Low rank matrix completion

presented by S. Dzhenzher, T. Garaev, O. Nikitenko,
A. Petukhov, A. Skopenkov, A. Voropaev*

Contents

1 Motivation and some main results

Remark (motivation; formally is not used later)

'Matrix completion is the task of filling in the missing entries of a partially observed matrix... One example is the movie-ratings matrix, as appears in the Netflix problem: Given a ratings matrix in which each entry (i, j) represents the rating of movie j by customer i, if customer i has watched movie j and is otherwise missing, we would like to predict the remaining entries in order to make good recommendations to customers on what to watch next...' [MC] The remaining entries are predicted so as to minimize the *rank* of the completed matrix. All the required definitions (of rank etc.) are given below. For a brief overview of the history of this and related problems, see [MC, NKS], [Ko21, Remark 4].

Here for simplicity we consider matrices with entries in the set $\mathbb{Z}_2 = \{0, 1\}$ of all residues modulo 2 (with the sum and product operations). We present interesting elementary results in linear algebra. These results allow us to construct algorithms estimating minimal rank of partial matrices for the particular case of filling the diagonal (Proposition 1.1 and Theorems

*S. Dzhenzher, A. Skopenkov: Moscow Institute of Physics and Technology. T. Garaev: Moscow State University. O. Nikitenko: Altay Technical University (Barnaul). A. Petukhov: Institute for Information Transmission Problems (Moscow). A. Skopenkov: Independent University of Moscow, https://users.mccme.ru/skopenko/. A. Voropaev: (Moscow).

We are grateful to E. Kogan for allowing us to use his text [Ko21], to V. Retinskiy and Ya. Abramov for useful discussions, to D. Deomidov and F. Nilov for translating parts of the text, to A. Ryabichev and MCCME publishing house for allowing us to use figures they prepared.

1.3, 1.4, see also Proposition 1.2). Then we consider a more complicated problem. Instead of relations $M_{ij} = a_{ij}$ for some elements M_{ij} of matrix M (where a_{ij} are given numbers) we consider more complicated relations on matrix elements. We estimate minimal rank of matrices with such relations (Theorems 5.1, 7.1.)

These results have applications to embeddings of graphs in surfaces (including embedding modulo 2, see §4), and of k-dimensional 'hypergraphs' to $2k$-dimensional surfaces, see [KS21, KS21e, DS22]. In particular, Theorems 1.3 and 1.4 give polynomial (in the number of edges) algorithms recognizing 'weak realizability' of 'graphs with rotations' on non-orientable surfaces (see the remark on weak realizability below).

Denote by $\mathbb{Z}_2^{s\times n} = (\mathbb{Z}_2^s)^n$ the set of all $s \times n$ matrices with entries in $\mathbb{Z}_2$.

Proposition 1.1. *(a) For a symmetric matrix with $\mathbb{Z}_2$-entries the following conditions are equivalent:*

• *some entries on the main diagonal can be changed so that in the resulting matrix all non-zero rows are equal;*

• *it is impossible to make the same permutation of rows and of columns*[1] *so that the upper left square will be one of the submatrices*

$$\begin{pmatrix} * & 1 & 1 \\ 1 & * & 0 \\ 1 & 0 & * \end{pmatrix} \quad or \quad \begin{pmatrix} * & 1 & 0 & 0 \\ 1 & * & 0 & 0 \\ 0 & 0 & * & 1 \\ 0 & 0 & 1 & * \end{pmatrix},$$

*where by * are denoted arbitrary (possibly different) elements.*

(b) There is an algorithm with the complexity of $O(n^2)$ deciding for a matrix $M \in \mathbb{Z}_2^{n\times n}$ whether some entries on the main diagonal can be changed so that in the resulting matrix all non-zero rows are equal.

You can submit separately your solution of the 'only if' implication of part (a).

The algorithmic results in this text could be omitted by theoretically-minded students because they are easy corollaries of mathematical results. The *complexity* of an algorithm is the number 'elementary' steps in this algorithm. An algorithm has complexity $O(f(n))$ if there is $C > 0$ such that the complexity does not exceed $Cf(n)$ for any n.

A square matrix $M \in \mathbb{Z}_2^{n\times n}$ is called **degenerate** if the sum of its several columns (a non-zero number of columns) is the zero column (i. e., the column consisting of zeros only). A matrix is called **non-degenerate** otherwise. Introductory problems on degenerate matrices useful for the following result are presented in §2.

Proposition 1.2. *(a) For any matrix $M \in \mathbb{Z}_2^{n\times n}$ some entries on the main diagonal can be changed so that the resulting matrix would be degenerate.*

(b) The same with 'degenerate' replaced by 'non-degenerate'.

The **rank** $\operatorname{rk} M$ of a matrix $M \in \mathbb{Z}_2^{s\times n}$ is the maximal number of columns of M none of whose sums is zero. (This is the 'dimension' of the 'vector space' formed by the columns of the matrix.) Introductory problems on rank useful for the following results are presented in §3.

For a matrix $M \in \mathbb{Z}_2^{n\times n}$ let $R(M)$ be the minimal rank of all the matrices obtained by changing some entries on the main diagonal of M.

A matrix is said to be **diagonal** if all its entries outside of the main diagonal are zeros.

[1] This means that the rows and columns are numbered by $1, \dots, n$ (where $n = 3, 4$) and the permutation of the set $[n]$ is applied both to the rows and to the columns.

Theorem 1.3. *(a') To make a square matrix of rank k out of a square matrix of rank n by changing some diagonal entries, one needs to change at least $|n-k|$ entries.*

(a) For any non-degenerate matrix $M \in \mathbb{Z}_2^{n\times n}$ the inequality $R(M) \le k$ is equivalent to the existence of a diagonal matrix D with at most k zeroes on the main diagonal such that $\operatorname{rk}(M+D) \le k$.

(b) For any fixed k there is an algorithm with the complexity of $O(n^{k+3})$ deciding for a matrix $M \in \mathbb{Z}_2^{n\times n}$ whether $R(M) \le k$.

The **identity matrix** E is the diagonal matrix whose diagonal elements are units.

Theorem 1.4. *(a) For any non-degenerate matrix $M \in \mathbb{Z}_2^{n\times n}$ and diagonal matrix $D \in \mathbb{Z}_2^{n\times n}$ we have $2\operatorname{rk}(M+D) \ge \operatorname{rk}(M+E)$.*

(b) There is an algorithm with the complexity of $O(n^4)$ calculating for a matrix $M \in \mathbb{Z}_2^{n\times n}$ a number k such that $k/2 \le R(M) \le k$.

Remark (weak realizability; formally is not used later)

A *hieroglyph* on n letters is an unoriented cyclic letter sequence of length $2n$ such that each letter from the sequence appears in the sequence twice.

Take a hieroglyph on n letters. Take a convex polygon with $2n$ sides. Put the letters in the hieroglyph on the sides of the convex polygon in the nonoriented cyclic order. For each letter glue the ends of a ribbon to the pair of sides corresponding to the letter so that the glued ribbons are pairwise disjoint. The ribbons can be either twisted or not twisted. Call the resulting surface a *disk with ribbons* corresponding to the hieroglyph (see Figure 1).

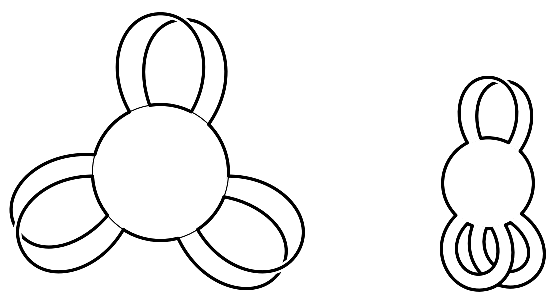

Figure 1: Disk with ribbons corresponding to the hieroglyph *aabbcc* (left) and *aabcbc* (right)

A hieroglyph H is called *weakly realizable* on the Möbius band if some disk with ribbons corresponding to H can be cut out of the Möbius band. Analogously one defines weak realizability on the Klein bottle and other non-orientable surfaces.

Two letters a, b in a hieroglyph H *overlap in* H if they interlace in the cyclic sequence of the hieroglyph (i. e., if they appear in the sequence in the order *abab* but not *aabb*). Define the *overlap matrix* $M(H) \in \mathbb{Z}_2^{n\times n}$ of a hieroglyph H as follows. Put zeros on the main diagonal. Put 1 in the cell (i,j) for $i \ne j$ if the letters i, j overlap in H, and put 0 otherwise.

Hieroglyph H is weakly realizable on the Möbius band if and only if $R(M(H)) \le 1$.

See more in [Bi20], [Ko21, Appendix], [Sk20, §2].

Recommendations for participants

If a mathematical statement is formulated as a problem, then the objective is to prove this statement. (Open-ended questions are called challenges or riddles; here one must come up with a clear wording, and a proof.) If a problem is named 'theorem' ('lemma', 'corollary', etc.), then this statement is considered to be more important. Usually we *formulate* beautiful or important statement *before* giving a sequence of results (lemmas, assertions, etc.) which constitute its *proof*. We give hints on that after the statements but we do not want to deprive you of the pleasure of finding the right moment when you finally are ready to prove the statement. In general, if you are stuck on a certain problem, try looking at the next ones. They may turn out to be helpful. *Remarks* and problems marked by star are not used in the sequel. Important definitions are highlighted in **bold** for easy navigation. You are welcomed to *consult* the jury on any questions on the project. If you successfully work on the project, you can get interesting *extra problems*.

For every solution **written for a user** marked with either '+' or '+.' a student (or a group of students) gets a 'bean' (see recommendations in p. 3, 'How to write a proof for a user' of https://www.mccme.ru/circles/oim/multicomb.pdf). The jury may also award extra beans for beautiful solutions, solutions of hard problems, or solutions typeset in TEX. The jury has infinitely many beans. Every participant (or team) initially has 1 bean. You may submit a solution **in oral form** or as **written for a developer**; you lose a bean with every 5 attempts (successful or not).

Please notify us if you already know solutions of several problems. If you confirm your knowledge by presenting some solutions, you will be allowed not to receive plus-marks for the problems, but to use them in solutions of other problems.

(In the project the tasks were separated as follows: the tasks before and including Theorem 4.3, then some solutions to simple tasks from §§1–2, then other tasks and solutions.)

2 Degenerate matrices

2.1. *(a1-a4) Which of the following matrices are degenerate?*

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.2. *(a) Degeneracy is not changed under permutation of columns (rows).*

(b) Degeneracy is not changed under adding one column (row) to another.

(c) Any matrix can be changed to a diagonal matrix by transformations from (a,b).

(d) A matrix is degenerate if and only if it cannot be changed by transformations from (a,b) to the identity matrix.

(f) A square matrix is degenerate if and only if the sum of its several rows (a non-zero number of rows) is the zero row.

(g) There is an algorithm with the complexity of $O(n^3)$ checking the degeneracy of an $n \times n$ matrix.

For a matrix $M \in \mathbb{Z}_2^{n\times n}$ define $\det M := 0$ if M is degenerate, and $\det M := 1$ otherwise. This is called the *determinant* of M. Another notation is

$$\det \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} = \begin{vmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{vmatrix}, \quad \det M = \begin{vmatrix} M_{1,1} & \dots & M_{1,n} \\ \vdots & \ddots & \vdots \\ M_{n,1} & \dots & M_{n,n} \end{vmatrix}.$$

2.3. *(a)* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + bc.$

(b) $\det(a_1 + b_1, a_2, \dots, a_n) = \det(a_1, a_2, \dots, a_n) + \det(b_1, a_2, \dots, a_n)$. *Here and below $a_j, b_1 \in \mathbb{Z}_2^n$ are columns of length n.*

(c) $\det(a_1,\dots,a_n) = \sum\limits_{i=1}^n a_{i,n}\det(a_1^-,\dots,a_{i-1}^-,a_{i+1}^-,\dots,a_n^-)$, *where every column* $a_i^- \in \mathbb{Z}_2^{n-1}$ *is obtained from the column* a_i *by deleting the last coordinate.*

(d)* $\det M = \sum\limits_{\sigma\in S_n}\prod\limits_{i=1}^n M_{i,\sigma(i)}$, *where* S_n *is the set of all permutations (i. e., 1–1 correspondences)* $\sigma : [n] \to [n]$.

2.4. *(a1-a4) For every matrix of Problem 2.1 find if some entries on the main diagonal can be changed so that the resulting matrix would be degenerate.*

(b1-b4) The same with 'degenerate' replaced by 'non-degenerate'.

Lemma 2.5. *(a) Let* $M \in \mathbb{Z}_2^{n\times n}$ *be a matrix with zeros on the main diagonal. Define the sequence* $M^{(i)}$, $i = 0,1,2,\dots,n$ *recursively as follows:*

- $M^{(0)} := M$, *and*
- $M^{(i)}$ *is the result of replacing in* $M^{(i-1)}$ *the element* $M^{(i-1)}_{i,i} = 0$ *by* $1+\delta_i$, *where* $\delta_0 = 0$ *and* $\delta_i := \det M^{(i-1)}_{[i]\times[i]}$ *is the determinant of the left upper* $i\times i$*-corner submatrix of* $M^{(i-1)}$.

Then the matrix M_n *is non-degenerate.*

(b) There is an algorithm with the complexity of $O(n^4)$ *which for a matrix* $M \in \mathbb{Z}_2^{n\times n}$ *finds some numbers from* $\mathbb{Z}_2$ *to replace the entries on the main diagonal of* M *so that the resulting matrix is non-degenerate.*

3 The rank of a matrix

3.1. *(a1-a4) Find* $\operatorname{rk} M$ *for*

$$M = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix},\quad \begin{pmatrix}1&1&1&0\\1&0&1&0\\1&1&1&1\end{pmatrix},\quad \begin{pmatrix}0&1&1&1\\0&0&1&1\\1&0&0&1\end{pmatrix}.$$

(b1-b4) Find $R(M)$ *for the matrices from Problem 2.1.*

3.2. *Take a matrix* $M \in \mathbb{Z}_2^{s\times n}$.

(a) One can choose $\operatorname{rk} M$ *columns of* M *such that every column is the sum of some chosen columns.*

(b) Assume that there are k *columns (not necessarily of* M*) such that every column of* M *is the sum of some of them. Then* $\operatorname{rk} M \le k$.

(c) The rank of a submatrix does not exceed the rank of a matrix.

3.3. *(a) A permutation of columns (or of rows) does not change the rank of a matrix.*

(b) Adding one column to another one (or one row to another one) does not change the rank of a matrix.

(c) The rank of a matrix equals to the maximal number of its rows none of whose sums is zero.

(d) The rank of a matrix equals to the maximal size of its non-degenerate square submatrix.

A square matrix with $\mathbb{Z}_2$-entries is called **even** if all the entries on the main diagonal are zeros.

3.4. *(a) For* $M \in \mathbb{Z}_2^{s\times n}$ *all non-zero rows are equal if and only if* $\operatorname{rk} M \le 1$.

(b) For a symmetric matrix $M \in \mathbb{Z}_2^{n\times n}$ *all non-zero rows are equal if and only if by same permutation of rows and of columns it is possible to obtain a matrix whose upper left square is filled by ones, and all other elements are zeros.*

(c) The rank of any non-zero symmetric even matrix is greater than one.

3.5. *The number* $R(M)$ *is not necessarily preserved by*

(a) permutation of columns;

(b) adding one column to another one.

3.6. *(a) There is an algorithm with the complexity of* $O(n^3)$ *which calculates the rank of a matrix from* $\mathbb{Z}_2^{s\times n}$, $s \le n$.

(b) For a fixed integer* k *there is an algorithm with the complexity of* $O(n^2)$ *deciding for* $M \in \mathbb{Z}_2^{s\times n}$, $s \le n$, *whether* $\operatorname{rk} M \le k$.

Lemma 3.7. *Let* M, D *be matrices of the same size with entries in* $\mathbb{Z}_2$. *Then*

(a) $\operatorname{rk}(M + D) \le \operatorname{rk} M + \operatorname{rk} D$;

(b) $\operatorname{rk}(M + D) \ge \operatorname{rk} M - \operatorname{rk} D$.

3.8. *There is an algorithm with the complexity of* $O(n^{k+3})$ *finding for* $M \in \mathbb{Z}_2^{n\times n}$ *a diagonal matrix* D *such that*

(a) $\operatorname{rk}(M + D) \le k$; *(b)* $\operatorname{rk}(M + D) = k$

under the assumption that such a matrix D *exists.*

3.9. *Is it correct that for any* $m, k \le n$ *and a matrix* $M \in \mathbb{Z}_2^{n\times n}$ *of rank* m, *if a matrix of rank* k *can be obtained by changing some entries on the diagonal of* M, *then this can be done by changing exactly* $|m - k|$ *entries?*

3.10. *(a,b,c) Find the number of matrices of rank* k *in* $\mathbb{Z}_2^{n\times n}$ *for* $k = 0, 1, 2$.

4 Modulo 2 embeddings of graphs to surfaces

This section is formally not used later, but serves as an additional motivation for §5.

Denote by S the torus, or sphere with handles, or the Möbius band, or the Klein bottle, or a 2-dimensional surface. Their simple definitions can be found e. g. in §2.1 of

[Sk20]=https://www.mccme.ru/circles/oim/obstructeng.pdf

Below graph drawings on S may have self-intersections. An *embedding* is a graph drawing without self-intersections.

4.1. *There are embeddings (a1,a2,a3) of* K_5, K_6, K_7 *in the torus;*

(b1,b2) of K_5, K_6 *in the Möbius band;*

(c) of K_8 *in the sphere with two handles;*

(d) of K_m *in the sphere with some number of handles (depending on* m*).*

Remark. *In this problem you need to give not rigorous proofs, but large, comprehensible, and preferably beautiful pictures.*

4.2. *The graph* K_5 *can be drawn in the plane so that the drawings (i. e., the images of) every two non-adjacent edges intersect at an even number of points.*

A *self-intersection point* of a drawing is a point on the drawing to which corresponds more than one point of the graph itself.

A graph drawing is said to be **general position** if

• to every self-intersection point there corresponds exactly two points of the graph;
• every drawing of a vertex is not a self-intersection point,
• the drawing has finitely many self-intersection points, and

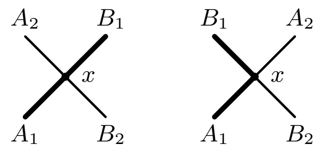

Figure 2: A transverse intersection and a non-transverse intersection

• at every such point the self-intersection is transverse (Figure 2)[2].

A general position graph drawing is a $\mathbb{Z}_2$**-embedding** if the drawings of every two non-adjacent edges intersect at an even number of points.

Remark. Let S be either the plane or the torus or the Möbius band. If a graph has a $\mathbb{Z}_2$-embedding to S, then the graph has an embedding in S. However, there is a graph having a $\mathbb{Z}_2$-embedding to the sphere with 4 handles but not an embedding in the sphere with 4 handles. See references in [Bi21, Remark 1.3.b,c].

Theorem 4.3. *If graph K has a $\mathbb{Z}_2$-embedding to the sphere with g handles, then*
(a) $g \geq (m-4)/3$ for $K = K_m$.
(b) $g \geq (m-5)^2/16$ for $K = K_m$.
(c) $g \geq (n-2)^2/4$ for $K = K_{n,n}$.

Theorem 4.3 is proved by showing that on a surface to which a large graph has a $\mathbb{Z}_2$-embedding, the intersections of closed curves are sufficiently complicated (in the sense of rank of certain matrix; cf. Assertion 4.5). More precisely, Theorem 4.3.a [PT19] follows by Theorems 4.4 and 5.1.b together with Assertion 4.5 (all below). Theorem 4.3.b follows by Theorem 4.3.c (prove!). Theorem 4.3.c is proved in [FK19], see a well-structured exposition in [DS22]. Analogously, Assertions 4.5 and 5.2 (together with Theorem 4.4 and its non-orientable analogue) imply the non-$\mathbb{Z}_2$-embeddability of K_8 to the torus, and of K_7 to the Möbius band (or even of K_7 to the Klein bottle; the non-embeddability of K_7 to the Klein bottle does not follow from the Euler inequality). Analogous non-embeddabiliy result in higher dimensions follows by Theorem 7.1.

Denote by $|X|_2 \in \mathbb{Z}_2$ the parity of the number of elements in a finite set X.

Closed curves $\gamma_1, \dots, \gamma_p$ on S are said to be in **general position** if the graph drawing (of disjoint union of p cycles) formed by this curves is in general position. Their *intersection $p \times p$- matrix* G is defined as

$$G_{i,j} := \begin{cases} |\gamma_i \cap \gamma_j|_2, & i \neq j, \\ |\gamma_i \cap \gamma_i'|_2, & i = j, \end{cases}$$

[2]Strictly speaking, the transversality is only easy to define for PL (piecewise-linear) graph drawings. PL curves on the torus can be easily defined by regarding the torus as obtained from a rectangle by gluing. A *PL curve on the torus* is then a family of polygonal lines in the rectangle satisfying certain conditions (work out these conditions!). In a similar way, other surfaces S can be obtained from plane polygons by gluing. (For Möbius band and Klein bottle see [Sk20, §2.1]; for spheres with handles see visualization in `https://www.youtube.com/watch?v=G1yyfPShgqw` and in `https://www.youtube.com/watch?v=U5N5mg3MePM`.) This allows one to define PL curves on S. A graph drawing on S is called *PL* if the drawing of every edge is PL. Another formalizations are given in [Sk20, §4, §5].

where γ_i' is a curve close to γ_i in general position to γ_i. Please use the following 'Homology Betti Theorem' without proof.

Theorem 4.4. *For any closed general position curves $\gamma_1, \ldots, \gamma_p$ on*

(a) the sphere with g handles the rank of their intersection matrix does not exceed $2g$.

(b) the disk with m Möbius bands the rank of their intersection matrix does not exceed m.

Here the *disk with k Möbius bands* is the figure shown on the left of fig. 1. More precisely, the *disk with m Möbius bands* is the union of a disk and m pairwise disjoint ribbons having their ends glued to $2m$ pairwise disjoint arcs on the boundary circle of the disk (the ribbons do not have to lie in the plane of the disk) so that

- the orientations of the ends of each ribbon given by an orientation of the boundary circle of the disk have 'the same direction along the ribbon', and
- the ribbons are 'separated', i. e. there are m pairwise disjoint arcs A_i on the boundary circle of the disk such that the ends of the i-th ribbon are glued to two disjoint arcs contained in A_i, $i = 1, 2, \ldots, m$.

You can make approximations by general position drawings at an intuitive level.

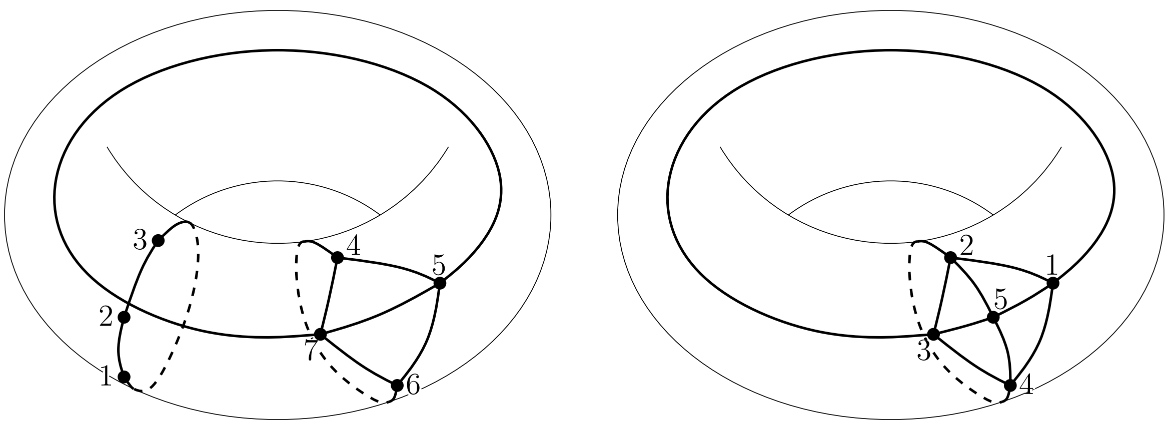

Figure 3: Left: K_3 and K_4 on the torus. Right: K_5 on the torus

4.5. *Take any embedding (or $\mathbb{Z}_2$-embedding) $f\colon K_n \to S$. Take any map $f'\colon K_n \to S$ in general position to f, and close to f. For any pairwise different numbers $i, j, k \in [n]$ denote by $\langle ijk\rangle$ the cycle of length 3 in K_n passing through i, j, k. Let*

$$ijk \wedge pqr := \left|f\langle ijk\rangle \cap f'\langle pqr\rangle\right|_2.$$

Then

(4.5.1) $123 \wedge 456 = 0$.

(4.5.2) $123 \wedge 456 + 123 \wedge 567 + 123 \wedge 467 + 123 \wedge 457 = 0$.
$123 \wedge 345 + 123 \wedge 346 + 123 \wedge 356 + 123 \wedge 456 = 0$.
$123 \wedge 234 + 123 \wedge 235 + 123 \wedge 245 + 123 \wedge 345 = 0$.
$123 \wedge 123 + 123 \wedge 124 + 123 \wedge 134 + 123 \wedge 234 = 0$.

See Figure 3, left. For one formula covering these four formulas see the linear dependence property in §5.

(4.5.3) $125 \wedge 345 + 135 \wedge 245 + 145 \wedge 235 = 1$.

See Figure 3, right. Hint: deduce from (B) below.

Remark. (A) For any pairwise distinct points A_1, A_2, A_3, A_4 in the line there is exactly one 'intertwined' coloring into two colors.

(B) For any pairwise distinct points A_1, A_2, A_3, A_4 on the circle

$$|A_1A_2 \cap A_3A_4| + |A_1A_3 \cap A_2A_4| + |A_1A_4 \cap A_2A_3| = 1.$$

(B') For any 'general position' map $f\colon K_5 \to \mathbb{R}^2$ the number of intersection points in $\mathbb{R}^2$ formed by images of disjoint edges is odd.

A simple deduction of $(A) \Rightarrow (B')$ is presented in [Sk14] (for the linear case; for the PL case the deduction is analogous). Observe that (B') does not follow from Euler formula for planar graphs. Analogously, the non-$\mathbb{Z}_2$-embeddability to surfaces (unlike the non-embeddability) does not follow from the Euler inequality for surfaces [Sk20, §2.4].

5 Rank of matrix with relations

We shorten $\{i\}$ to i. An $\binom{[m]}{3}$*-matrix* is a symmetric square matrix with $\mathbb{Z}_2$-entries whose rows and whose columns correspond to all 3-element subsets of $[m]$, and for which the following properties hold:

(triviality) $A_{P,Q} = 0$ if $P \cap Q = \emptyset$;

(linear dependence) for each 4-element and 3-element subsets $F, P \subset [m]$

$$\sum_{i\in F} A_{F-i,P} = 0.$$

(non-triviality) for each $i \in [m]$ and 4-element subset $F \subset [m] - i$ we have $A_{F,i} = 1$, where

$$A_{F,i} := \sum_{\{X,Y\}\ :\ F\cup i = X\cup Y,\ X\cap Y = i,\ |X|=|Y|=3} A_{X,Y} = \sum_{\{\sigma,\tau\}\ :\ F=\sigma\sqcup\tau,\ |\sigma|=|\tau|=2} A_{i\sqcup\sigma, i\sqcup\tau}.$$

By Assertion 4.5, an $\binom{[m]}{3}$-matrix is constructed from a $\mathbb{Z}_2$-embedding $f\colon K_m \to S$ to a surface. Indeed, set $A_{\{i,j,k\},\{p,q,r\}} := ijk \wedge pqr$. If the surface S is orientable, then the constructed matrix A is even (i. e., $A_{P,P} = 0$ for each 3-element subset $P \subset [m]$).

Theorem 5.1. *(a) If A is an $\binom{[m]}{3}$-matrix, then* $\operatorname{rk} A \ge \dfrac{m-4}{3}$.

(b) If, moreover, A is even, then $\operatorname{rk} A \ge \dfrac{2(m-4)}{5}$.

You can submit separately proofs of the following particular cases of Theorem 5.1. For more strong estimates see §7.

5.2. *(a)* There are no $\binom{[7]}{3}$-matrices of rank* 1.

(b) There are no even $\binom{[8]}{3}$-matrices of rank smaller than* 3.

You can deduce Theorem 5.1 from Proposition 5.7.a,b.

The following assertion is not used in the sequel. In its proof you do not need to explicitly give the matrix, just describe the construction. We only know a proof using Assertions 4.1, 4.5, and Theorem 4.4.

5.3. *(a) There is a non-zero* $\binom{[4]}{3}$*-matrix.*

(b) There is a $\binom{[5]}{3}$*-matrix.*

(c) For any $m \ge 5$ *there is an* $\binom{[m]}{3}$*-matrix.*

(a',b',c') The same for even matrices.

(d) There is a $\binom{[5]}{3}$*-matrix of rank* 1*.*

(e) There is an even $\binom{[5]}{3}$*-matrix of rank* 2*.*

(f) There is a $\binom{[6]}{3}$*-matrix of rank* 1*.*

(g) There is an* $\binom{[8]}{3}$*-matrix of rank greater than* 2*.*

5.4. *Let* A' *be the square matrix of size* $\binom{m-1}{3}$ *obtained from an* $\binom{[m]}{3}$*-matrix by deleting rows and columns corresponding to all subsets containing* m*. Then* A' *is an* $\binom{[m-1]}{3}$*-matrix.*

5.5. *(a) Let* B *be the square matrix of size* $\binom{m-3}{3}$ *obtained from an* $\binom{[m]}{3}$*-matrix* A *by deleting rows and columns corresponding to subsets containing at least one element of* $X := \{m, m-1, m-2\}$*. If* $A_{X,X} = 1$*, then* $\operatorname{rk} A > \operatorname{rk} B$*.*

(b) Let C *be the square matrix obtained from an* $\binom{[m]}{3}$*-matrix* A *by deleting rows and columns corresponding to subsets containing at least one element of certain* 3*-element subsets* $X, Y \subset [m]$*. If* $A_{X,X} = A_{Y,Y} = 0$ *and* $A_{X,Y} = 1$*, then* $\operatorname{rk} A \ge \operatorname{rk} C + 2$*.*

Denote by r_m the minimal rank of an $\binom{[m]}{3}$-matrix. Denote by $\widetilde{r_m}$ the minimal rank of an even $\binom{[m]}{3}$-matrix. Clearly, $r_m = \widetilde{r_m} = 0$ for $m \le 4$, and $r_m \le \widetilde{r_m}$. The non-triviality implies that $r_5, \widetilde{r_5} \ge 1$. Theorem 5.1 asserts that $r_m \ge \dfrac{m-4}{3}$ and $\widetilde{r_m} \ge \dfrac{2(m-4)}{5}$.

5.6. *(a,b) Find* r_5, r_6 *and* $\widetilde{r}_5, \widetilde{r}_6, \widetilde{r}_7$*.*

(c) Both sequences $r_m, \widetilde{r_m}$ *are non-decreasing.*

Proposition 5.7. *(a)* $r_m \ge \min\{r_{m-3}+1, \widetilde{r_m}\}$ *(more precisely, either* $r_m = \widetilde{r_m}$ *or* $r_m \ge r_{m-3}+1$*);*

(b) $\widetilde{r_m} \ge \widetilde{r_{m-5}} + 2$*.*

6 Classification of symmetric bilinear forms

Fix a symmetric matrix $A \in \mathbb{Z}_2^{n\times n}$. For $U, V \in \mathbb{Z}_2^n$ let

$$A(U,V) = U \cdot_A V := \sum_{i,j=1}^{n} A_{i,j} U_i V_j \quad (= U^T A V).$$

A **basis** of $\mathbb{Z}_2^n$ is an inclusion-minimal ordered set of vectors such that every vector from $\mathbb{Z}_2^n$ is the sum of some vectors from this set.

Theorem 6.1. *For* $n = 2$ *there is a basis* X_1, X_2 *of* $\mathbb{Z}_2^2$ *and numbers* $\gamma_1, \gamma_2 \in \mathbb{Z}_2$ *such that either*

(i) for any $a_1, a_2, b_1, b_2 \in \mathbb{Z}_2$ *we have*

$$(a_1X_1 + a_2X_2) \cdot_A (b_1X_1 + b_2X_2) = \gamma_1 a_1 b_1 + \gamma_2 a_2 b_2, \quad \textit{or}$$

(ii) for any $a_1, a_2, b_1, b_2 \in \mathbb{Z}_2$ *we have*

$$(a_1X_1 + a_2X_2) \cdot_A (b_1X_1 + b_2X_2) = a_1 b_2 + a_2 b_1.$$

Recall that problems stated after theorems are hints to proofs of the theorems.

6.2. *Assume that $n = 2$, $X \in \mathbb{Z}_2^2$ and $X \cdot_A X = 1$.*

(a) For any $P \in \mathbb{Z}_2^2$ there is $\lambda_{X,P} \in \mathbb{Z}_2$ such that for $P_X := P + \lambda_{X,P} X$ we have $P_X \cdot_A X = 0$.

(b) There is a basis $X_1 = X$, X_2 of $\mathbb{Z}_2^2$ and numbers $\gamma_1 = 1, \gamma_2 \in \mathbb{Z}_2$ such that the property (i) of Theorem 6.1 holds.

6.3. *Assume that $n = 2$, $X, Y \in \mathbb{Z}_2^2$ and $X \cdot_A Y = 1$, $X \cdot_A X = Y \cdot_A Y = 0$. Then $X_1 := X$, $Y_1 := Y$ is a basis of $\mathbb{Z}_2^2$ such that the property (ii) of Theorem 6.1 holds.*

Theorem 6.4. *For $n = 3$ there is a basis X_1, X_2, X_3 of $\mathbb{Z}_2^3$ and numbers $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}_2$ such that either*

(i) for any $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{Z}_2$ we have

$$(a_1X_1 + a_2X_2 + a_3X_3) \cdot_A (b_1X_1 + b_2X_2 + b_3X_3) = \gamma_1 a_1 b_1 + \gamma_2 a_2 b_2 + \gamma_3 a_3 b_3, \quad \text{or}$$

(ii) for any $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{Z}_2$ we have

$$(a_1X_1 + a_2X_2 + a_3X_3) \cdot_A (b_1X_1 + b_2X_2 + b_3X_3) = a_1b_2 + a_2b_1 + \gamma_3 a_3 b_3.$$

6.5. *Assume that $X, Y \in \mathbb{Z}_2^3$ and $X \cdot_A Y = 1$, $X \cdot_A X = Y \cdot_A Y = 0$.*

(a) For any $P \in \mathbb{Z}_2^3$ there are $\lambda_{X,Y,P}, \lambda_{Y,X,P} \in \mathbb{Z}_2$ such that for $P_{X,Y} := P + \lambda_{X,Y,P} Y + \lambda_{Y,X,P} X$ we have $P_{X,Y} \cdot_A X = P_{X,Y} \cdot_A Y = 0$.

(b) There is a basis $X_1 = X$, $X_2 = Y$, X_3 of $\mathbb{Z}_2^3$ and a number $\gamma_3 \in \mathbb{Z}_2$ such that the property (ii) of Theorem 6.4 holds.

Theorem 6.6. *There are k, l and a basis $X_1, Y_1, \ldots, X_k, Y_k, Z_1, \ldots, Z_{n-2k}$ of $\mathbb{Z}_2^n$ such that $2k + l \le n$ and for any $a, a', b, b' \in \mathbb{Z}_2^k$ and $c, c' \in \mathbb{Z}_2^{n-2k}$ we have*

$$(a_1X_1 + b_1Y_1 + \ldots + a_kX_k + b_kY_k + c_1Z_1 + \ldots + c_{n-2k}Z_{n-2k}) \cdot_A$$

$$\cdot_A (a_1'X_1 + b_1'Y_1 + \ldots + a_k'X_k + b_k'Y_k + c_1'Z_1 + \ldots + c_{n-2k}'Z_{n-2k}) =$$

$$= a_1b_1' + a_1'b_1 + \ldots + a_kb_k' + a_k'b_k + c_1c_1' + \ldots + c_lc_l'.$$

If A is even, then $l = 0$.

6.7. *Assume that $X \in \mathbb{Z}_2^n$ and $X \cdot_A X = 1$.*

(a) State and prove the n-dimensional analogue of Assertion 6.2.a.

(b) There is a basis $X, E_1, \ldots, E_{n-1}$ of $\mathbb{Z}_2^n$ and a symmetric matrix $B \in \mathbb{Z}_2^{(n-1)\times(n-1)}$ such that for any $a, b \in \mathbb{Z}_2$ and $\lambda, \mu \in \mathbb{Z}_2^{n-1}$ we have

$$(aX + \lambda_1E_1 + \ldots + \lambda_{n-1}E_{n-1}) \cdot_A (bX + \mu_1E_1 + \ldots + \mu_{n-1}E_{n-1}) = ab + \lambda \cdot_B \mu.$$

6.8. *Assume that $X, Y \in \mathbb{Z}_2^n$ and $X \cdot_A Y = 1$, $X \cdot_A X = Y \cdot_A Y = 0$.*

(a) State and prove the n-dimensional analogue of Assertion 6.5.a.

(b) There is a basis $X, Y, E_1, \ldots, E_{n-2}$ of $\mathbb{Z}_2^n$ and a symmetric matrix $B \in \mathbb{Z}_2^{(n-2)\times(n-2)}$ such that for any $a_X, a_Y, b_X, b_Y \in \mathbb{Z}_2$ and $\lambda, \mu \in \mathbb{Z}_2^{n-2}$ we have

$$(a_XX + a_YY + \lambda_1E_1 + \ldots + \lambda_{n-2}E_{n-2}) \cdot_A (b_XX + b_YY + \mu_1E_1 + \ldots + \mu_{n-2}E_{n-2}) = a_Xb_Y + a_Yb_X + \lambda \cdot_B \mu.$$

7 Rank of matrix with relations: generalization

The following results are 'higher-dimensional' (and more strong) generalizations of Theorem 5.1, Assertions 5.4 and 5.5, and Proposition 5.7. They give a simplified well-structured exposition of [PT19, Theorem 1].

An $\binom{[m]}{l}$-*matrix* is a symmetric square matrix with $\mathbb{Z}_2$-entries whose rows and whose columns correspond to all l-element subsets of $[m]$, and for which (triviality) and the following properties hold:

(linear dependence) for each $(l+1)$-element and l-element subsets $F, P \subset [m]$

$$\sum_{i\in F} A_{F-i,P} = 0.$$

(non-triviality) for each $i \in [m]$ and $(2l-2)$-element subset $F \subset [m]-i$ we have $A_{F,i} = 1$, where

$$A_{F,i} := \sum_{\{X,Y\}\ :\ F\cup i = X\cup Y,\ X\cap Y = i,\ |X|=|Y|=l} A_{X,Y} = \sum_{\{\sigma,\tau\}\ :\ F=\sigma\sqcup\tau,\ |\sigma|=l-1} A_{i\sqcup\sigma, i\sqcup\tau}.$$

Analogously to Assertion 4.5, an $\binom{[m]}{l}$-matrix is constructed by a $\mathbb{Z}_2$-embedding of the $(l-1)$-dimensional skeleton of the $(m-1)$-dimensional simplex to a $2(l-1)$-dimensional manifold.

Theorem 7.1. *Suppose $l \ge 3$ and A is an $\binom{[m]}{l}$-matrix.*

(a) Then $\operatorname{rk} A \ge \dfrac{m-2l+2}{l-1}$. *(b) If, moreover, A is even, then* $\operatorname{rk} A \ge \dfrac{2(m-2l+2)}{l}$.

You can deduce Theorem 7.1 from Propositions 7.4.a,b.

7.2. *Let A' be the square matrix of size $\binom{m-1}{l}$ obtained from an $\binom{[m]}{l}$-matrix by deleting rows and columns corresponding to all subsets containing m. Then A' is an $\binom{[m-1]}{l}$-matrix.*

7.3. *Let A be an $\binom{[m]}{l}$-matrix and $X := \{m-l+1, m-l+2, \dots, m\}$.*

(a,b') Let B be the square matrix of size $\binom{m-l}{l}$ obtained from A by deleting rows and columns corresponding to subsets containing at least one of the elements of X.

If $A_{X,X} = 1$, then $\operatorname{rk} A > \operatorname{rk} B$.

If $A_{X,X} = A_{Y,Y} = 0$ and $A_{X,Y} = 1$ for some $Y \subset [m]$, then $\operatorname{rk} A \ge \operatorname{rk} B + 2$.

(b) Let C be the square matrix obtained from A by deleting rows and columns corresponding to subsets containing at least one element of X or of certain l-element subset $Y \subset [m]$. If $A_{X,X} = A_{Y,Y} = 0$ and $A_{X,Y} = 1$, then $\operatorname{rk} A \ge \operatorname{rk} C + 2$.

(a') For l-element subsets $P, Q \subset [m-l+1]$ define

$$D_{P,Q} := A_{P,Q} + A_{P,X}A_{Q,X}.$$

If $A_{X,X} = 1$, then $\operatorname{rk} D < \operatorname{rk} A$ *and D is an $\binom{[m-l+1]}{l}$-matrix.*

Assertions 7.3.a,b are only required to illustrate the idea of Assertions 7.3.a',b' by proving much easier results giving estimates $\operatorname{rk} A \ge \dfrac{m-2l+2}{l}$ and, for A even, $\operatorname{rk} A \ge \dfrac{2(m-2l+2)}{2l-1}$.

Denote by r_m the minimal rank of an $\binom{[m]}{l}$-matrix. Denote by $\widetilde{r_m}$ the minimal rank of an even $\binom{[m]}{l}$-matrix. Clearly, $r_m = \widetilde{r_m} = 0$ for $m \le 2l-2$, both sequences $r_m, \widetilde{r_m}$ are non-decreasing, and $r_m \le \widetilde{r_m}$. The non-triviality implies that $r_{2l-1}, \widetilde{r_{2l-1}} \ge 1$. Theorem 7.1 asserts that $r_m \ge \dfrac{m-2l+2}{l-1}$ and $\widetilde{r_m} \ge \dfrac{2(m-2l+2)}{l}$.

Proposition 7.4. *(a)* $r_m \geq \min\{r_{m-l+1}+1, \widetilde{r_m}\}$ *(more precisely, either* $r_m = \widetilde{r_m}$*, or* $r_m \geq r_{m-l+1}+1$*);*

(b) $\widetilde{r_m} \geq \widetilde{r_{m-l}} + 2$.

Proof of Proposition 7.4.a also uses an algebraic version (b) of the higher-dimensional analogue of the following result (a).

Proposition 7.5. *(a) Denote by* $X = \binom{\binom{[5]}{2}}{2}$ *the set of unordered pairs of* 2*-element subsets of* $[5]$*. For any* $i \in [5]$ *and a partition* $[5] - i = \sigma \sqcup \tau$ *into disjoint* 2*-element sets denote*

$$T_{i,\{\sigma,\tau\}} := \big\{\{\alpha,\beta\} \in X \ : \ \alpha \subset \sigma \sqcup i,\ \beta \subset \tau \sqcup i\big\}.$$

Denote by A_i *the sum modulo 2 (i. e., the symmetric difference) of sets* $T_{i,\{\sigma,\tau\}}$ *over all non-ordered partitions* $[5] - i = \sigma \sqcup \tau$ *as above. Then*

$$A_i = \big\{\{\alpha,\beta\} \in X \ : \ \alpha \cap \beta = \emptyset\big\}$$

and so is independent of i*.*

(b) Let A *be a symmetric square matrix with* $\mathbb{Z}_2$*-entries whose rows and whose columns correspond to all* l*-element subsets of* $[m]$*. If* A *satisfies the linear dependence property (from the definition of an* $\binom{[m]}{l}$*-matrix), then* $A_{F,i}$ *depends only on* $F \sqcup i$ *not on* (F, i)*.*

Hints and solutions to some problems

See proof of Proposition 1.1 in [Bi20].

1.2. (a) Change the numbers on the main diagonal of M so that the sum of the entries in each row is even. The resulting matrix is degenerate.

(b) Use induction. See Lemma 2.5.b.

1.3. (a) Any matrix formed as a result of putting numbers from $\mathbb{Z}_2$ in the elements on the main diagonal of M can be uniquely represented as the sum $M+D$ where D is a diagonal matrix. By Lemma 3.7.b for every diagonal matrix D with more than k zeros on the main diagonal we have $\operatorname{rk}(M+D) \geq \operatorname{rk} M - \operatorname{rk} D > n-(n-k) = k$.

(b) The algorithm of (b) is constructed using (a) and Lemma 2.5.b. The algorithm given by Lemma 2.5.b has complexity $O(n^4)$. There is an algorithm searching through all diagonal $n\times n$ matrices with $\leq k$ zeroes on the main diagonal with the complexity of

$$O\left(n\binom{n}{0}+n\binom{n}{1}+\ldots+n\binom{n}{k}\right) \overset{(*)}{=} O\left((k+1)n\binom{n}{k}\right) = O\left(n\cdot n^k\right) = O\left(n^{k+1}\right).$$

Here (*) holds because we may assume that $n\geq 2k$. Thus, by Assertion 3.6.b the complexity of the whole algorithm is $O(n^4)+O(n^{k+1}n^2)=O(n^{k+3})$ (since $k\geq 1$).

1.4. (a) Denote by n the number of columns of M and of D. By Lemma 3.7.b we have

$$\begin{aligned} 2\operatorname{rk}(M+D) &= \operatorname{rk}(M+D)+\operatorname{rk}((M+E)+(E+D)) \\ &\geq (\operatorname{rk} M-\operatorname{rk} D)+(\operatorname{rk}(M+E)-\operatorname{rk}(E+D)) \\ &= n-\operatorname{rk} D+\operatorname{rk}(M+E)-(n-\operatorname{rk} D)=\operatorname{rk}(M+E). \end{aligned}$$

(b) Let M_n be the matrix obtained by applying the algorithm of Lemma 2.5.b to the matrix M. Let $k:=\operatorname{rk}(M_n+E)$. We have $R(M)=\operatorname{rk}(M+D)$ for some diagonal matrix D. Hence by (a) $k/2\leq R(M)\leq k$ as required.

The number k can be computed in time $O(n^3)$. Hence the total complexity of the algorithm is $O(n^4)+O(n^3)=O(n^4)$.

2.1. *Answers:* A_1, A_2 and A_3 are degenerate, while A_4 is non-degenerate.

2.2. Hints: (a)-(b) track the maximal non-degenerate submatrix; (c) use induction.

Part (a) is clear.

(b) For a matrix M denote by $\operatorname{row}_{i\to i+j}M$ the matrix obtained from M by replacing the ith row by the sum of the ith row and the jth row. The matrix $\operatorname{col}_{i\to i+j}M$ is defined similarly.

It is clear that to prove part (b) we have to show that the matrices M, $\operatorname{row}_{i\to i+j}M$ and $\operatorname{col}_{i\to i+j}M$ are degenerate or not simultaneously.

Next, observe that $\operatorname{row}_{i\to i+j}\operatorname{row}_{i\to i+j}M = M = \operatorname{col}_{i\to i+j}\operatorname{col}_{i\to i+j}M$. Thus it suffices to prove that if M is degenerate then both $\operatorname{row}_{i\to i+j}M$ and $\operatorname{col}_{i\to i+j}M$ are degenerate.

Assume that the sum of columns $c_1,c_2,\ldots,c_s$ of M equals zero.

Then the sum of the 'same' columns of $\operatorname{row}_{i\to i+j}M$ equals zero.

If $i\notin\{c_1,\ldots,c_s\}$ then the sum of 'the same' columns of $\operatorname{col}_{i\to i+j}M$ equals zero. If $i,j\in\{c_1,\ldots,c_s\}$ then the sum of columns indexed by $\{c_1,\ldots,c_s\}-\{j\}$ of $\operatorname{col}_{i\to i+j}M$ equals zero. If $i\in\{c_1,\ldots,c_s\}$ and $j\notin\{c_1,\ldots,c_s\}$ then the sum of columns indexed by $\{c_1,\ldots,c_s,j\}$ of $\operatorname{col}_{i\to i+j}M$ equals zero.

This completes the proof of (b).

(c) We will show explicitly how to produce a diagonal matrix out of M.

If all entries of M are 0 then M is already diagonal. If M has a non-zero entry then we place this entry in the top-left corner by permuting the row of this entry with the top row, and the column of this entry with the left column. For the obtained matrix add the top row to other rows and the left column to other columns. All entries in the left column and the top row except the the top-left entry become zeros. Delete the top row and the left column of the obtained matrix.

Repeat the procedure inductively for the obtained submatrix. In the end this will produce a diagonal matrix.

(d) By part (c) we can change M into a diagonal matrix using transformations from parts (a, b); also M is degenerate if and only if the new matrix is degenerate. It remains to mention that a diagonal matrix is non-degenerate iff it is the identity matrix.

(f) This follows from (a)-(d).

(g) The algorithm is constructed in the solution of part (c). The algorithm has n major steps, a single major step is described in the second paragraph of (c). Each major step requires at most one permutation of rows, at most one permutation of columns and up to $2n$ additions of rows and columns. Thus the complexity of the whole algorithm is $O(n)+n\cdot O(n^2)=O(n^3)$.

2.3. (a) The formula follows because a matrix from $\mathbb{Z}_2^{2\times 2}$ is degenerate if and only if either it has a zero row, or it has a zero column, or rows are the same and columns are the same (in the latter case all entries are ones).

Alternatively, here are all matrices from $\mathbb{Z}_2^{2\times 2}$ up to permutations of rows and columns:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

The first two are non-degenerate, and the other are degenerate. It is easy to verify the formula for each of them.

(d) This follows by (b,c).

Here is an alternative direct proof. Consider $n\times n$ chessboard. A *rook placement* for such a chessboard is a placement of n rooks on that board with the condition that they do not beat each other. A *rook M-placement* for such a chessboard is a rook placement such that all rooks are staying on cells corresponding to unit entries of M. Denote by $\det^* M$ the parity of the amount of rook M-placements. Then (d) can be restated as follows: $\det M=\det^* M$. This follows because

- transformations of 2.2.a, 2.2.b preserve $\det^* M$, and
- $\det M'=\det^* M'$ for a diagonal matrix M'.

2.4. For any degenerate matrix of Problem 2.1 we show how to change entries on the main diagonal to make it non-degenerate; we show the opposite for A_4:

$$A_1\to\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad A_2,A_3\to\begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix},\quad A_4\to\begin{pmatrix}0&1&1\\0&1&1\\1&0&0\end{pmatrix}.$$

2.5. (a) In the following paragraph we prove by induction on $i\ge 1$ that the determinant $\Delta_i := \det M^{(i)}_{[i]\times[i]}$ of the left upper $i\times i$-corner submatrix of $M^{(i)}$ is equal to 1. Then $\det M^{(n)}=\Delta_n=1$.

Base $i=1$ follows because $\Delta_1=1+\delta_0=1$. Let us prove the inductive step $i-1\to i$. Apply the decomposition formula for the determinant Δ_i by the last row of the corresponding submatrix of $M^{(i)}$ (Assertion 2.3.c). Since $M^{(i-1)}_{i,i} = M_{i,i} = 0$ and $\Delta_{i-1} = 1$, we have $\Delta_i = \delta_i + (1+\delta_i)\Delta_{i-1} = 1$.

(b) The algorithm is constructed by (a). The algorithm is essentially a computation of the determinants of n square submatrices of sizes $1, 2, \ldots, n$. Hence by Assertion 2.2.g its complexity is $O(1^3 + 2^3 + \ldots + n^3) = O(n \cdot n^3) = O(n^4)$.

3.1. *Answers:* (a1) 0; (a2) 1; (a3), (a4) 3; (b1) 0; (b2), (b3) 1; (b4) 2.

3.2. Hint to (b): find the number of sums of k columns.

Part (a) follows from the definition of $\operatorname{rk} M$.

(b) By definition of rk the number of different sums of columns of M is $2^{\operatorname{rk} M}$. On the other hand the number of such sums does not exceed 2^k. Therefore $2^k \geq 2^{\operatorname{rk} M}$, hence $k \geq \operatorname{rk} M$.

3.3. The proofs of are similar to the proofs of Assertion 2.2.

3.4. Part (a) is clear.

(b) If for a non-zero symmetric matrix M there exists such a permutation of rows and columns, then $\operatorname{rk} M = 1$ by Assertion 3.2.b.

We now take a symmetric matrix M of rank 1. As the required permutation we can take any permutation mapping non-zero rows of M to the first rows. Indeed, take any non-zero rows i, j. If $M_{i,j} = 0$ then there exists a non-zero row k such that $M_{i,k} = 1$. Hence the jth and the kth rows are distinct non-zero rows of the matrix M of rank one. A contradiction. Hence $M_{i,j} = 1$.

(c) Pick a nonzero row and apply the above argument.

3.5. (a) $R\begin{pmatrix}1&0&1\\0&0&1\\0&1&0\end{pmatrix} = 2, \quad R\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} = 1.$

(b) $R\begin{pmatrix}1&0&1\\0&1&1\\0&1&0\end{pmatrix} = 2, \quad R\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} = 1.$

3.6. Hint for (a): cf. Assertion 2.2.

(a) The algorithm from the proof of 2.2.c provides a diagonal matrix of the same rank, and has the required complexity. The rank of a diagonal matrix is equal to the number of non-zero entries in it.

(b) We shall construct a set S_k of columns such that

- these columns constitute a non-degenerate submatrix;
- the first k columns of the matrix M are sums of several columns from the set S_k.

If $|S_k| > r$ for some $k = 1, \ldots, n$ then the answer is 'NO'. If for all $k = 1, \ldots, n$ we have $|S_k| \leq r$ then the answer is 'Yes'. The answer is correct because $|S_1| \leq |S_2| \leq \ldots \leq |S_n|$, and because $|S_n| > r$ is equivalent to $\operatorname{rk} M > r$.

Set $S_1 := \emptyset$ if the first column of M is 0 then, and $S_1 := \{1\}$ otherwise.

Let us define S_{k+1} from S_k. We form the set of all sums of columns of M with indices from S_k (it takes $O(n)$ operations because $|S_k| \leq r$). Then we compare the $(k+1)$st column of M with all sums from this set (this will take at most $2^r O(n) = O(n)$ operations). If the $(k+1)$st column of M equals to at least one of the sums then $S_{k+1} := S_k$. Otherwise we set $S_{k+1} := S_k \cup \{k+1\}$.

It is easy to verify that the total complexity of the algorithm is $O(n^2)$.

3.7. Part (b) follows from (a) because

$$\operatorname{rk} M = \operatorname{rk}(M + D + D) \leq \operatorname{rk}(M + D) + \operatorname{rk} D \quad \Rightarrow \quad \operatorname{rk}(M + D) \geq \operatorname{rk} M - \operatorname{rk} D.$$

We now prove (a). Choose columns from Assertion 3.2.a for M and for D. Then every column of $M+D$ is the sum of some of chosen $\operatorname{rk} M + \operatorname{rk} D$ columns. By Assertion 3.2.b $\operatorname{rk}(M+D) \le \operatorname{rk} M + \operatorname{rk} D$.

3.8. The statement and the proof of Assertion 3.8 are similar to the statement and the proof of Theorem 1.3.b.

3.9. Answer: no. If M is a 3×3-matrix given next and $k=1$ then the statement is false.

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

3.10. *Answers:* (a) 1; (b) $(2^n-1)^2$; (c) $(2^n-1)^2(2^n-2)^2/6$.

(a) There exists only one matrix of rank 0, the matrix all whose entries are zeroes.

(b) For the matrix of rank 1 all columns containing a non-zero entry are the same. Hence such matrices are in 1–1 correspondence with ordered pairs formed by

• a non-empty subset of the set of columns ('non-zero columns'), and

• a nonzero vector in $v \in \mathbb{Z}_2^n$ ('column vector').

Therefore there are $(2^n-1)^2$ such matrices.

(c) Fix a matrix M of rank 2. Then there exists a pair (v,w) of columns of M forming a non-degenerate matrix. Any other column is either 0 or v, or w, or $v+w$ (see Assertion 3.2). This set $S = S_M$ of four vectors does not depend on a choice of the two columns v, w; we call it the *column span* of M. (This is a 2-dimensional vector subspace of $\mathbb{Z}_2^n$.)

Each column span is defined by any ordered pair of non-zero vectors in it. Each column span contains exactly 6 such ordered pairs. Hence there are $(2^n-1)(2^n-2)/6$ column spans. In the following paragraph we prove that there are exactly $(2^n-1)(2^n-2)$ matrices of rank 2 for a given column span. Hence there are $(2^n-1)^2(2^n-2)^2/6$ matrices of rank 2.

First proof. To a matrix M there correspond the set X of columns of M equal to v or to $v+w$, and the set Y of columns of M equal w or $v+w$. Since $\operatorname{rk} M = 2$, both sets are non-empty and $X \ne Y$. Moreover, matrix M can be reconstructed from X, Y. There are $(2^n-1)(2^n-2)$ pairs (X,Y) of distinct non-empty subsets.

Second proof. For a given 4-element set $S = \{0, v, w, v+w\}$, regard a matrix M with the column span S as a map ϕ_M from the set $[n]$ of columns to S. We have $\operatorname{rk} M = 2$ if and only if

(*) the image of ϕ_M contain at least two of vectors $v, w, v+w$.

There are 4^n maps $[n] \to S$. There are 2^n maps $[n] \to \{0, v\}$. The same holds for $\{0,v\}$ replaced either by $\{0,w\}$ or by $\{0, v+w\}$. There is only one map $[n] \to \{0\}$. Hence there are exactly $4^n - 3 \cdot 2^n + 2 = (2^n-1)(2^n-2)$ maps satisfying the condition (*).

Remark. More generally, the number of matrices of rank k in $\mathbb{Z}_2^{m\times n}$ is equal to

$$\frac{2^{k(k-1)/2}\Pi_{i=0}^{k-1}(2^{m-i}-1)\Pi_{i=0}^{k-1}(2^{n-i}-1)}{\Pi_{i=0}^{k-1}(2^{k-i}-1)}.$$

See theorem 7.1.5 in [ACM29, p. 299] (this theorem is even more general; for our case of matrices over $\mathbb{Z}_2$ take $q=2$, $\mathrm{GF}(2) = \mathbb{Z}_2$).

4.1. (a1-a3) A beautiful realization of the graph K_5 on the torus is shown in Figure 4, left. Realizations of K_6 and K_7 are analogous, see Figure 4, middle.

Another solutions (whose idea works for (c,d)). Draw the graph K_5 in the plane with *one* self-intersection point. In a small neighborhood of the point, attach a handle and lift one of the edges 'bridgelike' over the other edge to the handle, see Figure 5, left.

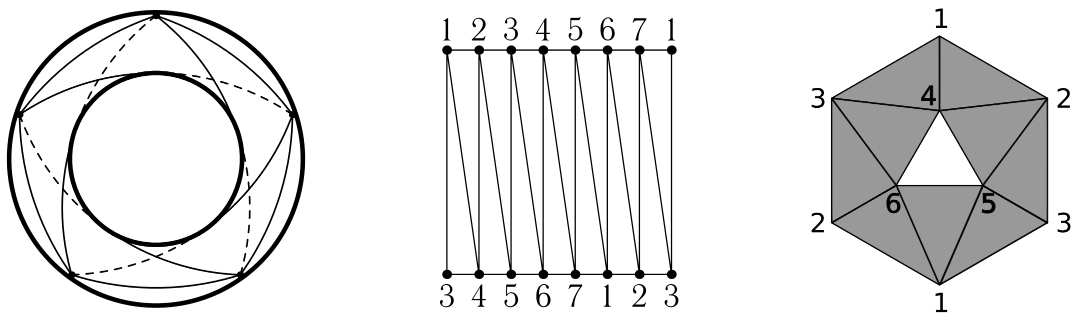

Figure 4: Realization of nonplanar graphs

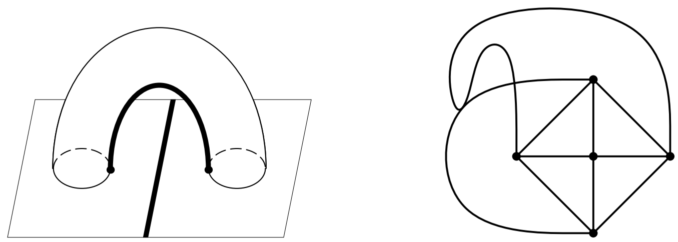

Figure 5: Left: resolving intersection by adding a handle.
Right: a 'non-general position even drawing' of K_5 in the plane

(b2) See Figure. 4, right.

4.2. See Fig. 5, right.

5.1. (b) Induction on m. The base $m \le 4$ is clear. By Assertion 5.7.b and induction hypothesis we have

$$\widetilde{r_m} \ge \widetilde{r_{m-5}} + 2 \ge \frac{2(m-5-4)}{5} + 2 = \frac{2(m-4)}{5}.$$

(a) Induction on m. The base $m \le 4$ is clear. By Assertion 5.7.a, Theorem 5.1.b and induction hypothesis we have

$$r_m \ge \min\{r_{m-3}+1, \widetilde{r_m}\} \ge \min\left\{\frac{m-3-4}{3}+1, \frac{2(m-4)}{5}\right\} = \frac{m-4}{3}.$$

5.3. Denote by A^f the $\binom{[m]}{3}$-matrix constructed via an embedding $f\colon K_m \to S$, see the beginning of Section 5.

(a) Consider the K_4-subgraph of K_5 with vertices 1, 2, 3, 4 together with its embedding to the torus defined by Figure 3, right side. Then A^f is a non-trivial $\binom{[m]}{3}$-matrix.

(b) Let f be the embedding of K_5 into the torus defined by the right side of Figure 3. Then A^f has the required property.

(c) By Assertion 4.1.d there exists an embedding $f\colon K_m \to S$ where S is a sphere with several handles. Then A^f has the required property.

(a', b', c') The matrices A^f from (a, b, c) respectively satisfy the needed conditions.

Parts (d, e, f) follow from Assertion 5.6.a, b.

5.4. See proof of Assertion 7.2 below.

5.5. (a, b) See proof of Assertions 7.3.a,b below.

5.6. (a) Recall that if $m \ge 5$ then every $\binom{[m]}{3}$-matrix is not a zero matrix. This implies that $r_5, r_6 \ge 1$. Let f be an embedding defined by Assertion 4.1.b1, b2. It follows from Theorem 4.4 that $\operatorname{rk} A^f \le 1$. Hence $r_5 = r_6 = 1$.

(b) Assertion 3.4.c implies that $\widetilde{r_5}, \widetilde{r_6}, \widetilde{r_7} \ge 2$. We have $\operatorname{rk} A^f = 2$ for f defined by Assertion 4.1.a1, a2, a3 and hence $\widetilde{r_5} = \widetilde{r_6} = \widetilde{r_7} = 2$.

(c) Let A be an $\binom{[m]}{3}$-matrix and let B be an $\binom{[m-1]}{3}$-submatrix of A introduced in Assertion 5.4. We have $\operatorname{rk} B \le \operatorname{rk} A$ and therefore $r_m \ge r_{m-1}$.

If A is even then B is even and therefore $\widetilde{r_m} \ge \widetilde{r_{m-1}}$.

5.7. (a) (Take $l = 3$.) Take an $\binom{[m]}{l}$-matrix A such that $\operatorname{rk} A = r_m$. If A is even, then $r_m = \widetilde{r_m}$, so we are done. Otherwise there is an l-element subset $X \subset [m]$ such that $A_{X,X} = 1$. Let B be the 'restriction' of A to l-element subsets of $[m] - X$.

Then

$$r_m = \operatorname{rk} A \ge \operatorname{rk} B + 1 \ge r_{m-l} + 1, \quad \text{where}$$

- the first inequality follows by Assertion 5.5.a;
- the second inequality holds because B is a $\binom{[m]-X}{l}$-matrix by Assertion 5.4.

(b) (Take $l = 3$.) Take an even $\binom{[m]}{l}$-matrix A such that $\operatorname{rk} A = \widetilde{r_m}$. By the non-triviality $A \ne 0$. Hence there are l-element subsets $X, Y \subset [m]$ such that $A_{X,Y} = 1$. Let C be the 'restriction' of A to l-element subsets of $[m] - X - Y$.

Then

$$\widetilde{r_m} = \operatorname{rk} A \ge \operatorname{rk} C + 2 \ge \widetilde{r_{m-2l+1}} + 2, \quad \text{where}$$

- the first inequality follows by Assertion 5.5.b;
- the second inequality holds because C is a $\binom{[m]-X-Y}{l}$-matrix by Assertion 5.4, and because $A_{X,Y} = 1$, so by the triviality $X \cap Y \ne \emptyset$, hence $|[m] - X - Y| \ge m - 2l + 1$.

6.7. (a) $\lambda_{X,P} = X \cdot_A P$.

6.8. (a) $\lambda_{X,Y,P} = X \cdot_A P$, $\lambda_{Y,X,P} = Y \cdot_A P$.

7.2. (For 5.4 take $l = 3$.) It is obvious that all the conditions for the mentioned submatrix are satisfied.

7.3. (a) (For 5.5.a take $l = 3$.) Let B' be the 'restriction' of A to X and to l-element subsets of $[m] - X$. Then

$$\operatorname{rk} A \ge \operatorname{rk} B' = \operatorname{rk} B + 1,$$

where equality holds because by the triviality $B'_{X,Z} = 0$ for any $Z \subset [m] - X$.

(b) (For 5.5.b take $l = 3$.) Let C' be the 'restriction' of A to X, Y and l-element subsets of $[m] - X - Y$. Then

$$\operatorname{rk} A \ge \operatorname{rk} C' = \operatorname{rk} C + 2,$$

where equality holds because by the triviality $C'_{X,Z} = C'_{Y,Z} = 0$ for any $Z \subset [m] - X - Y$.

(b') Take a basis of $\mathbb{Z}_2^{\binom{m}{l}}$ corresponding to l-element subsets of $[m]$. Define a bilinear form A on $\mathbb{Z}_2^{\binom{m}{l}}$ by setting $A(P,Q) := A_{P,Q}$ for basic vectors P, Q. Take any l-element set $P \subset [m]$. Let

$$\overline{P} = \overline{P}(X,Y) := P + A_{X,P}Y + A_{Y,P}X.$$

Recall that

$$A_{X,Y} = A_{Y,X} = 1 \quad \text{and} \quad A_{X,X} = A_{Y,Y} = 0. \tag{*}$$

Hence

$$A(\overline{P}, X) = A(\overline{P}, Y) = 0 \tag{**}$$

(i. e., $\overline{P}$ is the orthogonal projection of P to the orthogonal complement of $\langle X, Y \rangle$ with respect to A). By the triviality, for $P \subset [m] - X$ we have $\overline{P} = P + A_{Y,P}X$. Hence for every l-element sets $P, Q \subset [m] - X$ we have

$$A(\overline{P}, \overline{Q}) = A_{P,Q} + 0 + 0 + 0 = B_{P,Q}. \tag{***}$$

(I. e., B is the Gramian matrix with respect to A of the 'projections' $\overline{P}$ of l-element sets $P \subset [m]-X$.) Let B' be the Gramian matrix with respect to A of X, Y and the 'projections' $\overline{R}$ of l-element sets $R \subset [m]-X$. I. e., $B'_{P,Q} = A(\widehat{P}, \widehat{Q})$, where $\widehat{P} = P$ if $P \in \{X, Y\}$, and $\widehat{P} = \overline{P}$ otherwise ($\widehat{Q}$ is defined analogously). Then

- $B'_{X,Y} = B'_{Y,X} = 1$, $B'_{X,X} = B'_{Y,Y} = 0$ (by (*)),
- $B'_{X,P} = B'_{P,X} = B'_{Y,P} = B'_{P,Y} = 0$ for $P \neq X, Y$ (by (**)), and
- $B'_{P,Q} = B_{P,Q}$ for $P, Q \subset [m]-X$ (by (***)).

Hence $\operatorname{rk} B + 2 = \operatorname{rk} B' \le \operatorname{rk} A$.

(a') In this paragraph we prove that $\operatorname{rk} D < \operatorname{rk} A$. Take a basis of $\mathbb{Z}_2^{\binom{m}{l}}$ corresponding to l-element subsets of $[m]$. Define a bilinear form A on $\mathbb{Z}_2^{\binom{m}{l}}$ by setting $A(P,Q) := A_{P,Q}$ for basic vectors P, Q. Let P_X be the orthogonal projection of P to the orthogonal complement of X (with respect to A), i. e., $P_X := P + A_{P,X}X$. We have

$$A(P_X, Q_X) = A(P,Q) + A(A_{P,X}X, Q) + A(P, A_{Q,X}X) + A(A_{P,X}X, A_{Q,X}X) = \\ = A_{P,Q} + A_{P,X}A_{X,Q} + A_{P,X}A_{Q,X} + A_{P,X}A_{Q,X}A_{X,X} = A_{P,Q} + A_{P,X}A_{Q,X} = D_{P,Q}.$$

Then D is the Gramian matrix (with respect to A) of the projections of subsets of $[m-l+1]$. Let D' be the Gramian matrix (with respect to A) of X and the projections of subsets of $[m-l+1]$. We have $D_{P,Q} = D'_{P,Q}$ for all subsets $P, Q \subset [m-l+1]$. Furthermore, $D'_{X,P} = D'_{P,X} = 0$ for any basic vector $P \neq X$ and $D'_{X,X} = A_{X,X} = 1$. Thus $\operatorname{rk} D = \operatorname{rk} D' - 1 < \operatorname{rk} A$.

In this paragraph we prove that D satisfies the triviality property. If $P \cap Q = \emptyset$, then either $P \cap X = \emptyset$, or $Q \cap X = \emptyset$. Hence $D_{P,Q} = A_{P,Q} + A_{P,X}A_{Q,X} = 0 + 0 = 0$.

In this paragraph we prove that D satisfies the linear dependence property. For each $(l+1)$-element and l-element subsets $F, P \subset [m-l+1]$ we have

$$\sum_{i \in F} D_{F-i,P} = \sum_{i \in F} A_{F-i,P} + A_{P,X} \sum_{i \in F} A_{F-i,X} = 0.$$

In this paragraph we prove that D satisfies the non-triviality property. By Proposition 7.5.b for D, we may assume that $i \neq m-l+1$. Then for each summand $D_{i \sqcup \sigma, i \sqcup \tau}$ of $D_{F,i}$ at least one of the sets $i \sqcup \sigma, i \sqcup \tau$ does not contain $m-l+1$ and hence does not intersect X. Hence $D_{i\sqcup\sigma,i\sqcup\tau} = A_{i\sqcup\sigma,i\sqcup\tau} + A_{i\sqcup\sigma,X}A_{i\sqcup\tau,X} = A_{i\sqcup\sigma,i\sqcup\tau}$. Thus $D_{F,i} = A_{F,i} = 1$.

7.4. (a) Take an $\binom{[m]}{l}$-matrix A such that $\operatorname{rk} A = r_m$. If A is even, then $r_m = \widetilde{r_m}$, so we are done. Otherwise there is an l-element subset $X \subset [m]$ such that $A_{X,X} = 1$. Without loss of generality $X = \{m-l+1, m-l+2, \ldots, m\}$. Then by Assertion 7.3.a'

$$r_m = \operatorname{rk} A \ge \operatorname{rk} D + 1 \ge r_{m-l+1} + 1, \quad \text{where}$$

- D is the matrix defined in Assertion 7.3.a';
- the first inequality follows from Assertion 7.3.a';
- the second inequality holds because D is an $\binom{[m-l+1]}{l}$-matrix by Assertion 7.3.a'.

(b) Take an even $\binom{[m]}{l}$-matrix A such that $\operatorname{rk} A = \widetilde{r_m}$. By the non-triviality $A \neq 0$. Let X, Y, B be defined as in Assertion 7.3.b'. Then

$$\widetilde{r_m} = \operatorname{rk} A \ge \operatorname{rk} B + 2 \ge r_{m-l} + 2, \quad \text{where}$$

- the first inequality follows from Assertion 7.3.b';
- the second inequality holds because B is an even $\binom{[m]-X}{l}$-matrix by Assertion 7.2.

7.5. (a) It suffices to check that for each pair $\{\alpha,\beta\}$ the number of sets $T_{i,\{\sigma,\tau\}}$ containing $\{\alpha,\beta\}$ is odd if and only if $\alpha\cap\beta=\emptyset$ (hence this parity not depend on i). Clearly, $|\alpha\cap\beta|\le 2$.

Assume $|\alpha\cap\beta|=2$. Then $\{\alpha,\beta\}\notin T_{i,\{\sigma,\tau\}}$ for all i,σ,τ and hence $\{\alpha,\beta\}\notin A_i$ for all $i\in[5]$.

Assume $|\alpha\cap\beta|=1$. It suffices to consider the case $\alpha=\{1,2\}$, $\beta=\{1,3\}$. Then $\{\alpha,\beta\}\in T_{i,\{\sigma,\tau\}}$ iff $i=1$ and $\{\sigma,\tau\}$ is either $\big\{\{2,4\},\{3,5\}\big\}$ or $\big\{\{2,5\},\{3,4\}\big\}$ Therefore $\{\alpha,\beta\}\notin A_i$ for all $i\in[5]$.

Assume $|\alpha\cap\beta|=0$. It suffices to consider the case $\alpha=\{1,2\}$, $\beta=\{3,4\}$. Then $\{\alpha,\beta\}\in T_{i,\{\sigma,\tau\}}$ iff either

- $i=1$ and $\{\sigma,\tau\}=\big\{\{1,2,5\},\{1,3,4\}\big\}$, or
- $i=2$ and $\{\sigma,\tau\}=\big\{\{1,2,5\},\{2,3,4\}\big\}$, or
- $i=3$ and $\{\sigma,\tau\}=\big\{\{1,2,3\},\{3,4,5\}\big\}$, or
- $i=4$ and $\{\sigma,\tau\}=\big\{\{1,2,4\},\{3,4,5\}\big\}$, or
- $i=5$ and $\{\sigma,\tau\}=\big\{\{1,2,5\},\{3,4,5\}\big\}$.

Therefore $\{\alpha,\beta\}\in A_i$ for every $i\in[5]$.

(b) It suffices to prove that $A_{G\sqcup i,j}=A_{G\sqcup j,i}$ for each $i,j\in[m]$ and $(2l-3)$-element subset $G\subset[m]-i-j$. Denote $\overline{\sigma}:=\{i,j\}\sqcup\sigma$. Then

$$A_{G\sqcup j,i}+A_{G\sqcup i,j}\overset{(1)}{=}\sum_{\{(\sigma,\tau)\ :\ G=\sigma\sqcup\tau,\ |\sigma|=l-2\}}(A_{\overline{\sigma},i\sqcup\tau}+A_{\overline{\sigma},j\sqcup\tau})\overset{(2)}{=}$$

$$=\sum_{\{(\sigma,\tau)\ :\ G=\sigma\sqcup\tau,\ |\sigma|=l-2\}}\ \sum_{t\in\tau}A_{\overline{\sigma},\overline{\tau-t}}\overset{(3)}{=}\sum_{t\in G}\ \sum_{\{(\sigma,\nu)\ :\ G-t=\sigma\sqcup\nu,\ |\sigma|=l-2\}}A_{\overline{\sigma},\overline{\nu}}\overset{(4)}{=}0,\quad\text{where}$$

- equality (1) holds because $A_{G\sqcup j,i}$ is equal to the sum of the first summands $A_{\overline{\sigma},i\sqcup\tau}$, and $A_{G\sqcup i,j}$ is equal to the sum of the second summands $A_{\overline{\sigma},j\sqcup\tau}$;
- equality (2) holds by the linear dependence for $F=\overline{\tau}$, $P=\overline{\sigma}$;
- equality (3) is obtained by changes of the order of summation and of variable $\nu=\tau-t$;
- equality (4) holds because ordered decompositions (σ,ν) of $G-t$ into $(l-2)$-element subsets σ,ν split into pairs $\{(\sigma,\nu),(\nu,\sigma)\}$ and $A_{\overline{\sigma},\overline{\nu}}+A_{\overline{\nu},\overline{\sigma}}=0$.

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