(\frac{n(n-1)}{2}\right)^2.$$

If f is a $\mathbb{Z}_2$-embedding, then the following properties hold for any 4-cycles $P, Q \subset K_{n,n}$:

(*triviality*) $A_{P,Q} = 0$ if $P \cap Q = \emptyset$;

(*linear dependence*) If $P = X \oplus Y$ for 4-cycles $X, Y \subset K_{n,n}$, then $A_{P,Q} = A_{X,Q} \oplus A_{Y,Q}$.

In other words, if $P = ac'bd'$, $u \in [n]-\{a,b\}$, and $v \in [n]-\{c,d\}$ then

$$A_{P,Q} = A_{ac'ud',Q} + A_{uc'bd',Q} = A_{ac'bv',Q} + A_{av'bd',Q}.$$

[this follows from $ac'bd' = ac'ud' \oplus uc'bd' = ac'bv' \oplus av'bd'$]

(*non-triviality*) If $\{a,b,n\}, \{d,e,n\} \subset [n]$ are 3-element subsets, then

$$S_P A := A_{ad'nn',be'nn'} + A_{ae'nn',bd'nn'} = 1.$$

Theorem 5.1 (Fulek-Kynčl, arXiv:1903.08637)**.** *If A is a matrix with the above properties, then* $\operatorname{rk} A \geq \dfrac{(n-3)^2}{4}$.

Open Problem 5.2. *Find upper and lower estimations on the minimal $m = m(n)$ such that there is an $m \times s_{n,1}$ matrix Y such that the matrix $Y^T Y$ has the above properties.*

Known: $\dfrac{(n-3)^2}{4} \leq m \leq \dfrac{(n-2)^2}{4}$.

6 Quadratic estimation for the Kühnel problem

The classical Heawood inequality states that if the complete graph K_n on n vertices is embeddable into the sphere with g handles, then

$$g \geqslant \frac{(n-3)(n-4)}{12}.$$

Denote by Δ_n^k the union of k-faces of n-simplex.

Denote by S_g the connected sum of g copies of the Cartesian product $S^k \times S^k$ of two k-dimensional spheres.

A higher-dimensional analogue of the Heawood inequality is the Kühnel conjecture [Ku94, Conjecture B], cf. [DS22, Remark 1.2.b]. In a simplified form it states that *for every integer $k > 0$ there is $c_k > 0$ such that if Δ_n^k embeds into S_g, then*

$$g \geqslant c_k n^{k+1}.$$

Theorem 6.1 (Dzhenzher, S, arXiv:2208.04188)**.** *Take any (fixed) integer $k \geqslant 2$. If Δ_n^k PL embeds into S_g, then*

$$g \gtrsim \frac{n^2}{2^{k+1}(k+1)^2}$$

Remark 6.2. *(a) The linear estimate $2g \geqslant \dfrac{n-2k-1}{k+1}$ for Δ_n^k is proved in [PT19] (after a weaker linear estimate of [GMP+]); see [KS21] for simpler exposition.*

(b) The theorem is related to the generalized Heawood inequality [Ku94, Conjecture B], to to the low rank matrix completion problem (and thus to Netflix problem from machine learning), and to Radon and Helly type results [PT19, Theorem 2 and Corollary 3].

The theorem remains correct under the weaker assumption of $\mathbb{Z}_2$-embeddability.

For 2-element subsets $P_1, \dots, P_{k+1} \subset [n]$ define a **k-octahedron** to be $P = (P_1, \dots, P_{k+1})$. Let

$$s = s_{n,k} := \left(\frac{n(n-1)}{2}\right)^{k+1}.$$

An **(n,k)-matrix** is a symmetric matrix $A \in \mathbb{Z}_2^{s\times s}$ whose rows and whose columns correspond to all k-octahedra, and such that the following properties hold for any k-octahedra P, Q:

(*triviality*) $A_{P,Q} = 0$ if $P \cap Q = \emptyset$;

(*linear dependence*) for any $i \in [k+1]$ and 2-element subset $U \subset [n]$ such that $|P_i \cap U| = 1$ we have

$$A_{P,Q} + A_{(P_1,\dots,P_{i-1},U,P_{i+1},\dots,P_{k+1}),\ Q} = A_{(P_1,\dots,P_{i-1},(P_i \oplus U),P_{i+1},\dots,P_{k+1}),\ Q};$$

(In other words, denote $P^* = P_1 * \ldots * P_{k+1}$. If $P^* = X^* \oplus Y^*$ for k-octahedra X, Y, then $A_{P,Q} = A_{X,Q} \oplus A_{Y,Q}$.)

(*non-triviality*) $S_P A = 1$ if $P \subset [n-1]^{k+1}$, where

$$S_P A := \sum_{x \in P_1 \times \ldots \times P_k} A_{*(x_1,\dots,x_k,\min P_{k+1}),\,*(P_1 - x_1,\dots,P_k - x_k,\max P_{k+1})},$$

where for $\nu = (\nu_1, \dots, \nu_{k+1}) \in [n-1]^{k+1}$ we denote $*\nu := (\{\nu_1, n\}, \dots, \{\nu_{k+1}, n\})$.

E.g.

(i) $S_{[2]}A = A_{\{1,n\},\{2,n\}} = A_{*1,*2}$ for $k = 0$;

(ii) $S_{[2]^2}A = A_{*(1,1),\,*(2,2)} + A_{*(2,1),\,*(1,2)}$ for $k = 1$;

(iii) $S_{[2]^3}A = A_{*(1,1,1),\,*(2,2,2)} + A_{*(2,1,1),\,*(1,2,2)} + A_{*(1,2,1),\,*(2,1,2)} + A_{*(2,2,1),\,*(1,1,2)}$ for $k = 2$.

Theorem 6.3 (Dzhenzher, S, arXiv:2208.04188)**.** *If $n \geqslant 4$ and $A \in \mathbb{Z}_2^{s\times s}$ is an (n,k)-matrix, then* $\operatorname{rk} A \geqslant (n-3)^2/2^k$.

Open Problem 6.4. *Find upper and lower estimations on the minimal $m = m(n)$ such that there is an $m \times s_{n,k}$ matrix Y such that the matrix $Y^T Y$ is an (n,k)-matrix. Known:* $\dfrac{(n-3)^2}{2^k} \le m$.

7 Embeddability of general k-complexes to $2k$-manifolds

We present some results which generalize the Bikeev-Fulek-Kynčl-Schaefer-Stefankovič criteria for the $\mathbb{Z}_2$- and $\mathbb{Z}$-embeddability of graphs to surfaces, and which are related to the Harris-Krushkal-Johnson-Paták-Tancer criteria for the embeddability of k-complexes in $2k$-manifolds. For the history see [arXiv:2112.06636, Remarks 1.1.1b and 1.3.7].

Let K be a k-dimensional simplicial complex having n faces of dimension k, and M a closed $(k-1)$-connected PL $2k$-dimensional manifold.

Corollary 7.1 (Kogan, S, arXiv:2112.06636)**.** *For any odd $k \geq 3$ complex K embeds into M if and only if there are*

- *a skew-symmetric $n \times n$-matrix A with $\mathbb{Z}$-entries whose rank over $\mathbb{Q}$ does not exceed* $\operatorname{rk} H_k(M;\mathbb{Z})$,
- *a general position PL map $f : K \to \mathbb{R}^{2k}$, and*
- *a collection of orientations on k-faces of K*

such that for any nonadjacent k-faces σ, τ of K the element $A_{\sigma,\tau}$ equals to the algebraic intersection of $f\sigma$ and $f\tau$.

A general position PL map $h : K \to M$ is called a

- $\mathbb{Z}_2$**-embedding** if $|h\sigma \cap h\tau|$ is even for any pair σ, τ of non-adjacent k-faces.
- $\mathbb{Z}$**-embedding** if $h\sigma \cdot h\tau = 0$ for any pair σ, τ of non-adjacent k-faces (with some orientations).

Theorem 7.2. *(a; van Kampen-Shapiro-Wu, 1932-57) For any $k \geq 3$ a k-complex is PL embeddable into M if and only if the complex is $\mathbb{Z}$-embeddable to M.*

(b; Melikhov, 2006) For any $k \geq 2$ there is a k-complex $\mathbb{Z}_2$-embeddable but not $\mathbb{Z}$-embeddable to $\mathbb{R}^{2k}$.

(c; Fredman-Krushkal-Teichner, 1994) There is a 2-complex $\mathbb{Z}$-embeddable but not embeddable to $\mathbb{R}^4$.

It is unknown if the analogue of (a) is correct for $k = 1$.

Corollary 7.3 (Kogan, S, arXiv:2112.06636). *(a) There is an algorithm checking the $\mathbb{Z}_2$-embeddability of k-complexes to M.*

(b) For any s there is a k-complex having no $\mathbb{Z}_2$-embedding (and so no embedding) to the connected sum of s copies of $S^k \times S^k$. As an example one can take the disjoint union of $s+1$ copies of Δ^k_{2k+2}.

For $R = \mathbb{Z}$ or $\mathbb{Z}_2$ a bilinear form $q : V \times V \to R$ on a $\mathbb{Z}_2$-vector space or $\mathbb{Z}$-module V is called **even** if $q(v, v)$ is even for any $v \in V$, and is **odd** otherwise. A symmetric matrix with $\mathbb{Z}_2$- or $\mathbb{Z}$-entries is **even** (for the case of $\mathbb{Z}_2$ a.k.a. alternate) if its diagonal contains only even entries, and is **odd** otherwise. The **type** of a bilinear form or a symmetric matrix is its being even or odd.

Corollary 7.4 (Kogan, S, arXiv:2112.06636). *The $\mathbb{Z}_2$-embeddability of a given k-complex to M depends only on the rank and the type of the modulo 2 intersection form of M.*

Corollaries 7.3 and 7.4 are deduced from Theorem 7.5 below. Corollary 7.1 is deduced from the ($\mathbb{Z}$-version of) Theorem 7.5.

Theorem 7.5 (Kogan, S, arXiv:2112.06636). *The complex K has a $\mathbb{Z}_2$-embedding to M if and only if there are*

- *a symmetric square $n \times n$-matrix A with $\mathbb{Z}_2$-entries, and*
- *a general position PL map $f : K \to \mathbb{R}^{2k}$*

such that

- $A_{\sigma,\tau} = |f\sigma \cap f\tau| \mod 2$ *for any non-adjacent k-dimensional faces σ, τ of K,*
- $\operatorname{rk} A$ *does not exceed the rank of the modulo 2 intersection form $\cap_M$,*
- *A has only zeros on the diagonal, if $\cap_M$ is even,*
- *A has a non-zero diagonal entry, if $\cap_M$ is odd.*

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