Algebraic Topology From a Geometric Standpoint

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Abstract.

It is shown how main ideas, notions and methods of algebraic topology naturally appear in a solution of geometric problems. The main ideas are exposed in simple particular cases free of technical details. We keep algebraic language to a necessary minimum. So most of the book is accessible to beginners and non-specialists, although it contains beautiful non-trivial results. Part of the material is exposed as a sequence of problems, for which hints are provided. The book is intended for students, researchers, and teachers, who wish to know

• why what I learn or teach is interesting and useful?

• how the main idea of a result / proof / theory is exposed in simple terms?

• how is this idea elaborated to produce the result / proof / theory?

Here students could be undergraduate or postgraduate; with majors in mathematics, computer science or physics. All this would hopefully allow them to make their own useful discoveries (not necessarily in mathematics).

Other approaches to presenting this material can be found in other textbooks on algebraic topology.

We start from important visual objects of mathematics: graphs and vector fields on surfaces, continuous maps and their deformations. In §§1,2,5 basic theory of graphs on surfaces is exposed in a simplified way. In later sections I carry such a 'non-specialist', or 'user' or 'computer science' approach to topology pretty far. The appearing instruments include homology groups, obstructions and invariants, characteristic classes.

The book is based on decades of teaching topology courses in leading mathematical centers of Moscow (Moscow State University, Independent University of Moscow, Moscow Institute of Physics and Technology).

General information.

This is an updated English translation of a book published in Russian in 2015, 2020 by MCCME, Moscow. This publicly available part of the preprint is for personal or private reading only. It comprises contents and most of sections 1, 2, 5, 6, 8, 9 (observe that sections 5 and 6 are essentially independent of sections 3 and 4). The introduction and sections 3, 4 are currently in Russian.

Translated by I. Alexeev (§14), A. Balitskiy (§§5,6,12,13), M. Fedorov (§§10,11), D. Mamaev (§15), A. Nordskova (§16), A. Pratoussevitch (§§7,8,9), and N. Tsilevich (§§1,2); translation is edited by the author.

Early Russian versions of this book were available since 2008 at `https://arxiv.org/pdf/0808.1395.pdf` and `http://www.mccme.ru/circles/oim/obstruct.pdf` .

Publishing rights.

The translation is accepted for publication by ‘Moscow Lecture Notes’ of Springer in January, 2021. The translation was essentially rejected by Springer by sending an unacceptable publishing agreement, promising to make amends suggested by the author in May, 2021, and neither making amends nor informing the author that the amends are not accepted, by January, 2022 (in spite of the author’s monthly reminders). Thus no contract was signed, and this book is no longer submitted to Springer. See some details in the last pages of this file; *they could be useful for other authors considering to publish with Springer.*

This book was submitted to AMS in January, 2022. The Editor (following two reviews) suggested significant revision in June, 2023. The author expressed his wish to incorporate all the reviewers’ recommendations in June, 2023. However, in January and February 2024 the author is recommended to submit the book elsewhere, without AMS explicitly stating the formal rejection decision (upon multiple requests of the author).

Updates will be presented here.

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§ 10. Homology of higher-dimensional manifolds

§ 11. Further applications of homology

§ 12. Non-embeddability and non-immersibility

§ 13. Fiber bundles and their applications

§ 1. Graphs in the Plane

> Dass von diesem schwer lesbaren Buche noch vor Vollendung des ersten Jahrzehntes eine zweite Auflage notwendig geworden ist, verdanke ich nicht dem Interesse der Fachkreise...
>
> S. Freud. *Die Traumdeutung*, Vorwort zur zweiten Auflage[3]

1.1. Introduction and Main Results

In § 1.3 we prove basic results on graphs and map colorings in the plane, Assertions 1.1.1 and 1.3.2.

1.1.1. (a) A triangle is divided into finitely many convex polygons. They can be colored in six colors in such a way that any two polygons sharing a common boundary segment receive different colors.

(b)* The same for five colors.

(The famous Four Color Conjecture claims that four colors are enough, but its proof is much more involved.)

A graph is said to be *planar* (or *embeddable in the plane*) if it can be drawn in the plane without edges crossing. The basic notions of graph theory are recalled in § 1.2; a more rigorous definition of planarity is given in § 1.3.

Embeddability of graphs (or graphs with an additional structure) in the plane, torus, Möbius strip, and other surfaces (see § 2) is one of the main problems in topological graph theory [MT01].

Proposition 1.1.2. *There is an algorithm for deciding whether a graph is planar.* (See [Sk, footnote 4], [Sk18, footnote 7].)

One of the simplest (but slow) algorithms is constructed in §§ 1.5 and 1.6 (Assertion 1.1.2 follows from Assertions 1.6.1 (f) and 1.6.3 (a)). It is based on an important construction of *thickening*, which arises in many problems of topology and its applications (synonyms: graph with

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[3]If within ten years of the publication of this book (which is very far from being an easy one to read) a second edition is called for, this is not due to the interest taken in it by the professional circles... (S. Freud. *The Interpretation of Dreams*. Preface to the second edition.)

rotations, dessin [Ha, LZ, MT01]). The algorithm uses no nontrivial results (such as Kuratowski's theorem or Fáry's theorem; for the statements, as well as for a polynomial-time algorithm, see [Sk, § 1.2 'Algorithmic results on graph planarity']).

The proofs of these results illustrate applications of Euler's Formula 1.3.3 (c). (So, they are better postponed until the reader becomes familiar with it.) This formula is proved in § 1.4, where we also explain, in the language of algorithms, the nontriviality of this result ignored in some expositions.

1.2. Glossary of Graph Theory

The reader is probably familiar with the notions introduced below, but we give clear-cut definitions in order to fix the terminology (which can be different in other books).

A *graph* $G = (V, E)$ is a finite set $V = V(G)$ together with a set $E = E(G)$ of two-element subsets (i.e., unordered pairs of distinct elements). (A more precise term for the notion we have introduced is *graph without loops or multiple edges*, or *simple graph.*) Elements of the set V are called *vertices*, elements of the set E are called *edges.* Although edges are unordered pairs, in graph theory they are traditionally denoted by parentheses. Given an edge (a, b), the vertices a and b are called its *endpoints*, or *vertices.*

When working with graphs, it is convenient to use their drawings, e.g., in the plane or in the space (or, in more technical terms, maps of their geometric realizations to the plane or to the space, cf. §5.1). See Figs. 1.3.1, 1.3.2, 1.7.2 below. Vertices are represented by points. Every edge is represented by a polygonal line joining its endpoints. (But only the endpoints of polygonal lines represent vertices of the graph.) The polygonal lines are allowed to intersect, but their intersection points (other than the common endpoints) are not vertices. Importantly, a graph and a drawing of this graph are not the same. For example, Figs. 1.3.2 (middle and right), 1.3.1 show different drawings of the same graph (more exactly, of isomorphic graphs). Sometimes, not all vertices are shown in a drawing, see Figs. 1.2.1 and 1.6.2 (left).

The *path* P_n is the graph with vertices $1, 2, \dots, n$ and edges $(i, i+1)$, $i = 1, 2, \dots, n-1$. The *cycle* C_n is the graph with vertices

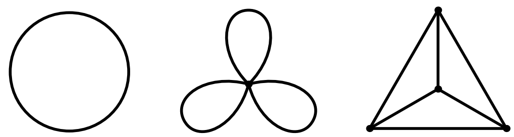

Figure 1.2.1. A cycle, a wedge of cycles, and the graph K_4

$1, 2, \ldots, n$ and edges $(1, n)$ and $(i, i+1)$, $i = 1, 2, \ldots, n-1$. (Do not confuse these graphs with a *path in a graph* and a *cycle in a graph*, which are defined below.)

The graph with n vertices any two of which are joined by an edge is called a *complete* graph and denoted by K_n. If the vertices of a graph can be partitioned into two sets so that no edge joins two vertices from the same set, then the graph is said to be *bipartite*, and the two sets of vertices are called its *parts*. By $K_{m,n}$ one denotes the bipartite graph with parts of size m and n that contains all the mn edges joining vertices from different parts. See Fig. 1.3.2.

Roughly speaking, a *subgraph* of a given graph is a part of this graph. Formally, a graph G is called a *subgraph* of a graph H if every vertex of G is a vertex of H and every edge of G is an edge of H. Note that two vertices of G joined by an edge in H are not necessarily joined by an edge in G.

A *path*[4] in a graph is a sequence $v_1e_1v_2e_2 \ldots e_{n-1}v_n$ such that for every i the edge e_i joins the vertices v_i and v_{i+1}. (The edges $e_1, e_2, \ldots, e_{n-1}$ are not necessarily pairwise distinct.) A *cycle* is a sequence $v_1e_1v_2e_2 \ldots e_{n-1}v_ne_n$ such that for every $i < n$ the edge e_i joins the vertices v_i and v_{i+1}, while the edge e_n joins the vertices v_n and v_1.

A graph is said to be *connected* if every pair of its vertices can be joined by a path, and *disconnected* otherwise. A graph is called a *tree* if it is connected and contains no simple cycles (i.e., cycles that do not pass twice through the same vertex). A *spanning tree* of a graph G is any subgraph of G that is a tree and contains all vertices of G. Clearly, every connected graph contains such a subgraph.

The definition of the operations of deleting an edge and deleting a vertex is clear from Fig. 1.2.2. The operation of *contracting an edge* (Fig. 1.2.2) deletes this edge from the graph, replaces its endpoints A and B with a vertex D, and replaces each edge from A or B to

[4] In graph theory, as opposed to topology, the term 'walk' is used.

a vertex C with an edge from D to C. (In contrast to the case of contracting an edge in a multigraph, each resulting edge of multiplicity greater than 1 is replaced with an edge of multiplicity 1.) For example, if the graph is a cycle with four vertices, then contracting any its edge results in a cycle with three vertices.

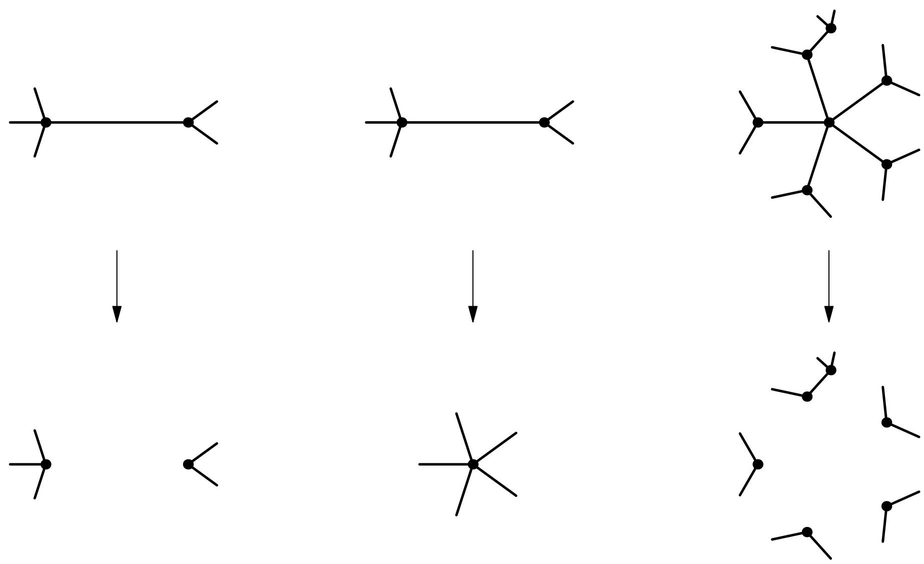

Figure 1.2.2. Deleting an edge $G - e$, contracting an edge G/e, and deleting a vertex $G - x$

In most of this book, one can use the notion of graph without loops or multiple edges. However, everything we have said is valid for the following generalization, which is even indispensable in some cases. A *multigraph* (or a *graph with loops and multiple edges*) is a square array (matrix) of nonnegative integers symmetric with respect to the main diagonal. The integer at the intersection of the ith row and jth column is interpreted as the number of edges (or the *multiplicity of the edge*) between the vertices i and j if $i \neq j$, and as the number of loops at the vertex i if $i = j$. An edge is said to be *multiple* if its multiplicity is greater than 1.

1.3. Graphs and Map Colorings in the Plane

A **plane graph** is a finite collection of non-self-intersecting polygonal lines in the plane such that any two of them meet only at their common endpoints (in particular, those with no common endpoints are disjoint).

The endpoints of the polygonal lines are called the *vertices* of the plane graph, and the polygonal lines themselves are its *edges*. Thus, to a plane graph there corresponds a graph (in the sense of § 1.2) for which the plane graph is a *plane drawing*. Sometimes, a plane graph is called just a graph, but this is not exactly correct, because one and the same graph can be drawn in the plane in different ways (if it can be drawn at all), see Fig. 1.3.1.

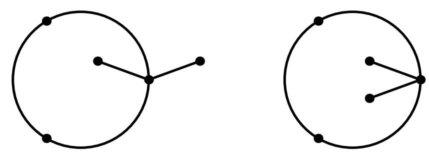

Figure 1.3.1. Different plane drawings of a graph

A graph is said to be **planar** if it can be represented by a plane graph.

1.3.1. The following graphs are planar:
(a) the graph K_5 without one edge (Fig. 1.7.2); (b) any tree;
(c) the graph of any convex polyhedron.

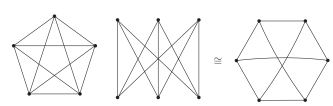

Figure 1.3.2. The nonplanar graphs K_5 and $K_{3,3}$

1.3.2. (a) The graph K_5 is not planar. (b) The graph $K_{3,3}$ is not planar.

(c) For every plane connected graph with V vertices and $E > 1$ edges, $E \leqslant 3V - 6$.

(d) Every plane graph contains a vertex with at most 5 incident edges.

A plane graph divides the plane into regions called its *faces*. Here is a rigorous definition.

A subset of the plane is said to be **connected** if any two its points can be joined by a polygonal line inside this set. (Caution: for subsets more general than those we consider here, the definition of connectedness is different!)

A **face** of a plane graph G is any of the connected parts into which the plane $\mathbb{R}^2$ is divided by the cuts along all the polygonal lines (= edges) of G, i.e., any maximal connected subset of $\mathbb{R}^2 - G$. Note that one of these parts is 'infinite'.

1.3.3. (a) Draw a plane graph G that has a face whose boundary contains three pairwise disjoint cycles.

(b) For every plane graph with $E > 1$ edges and F faces, $3F \leqslant 2E$.

(c)* **Euler's Formula.** *For every connected plane graph with V vertices, E edges, and F faces, $V - E + F = 2$.*

(d) Find a version of Euler's Formula for a plane graph with s connected components.

As to part (b), think about how many faces an edge belongs to and what is the smallest number of edges bounding a face.

The proof of Euler's Formula is given below. First, using this formula without proof, solve Problems 1.1.1 and 1.3.2.

1.4. Rigorous Proof of Euler's Formula

1.4.1. (a) We are given a non-closed non-self-intersecting polygonal line L in the plane and two points outside it. There is an algorithm for constructing a polygonal line that joins these points and does not intersect L.

(b) The same for a tree L in the plane whose edges are segments.

(c) If two segments are disjoint, then the distance between them is positive.

Hint. To construct the algorithms, use induction (or recursion). The induction step is based on deleting a pendant vertex. Cf. the construction of the regular neighborhood of a tree, see Fig. 1.6.3 (left) and the definition near this figure, [BE82, § 6], [CR, pp. 293—294]. Part (c) can be proved by looking at the possible relative positions of the segments.

The nontriviality of the algorithms from Problems 1.4.1 illustrates the nontriviality of the following assertions. (A similar remark applies to Assertion 1.4.3 (a) and Jordan's Theorem 1.4.3 (b).)

1.4.2. (a) Any non-closed non-self-intersecting polygonal line L in the plane $\mathbb{R}^2$ does not separate the plane, i.e., $\mathbb{R}^2 - L$ is connected.

(b) No tree in the plane separates the plane.

(c) Deleting an edge in a plane graph decreases the number of faces at most by 1.

(d) For any connected plane graph with V vertices, E edges, and F faces, $V - E + F \leqslant 2$.

Hint. Use the ideas from the solution of Problem 1.4.1.

1.4.3. (a) There is an algorithm that, given a closed non-self-intersecting polygonal line L in the plane and two points outside L, decides whether these points can be joined by a polygonal line that does not intersect L.

(The same is true even if a part of the given polygonal line outside some square containing the given points is deleted.)

(b) **Jordan's Theorem.** *Any closed non-self-intersecting polygonal line L in the plane $\mathbb{R}^2$ divides the plane into exactly two connected parts, i.e., $\mathbb{R}^2 - L$ is disconnected and is a union of two connected sets.*

Usually, by Jordan's Theorem one means a version of Theorem 1.4.3 (b) for *continuous curves* L, whose proof is much more involved [An03, Ch99]. While Theorem 1.4.3 (b) is sometimes called the *Piecewise Linear Jordan Theorem.*

A simple proof of Jordan's Theorem 1.4.3 (b) is given in [CR, pp. 292—295], see Remark 1.4.8. We present a similar, but slightly more complicated, proof. In return, it involves an interesting Intersection Lemma 1.4.4 and demonstrates the parity and general position techniques (Lemmas 1.4.5 and 1.4.6) useful for what follows.

Sketch of the proof of Jordan's Theorem 1.4.3 (b). The claim that the number of parts is at most 2 is simpler; it follows from Assertions 1.4.2 (b, c). Cf. [BE82, § 6], [CR, pp. 293—294].

The claim that the number of parts is greater than 1 is more difficult. To prove it, pick two points that are sufficiently close to a segment of the polygonal line L and symmetric with respect to this segment. Then

($*$) *it is these points that cannot be joined by a polygonal line that does not intersect L.*

This is implied by the following Intersection Lemma 1.4.4. □

Lemma 1.4.4 (intersection)**.** *Any two polygonal lines in a square joining different pairs of opposite vertices must intersect.*

The Intersection Lemma can be deduced from the following Parity Lemma 1.4.5 and Approximation Lemma 1.4.6 (a, b).

Several points in the plane are said to be **in general position** if no three of them lie on the same line and no three segments between them share a common interior point.

Lemma 1.4.5 (parity). *If the vertices of two closed plane polygonal lines are in general position, then the polygonal lines meet in an even number of points.*

Cf. the comments and proof in [Sk, § 1.3 'The intersection number for polygonal lines in the plane'].

A polygonal line $A_0 \dots A_n$ is said to be *vertex-wise* ε*-close* to a polygonal line $B_0 \dots B_m$ if $m = n$ and $|A_i - B_i| < \varepsilon$ for every $i = 0, 1, \dots, n$.

Lemma 1.4.6 (approximation). (a') *Take any* $\varepsilon > 0$ *and points* $A_1, \dots, A_n$ *in a square. Then there are points* $A'_1, \dots, A'_n$ *in the square such that the vertices of the square and* $A'_1, \dots, A'_n$ *are in general position, and* $|A_iA'_i| < \varepsilon$ *for any* $i = 1, \dots, n$.

(a) *Take any* $\varepsilon > 0$ *and polygonal lines* L_1, L_2 *in a square joining different pairs of opposite vertices. Then there exist polygonal lines* L'_1, L'_2 *in the square joining different pairs of opposite vertices such that the vertices of* L'_1, L'_2 *are in general position and* L'_1, L'_2 *are vertex-wise* ε*-close to* L_1, L_2.

(b') *For every pair of disjoint segments* XY *and* ZT *there is* $\alpha > 0$ *such that for any points* X', Y', Z', T' *in the plane, the inequalities* $|XX'|, |YY'|, |ZZ'|, |TT'| < \alpha$ *imply that the segments* $X'Y'$ *and* $Z'T'$ *are disjoint.*

(b) *If two polygonal lines* L_1, L_2 *do not intersect, then there exists* $\varepsilon > 0$ *such that any polygonal lines* L'_1, L'_2 *that are vertex-wise* ε*-close to* L_1, L_2 *do not intersect either.*

Sketch of the proof of Euler's Formula 1.3.3 (c). Induction on the number of edges outside a spanning tree. The induction base is Assertion 1.4.2 (b). The induction step follows from the fact that

(∗∗) *if deleting an edge from a plane graph results in a connected graph, then the number of faces decreases at least by* 1.

This can be proved analogously to the difficult part of Jordan's Theorem 1.4.3 (b) using the Intersection Lemma 1.4.4. □

The Intersection Lemma 1.4.4 is also useful for other results. It is often (e.g. in the following problem) more convenient to apply it instead of Jordan's Theorem 1.4.3 (b).

1.4.7. (a) Two bikers start at the same point moving northward and eastward, respectively. Both return (for the first time) to the initial point from south and west, respectively.

(b) Three bikers start at the same point moving westward, northward, and eastward, respectively. All of them arrive at another point from west, north, and east, respectively.

(a, b) Show that one of the bikers has crossed the track of another one. (See the middle pictures at Figs. 1.5.2 and 1.6.2 (left); the starting point is not counted as an intersection point of tracks; you may assume that the paths of the bikers are polygonal lines.)

Remark 1.4.8. (a) (on the proof of Jordan's Theorem 1.4.3 (b)) Jordan's Theorem is the special case of Euler's Formula 1.3.3 (c) for a graph that is a cycle. So deducing Jordan's Theorem from Euler's Formula would create a vicious circle.

The idea of the proof of claim $(*)$ is given in [CR, pp. 293—294], though the claim itself (i.e., the fact that $B \neq \varnothing$) is neither stated nor proved there. The argument uses simplified versions of the Parity Lemma (in the fifth paragraph at p. 293). At the beginning of the argument, one must pick a direction that is not parallel to any line passing through two vertices of the polygon (including nonadjacent ones); otherwise, in the fifth paragraph at p. 293, there arise more than two cases, contrary to what is stated.

The proof of claim $(*)$ given in [BE82, § 6] uses the Parity Lemma 1.4.5.

The proof of Jordan's Theorem in [Pr14′, pp. 19—20] is incomplete, because it uses without proof nontrivial facts similar to the Parity Lemma. More specifically, for the reader not familiar with Jordan's Theorem, the claim (given without proof) from the second proposition at p. 20 (as well as the fact from the first proposition at p. 20 that the parity changes continuously) seems to be more complicated than Jordan's Theorem itself, whose proof uses this claim.

(b) (on the proof of Euler's Formula 1.3.3 (c)) In a beginners' course, it is reasonable not to prove the above assertion $(**)$, which is geometrically obvious. One should only draw the reader's attention to the fact that this assertion is not proved, to algorithmic problems illustrating its nontriviality (cf. Problems 1.4.1 and 1.4.3 (a)), and to the remark about 'vicious circle' given in the solution of Problem 1.3.2 (a). Unfortunately, this assertion is not proved, and even not commented upon, in some expositions which claim to be rigorous[5]. This might give the wrong idea that the proof of Euler's Theorem does not use results close to Jordan's Theorem, and hence does not involve the corresponding difficulties.

1.5. Planarity of Disks with Ribbons

Consider a word of length $2n$ in which each of n letters occurs exactly twice. Take a convex polygon in the plane. Choose an orientation of the closed polygonal line that bounds it. Take $2n$ disjoint segments on this polygonal line corresponding to the letters of the word in the order they occur in it. For each letter, join (not necessarily in the plane) the two corresponding segments by a ribbon (i.e., a 'stretched' and 'creased' rectangle) so that different ribbons do not intersect each other. The **disk with ribbons** corresponding to the given word is the union of the constructed (two-dimensional) convex polygon and the ribbons[6].

A ribbon is said to be **twisted** if the arrows on the boundary of the polygon have the same direction 'when translated' along the ribbon, and **untwisted** if they have opposite directions (Fig. 1.5.1).

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[5]Here are two examples. In [Pr14′, proof of Theorem 1.6], it is not explained why "deleting one boundary edge decreases the number of faces by 1"; this fact is not simpler than Jordan's Theorem 1.4.3 (b), whose proof [Pr14′, p. 19—20] is nontrivial for a beginner and contains the gap described at the end of Remark 1.4.8. The proof of Euler's Formula in [Om18, Chapter 7, § 2] also includes neither explanations of a similar fact, no references to Jordan's Theorem (though the nontriviality of this theorem is discussed earlier).

[6]More precisely, a disk with ribbons is *any* shape obtained by this construction; cf. the remark before Problem 2.2.2. Still more precisely, it is the pair consisting of this union and the union of loops corresponding to the ribbons. This terminological distinction is not relevant for the realizability we study here, but it is important for calculating the number of disks with ribbons, see § 1.7 and [Sk, 'Orientability and classification of thickenings'].
This informal definition can be formalized using the notions of homeomorphism and gluing (§ 2.7 and Example 5.1.1.c); cf. § 1.7.

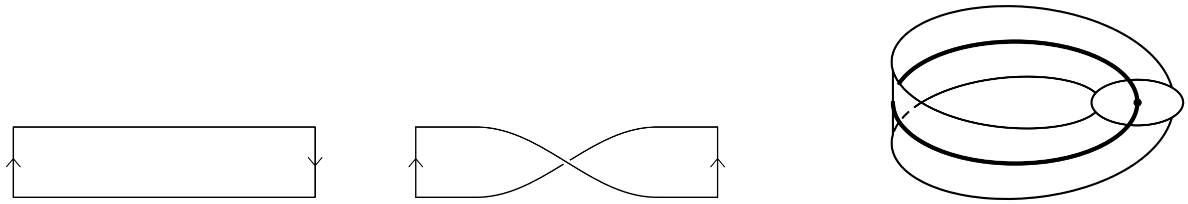


Figure 1.5.1. Left: arrows that have opposite directions 'when translated' along the ribbon. Right: a disk with a twisted ribbon (the Möbius strip)

For example, the annulus and the cylinder (Fig. 2.1.2 and the text before it) are disks with one untwisted ribbon, while the disk with n holes (Fig. 3.9.2) is a disk with n untwisted ribbons. For other examples of disks with untwisted ribbons, see Figs. 1.5.2 and 1.5.3.

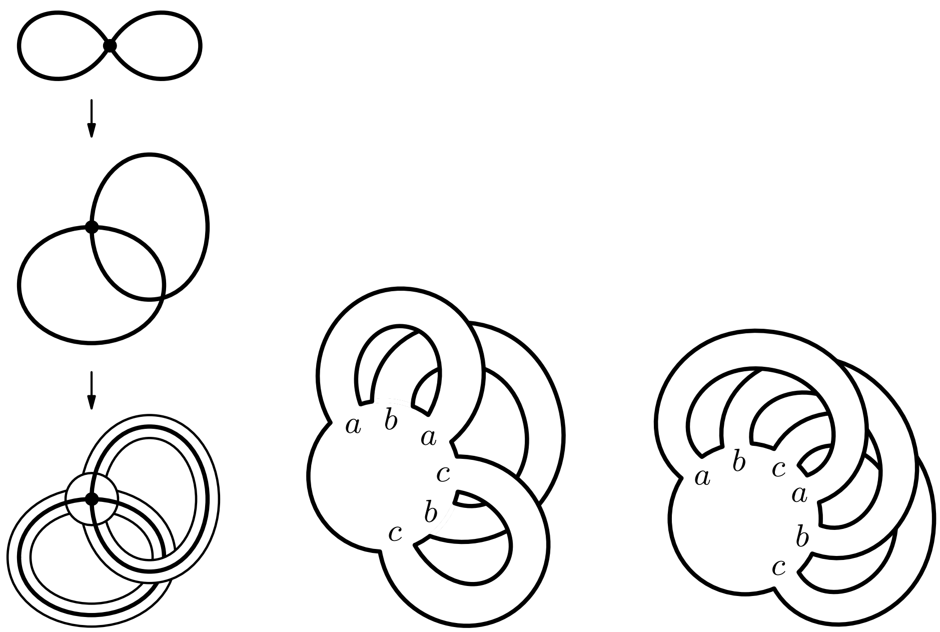


Figure 1.5.2. Left: the top picture shows a multigraph with one vertex and two loops, the middle one is a drawing of this multigraph in the plane, and the bottom one is the corresponding disk with untwisted ribbons; it corresponds to the word $(abab)$. Middle and right: the disks with three untwisted ribbons corresponding to the words $(abacbc)$ and $(abcabc)$.

Ribbons a and b in a disk with untwisted ribbons are said to **interlace** if the segments along which they are glued to the polygon

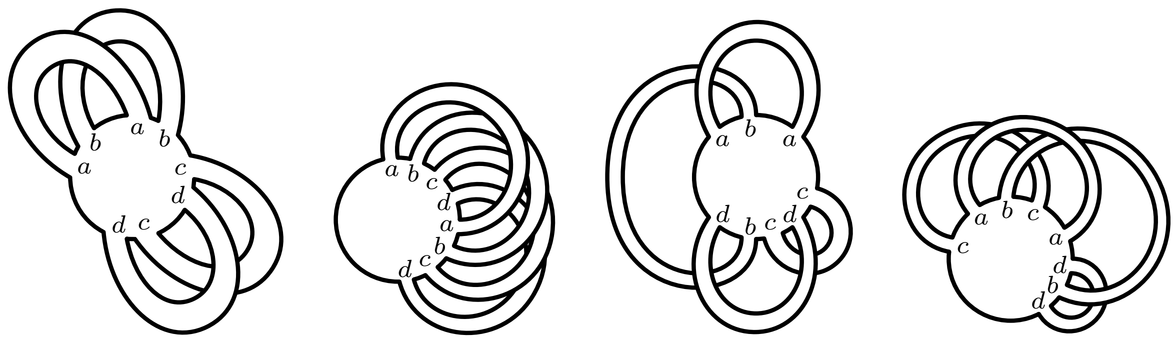


Figure 1.5.3. Disks with four untwisted ribbons (which cannot be realized on the torus)

alternate along its boundary, i.e., occur in the cyclic order ($abab$), and not ($aabb$).

Lemma 1.5.1. *A disk with untwisted ribbons can be cut out of the plane if and only if it has no interlacing ribbons.*

A **boundary circle** of a disk with ribbons is a connected part of the set of its points that it approaches 'from one side'. This informal definition is formalized in § 5.4. In Fig. 1.5.2 (middle and right), the boundary circles are shown in bold. For example, the disks with untwisted ribbons in Fig. 1.5.2 have one, two, and two boundary circles, respectively.

1.5.2. (a) How many boundary circles can a disk with two untwisted ribbons have (more precisely, find all F for which there exists a disk with two untwisted ribbons that has F boundary circles)?

(b) How many boundary circles do the disks with untwisted ribbons in Fig. 1.5.3 have?

(c) How many boundary circles can a disk with five untwisted ribbons have?

(d) Adding a non-twisted ribbon changes the number of boundary circles by ± 1.

1.5.3. (a) The number of boundary circles of a disk with n untwisted ribbons does not exceed $n + 1$.

(a') The number of boundary circles of a disk with n ribbons, of which at least one is twisted, does not exceed n.

(b) *Lemma.* For a disk with n untwisted ribbons, each of the assumptions of Lemma 1.5.1 is equivalent to the number of boundary circles being equal to $n+1$.

(c) Given a word of length $2n$ in which each of n letters occurs exactly twice, construct a graph with the number of connected components equal to the number of boundary circles of the disk with untwisted ribbons corresponding to this word. (Thus, this number can be found by computer without drawing a figure.)

1.6. Planarity of Thickenings

Given a graph with n vertices, consider the union of n pairwise disjoint convex polygons in the plane. On each of the closed polygonal lines bounding the polygons take disjoint segments corresponding to the edges incident to the corresponding vertex. For each edge of the graph, join (not necessarily in the plane) the corresponding two segments by a ribbon so that the ribbons do not intersect each other (Fig. 1.6.1). A **thickening** of the graph is the union of the constructed convex polygons and ribbons. The graph is called the *spine*, or the *thinning*, of this union. A remark similar to that in footnote 6 at the beginning of § 1.5 applies to this case as well.

Figure 1.6.1. Joining disks with a ribbon

A thickening is said to be **orientable** if the boundary circles of the polygons can be endowed with orientations so that every ribbon becomes untwisted, i.e., the arrows on the boundaries of the polygons have the *opposite direction* ‘when translated’ along the ribbon (Fig. 1.5.1, left). Note that each of the pictures in Fig. 1.6.1 can correspond to such a way of joining disks with ribbons. A thickening is said to be **non-orientable** if there are no such orientations.

For example, orientable thickenings of the graphs $K_{3,2}$ and $K_{3,3}$ are shown in Fig. 1.6.2.

A disk with ribbons (§ 1.5) is a thickening of a multigraph consisting of one vertex with several loops.

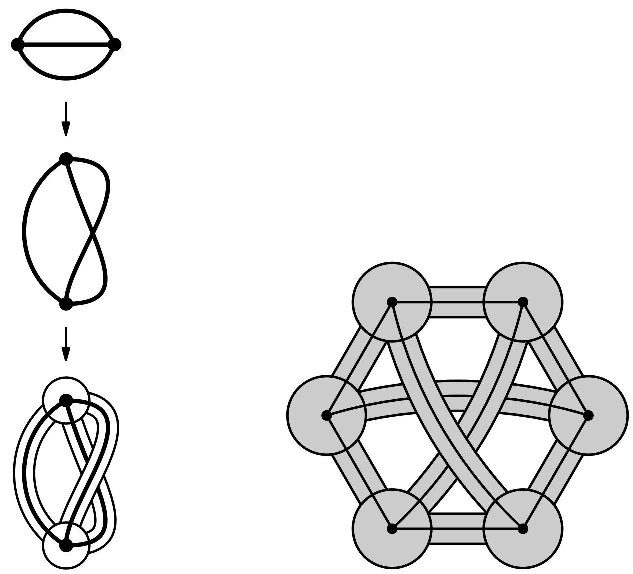


Figure 1.6.2. Left: the top picture shows the graph $K_{3,2}$, the middle one is a drawing of this graph in the plane, and the bottom one is the corresponding thickening.
Right: an oriented thickening of the graph $K_{3,3}$

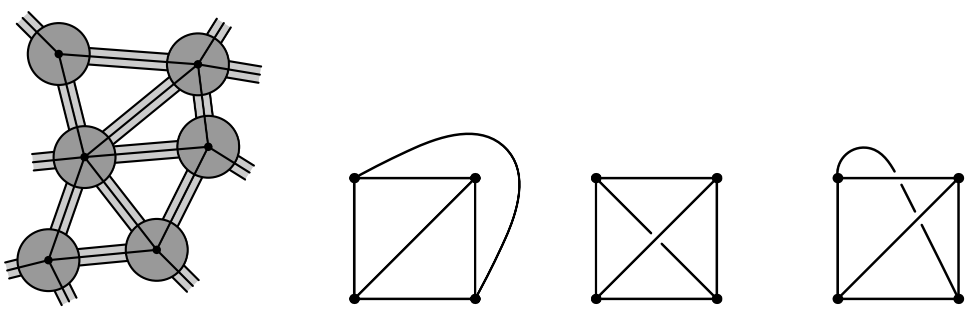


Figure 1.6.3. Left: the caps and ribbons (called clusters and pipes in [MT01]) form the regular neighborhood (thickening) of a graph on a surface.
Right: drawings of the graph K_4 in the plane

The **regular neighborhood** of a graph drawn in the plane (or on a surface, see § 2.1) without edges crossing is the union of caps and ribbons constructed as shown in Fig. 1.6.3 (left). For a rigorous definition, see § 5.4. The regular neighborhood of a graph G is an oriented thickening of G (Fig. 1.6.3 (left)). More generally, if we have a *general position* map of a graph G to the plane (or to a surface, see

§ 2.1), then we can construct an oriented thickening of G 'corresponding' to this map (Figs. 1.5.2 and 1.6.2 (left), Fig. 1.6.3 (right)).

An oriented thickening is said to be **planar** if it can be cut out of the plane.

1.6.1. (a) Every thickening of a tree is planar.

(b) Every orientable thickening of a cycle is planar.

(c) Every orientable thickening of a unicyclic graph is planar. (A graph is said to be *unicyclic* if it becomes a tree after deleting an edge.)

(d) Is the orientable thickening of the graph $K_{3,2}$ shown in Fig. 1.6.2 (left) planar?

(e) Which of the orientable thickenings of the graph K_4 (Fig. 1.6.3 (right)) are planar?

(f) A graph is planar if and only if it has a planar orientable thickening.

(g) A *rotation system* of a graph is an assignment to each vertex of an oriented cyclic order on the edges incident to this vertex. Every graph has finitely many rotation systems (moreover, there is an algorithm searching through those rotation systems).

Deciding the planarity of graphs reduces to deciding the planarity of orientable thickenings, see Assertion 1.6.1 (f, g).

1.6.2. (a) Define the operation of *contracting an edge* of a thickening so that it would give the operation of contracting an edge of a graph and preserve planarity.

(b) Draw the thickenings obtained from the thickenings of the graph K_4 (Fig. 1.6.3 (right)) by contracting the 'top horizontal' edge.

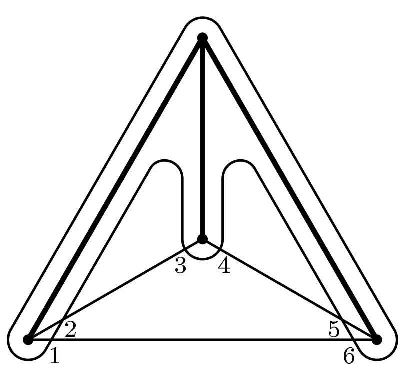


Figure 1.6.4. Walking around a spanning tree

Theorem 1.6.3. *(a) There is an algorithm for deciding the planarity of thickenings.*

(b) Each of the following conditions on an orientable thickening of a connected graph G is equivalent to the planarity of this thickening.

(I) *For every spanning tree T, going along the boundary of the thickening of T (Fig.* 1.6.4) *we obtain a cyclic sequence of edges not from T, in which every edge occurs twice; then any two edges in this sequence do not alternate, i.e., occur in the cyclic order* $(aabb)$, *and not* $(abab)$.

(E) *The number of boundary circles of the thickening is* $E - V + 2$, *where V and E are the numbers of vertices and edges.*

(Boundary circles of a thickening are defined analogously to boundary circles of a disk with ribbons.)

(S) *The thickening 'does not contain' the 'figure eight' and 'letter theta' subthickenings shown in Figs.* 1.5.2 *and* 1.6.2 *(left). (More precisely, the graph does not contain a subgraph* homeomorphic *to one of the graphs shown in the top pictures of these figures such that the* restriction *of the thickening to this subgraph is* homeomorphic *to one of the thickenings shown in the bottom pictures of these figures.)*

1.6.4. Every thickening

(a) of a tree has one boundary circle;

(c) of a connected graph with V vertices and E edges has at most $E - V + 2$ boundary circles.

1.6.5. Every non-orientable thickening of a connected graph with V vertices and E edges has at most $E - V + 1$ boundary circles.

Hint: Assertions 1.6.4.c and 1.6.5 follow from Assertions 1.5.3.a,a'.

1.7. Hieroglyphs and Orientable Thickenings*

In this subsection we give an interpretation of the constructions from §§ 1.5 and 1.6. A *representation* of a hieroglyph is a word of length $2n$ in which each of n letters occurs exactly twice. A *hieroglyph* is an equivalence class of such words up to renaming of letters and cyclic shift. Other names: chord diagram, one-vertex multigraph with rotations.

Hieroglyphs are drawn as shown in Figs. 1.5.2 (left) and 1.7.1, i.e., as families of loops in the plane with a common vertex. A cyclic order

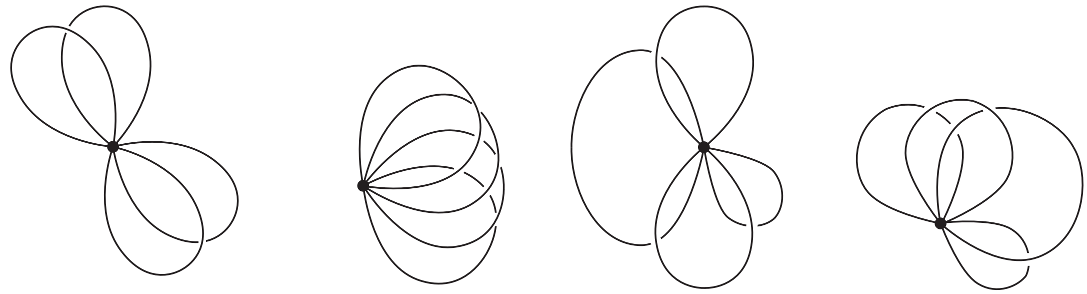


Figure 1.7.1. Hieroglyphs of four letters (this is the 'one–dimensional counterpart' of Fig. 1.5.3)

is determined by enumerating the segments incident to the vertex in the counterclockwise direction.

A hieroglyph can also be represented by a shape formed by $2n$ segments in the plane with a common vertex ('plane star with $2n$ rays) such that the segments meet only at the common vertex and are divided into pairs according to the word-hieroglyph. Joining the segments in each pair by a polygonal line (these polygonal lines are allowed to intersect each other), we obtain the previous representation.

A *disk with ribbons corresponding to a hieroglyph* is a disk with ribbons corresponding to any representation of the hieroglyph. Hence, a hieroglyph can also be defined as the unique corresponding disk with untwisted ribbons (§ 1.5). For example, Fig. 1.5.3 shows the disks with untwisted ribbons corresponding to the hieroglyphs in Fig. 1.7.1.

1.7.1. (a) How many three-letter hieroglyphs are there? (b) And four-letter hieroglyphs?

A *half-edge* in a graph is a 'half' of an edge. A loop of multiplicity k gives rise to $2k$ half-edges. A (one-dimensional) *orientable thickening* of a graph is this graph equipped with oriented cyclic orders on the half-edges incident to each vertex. See examples in Figs. 1.6.2 and 1.6.3 (right).

In § 1.6 we have given an 'equivalent two-dimensional definition' of an orientable thickening. It is more complicated due to being two-dimensional (rather than one-dimensional), but it is this definition that arises in other areas of mathematics. Besides, it is sometimes more convenient to work with.

§ 2. Intuitive Problems About Surfaces

> Wissen war ein bisschen Schaum, der über eine Woge tanzt. Jeder Wind konnte ihn wegblasen, aber die Woge blieb.
>
> E. M. Remarque. *Die Nacht von Lissabon*[7]

In § 2.1 we recall the definitions of basic surfaces. The reader may omit this subsection and return to it when necessary. Subsection 2.2 contains intuitive problems about cutting surfaces and cutting out of surfaces. Here we state Riemann's and Betti's Theorems 2.3.5, which are used to prove than a surface cannot be cut out of another surface. Subsection 2.4 contains basic results about graphs and map colorings on surfaces (Theorems 2.4.4, 2.4.5 (b), 2.4.7). They are similar to the results from §§ 1.1 and 1.3 about graphs and map colorings in the plane. The proofs involve an analog of Euler's Formula, namely, Euler's Inequality 2.5.3 (a). This inequality is proved in § 2.5 together with Riemann's Theorem 2.3.5 (a). In § 2.6, an algorithm is constructed for deciding whether a graph can be realized on a given surface (i.e., Theorem 2.4.5 (b) is proved). In § 2.7 we informally introduce and study the notion of topological equivalence of surfaces. In particular, Assertions 2.7.7 (b) and 2.7.9 (b) demonstrate one of the main ideas of the proof of Theorem 5.6.1 on classification of surfaces. Subsection 2.8 contains versions of the previous examples and results for non-orientable surfaces.

2.1. Examples of Surfaces

If you are not familiar with Cartesian coordinates in the space, then at the beginning of the book you may omit coordinate definitions and work with intuitive descriptions and drawings (given after coordinate definitions).

---

[7] Knowledge was a speck of foam dancing on top of a wave. Every gust of wind could blow it away; but the wave remained. (E. M. Remarque. *The Night in Lisbon*)

The **sphere** S^2 is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 1$:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}.$$

This is the same thing as the set of all points (x, y, z) of the form

$$(\cos\varphi \cos\psi, \sin\varphi \cos\psi, \sin\psi).$$

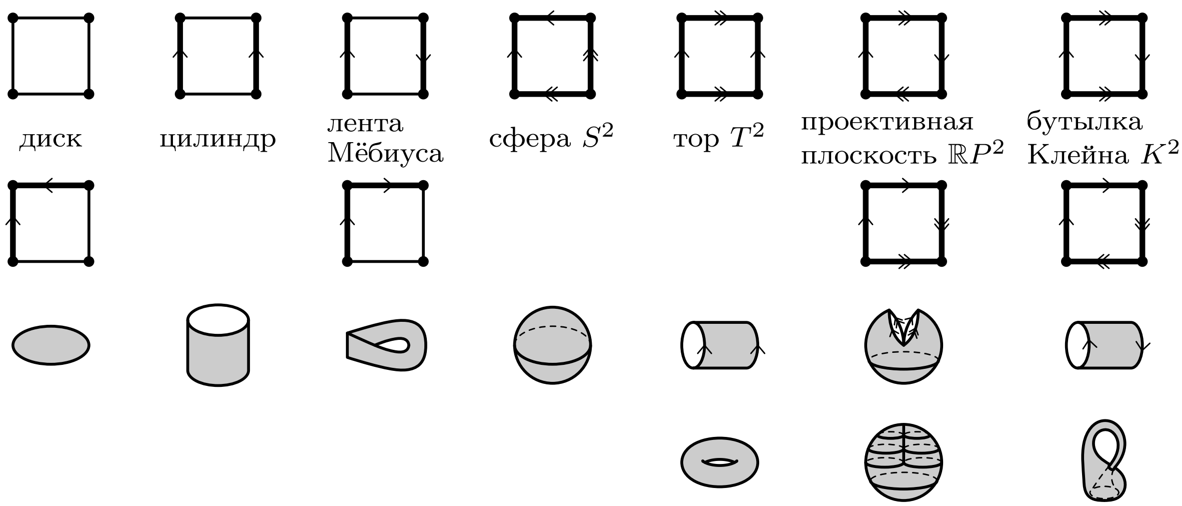


Figure 2.1.1. The surfaces obtained by gluing together sides of a rectangle

In what follows, by a rectangle we mean a two-dimensional part of the plane (and not its boundary), and 'gluing' includes a 'continuous deformation' that drags the points to be glued to each other.

The sphere is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AD}$, $\overrightarrow{CB}$ and $\overrightarrow{CD}$ with the directions indicated in the picture (the fourth column in Fig. 2.1.1).

The **annulus** is the set $\{(x, y) \in \mathbb{R}^2 \;:\; 1 \leqslant x^2 + y^2 \leqslant 2\}$ (Fig. 6.3.1). The **lateral surface of a cylinder** (Fig. 2.1.2 (right)) is the set

$$\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = 1, 0 \leqslant z \leqslant 1\}.$$

Each of these shapes is obtained from a rectangle $ABCD$ by 'gluing together' the pair of opposite sides $\overrightarrow{AB}$ and $\overrightarrow{DC}$ 'with the same direction' (the second column in Fig. 2.1.1).

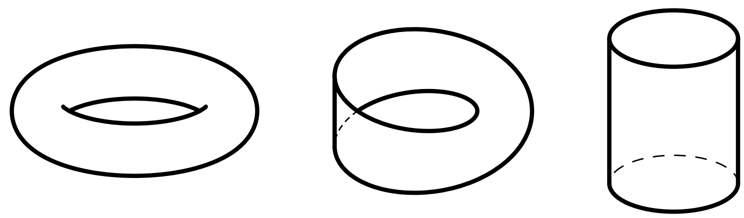


Figure 2.1.2. The torus, Möbius strip, and lateral surface of a cylinder

The **torus** T^2 is the shape obtained by rotating the circle $(x-2)^2+y^2=1$ about the Oy axis (Fig. 2.1.2 (left)).

The torus is the 'surface of a doughnut'. It is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of opposite sides $\overrightarrow{AB}$ and $\overrightarrow{DC}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ 'with the same direction' (the fifth column in Fig. 2.1.1).

The **Möbius strip** is the set of points in $\mathbb{R}^3$ swept by a bar of length 1 rotating uniformly about its center as this center moves uniformly along a circle of radius 9 when the bar makes half a turn (Fig. 2.1.2 (middle)).

The Möbius strip is obtained from a rectangle $ABCD$ by 'gluing together' two opposite sides $\overrightarrow{AB}$ and $\overrightarrow{CD}$ 'with opposite directions' (the third column in Fig. 2.1.1).

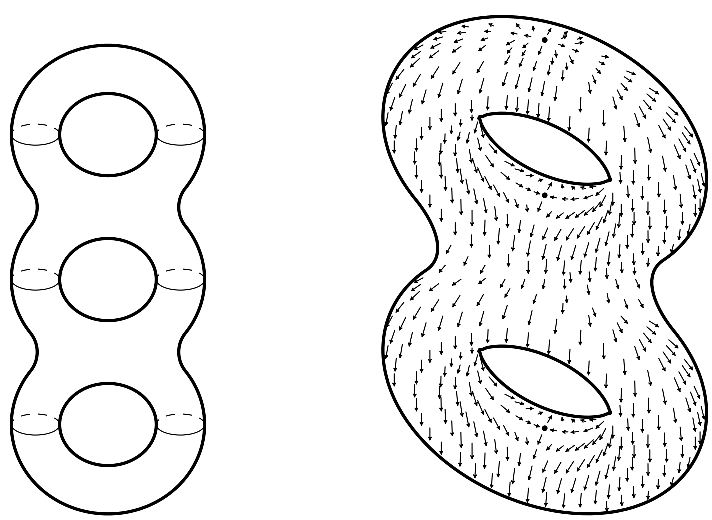


Figure 2.1.3. The spheres with two and three handles

The **sphere with g handles** S_g, where $g \geqslant 1$, is the set of points $(x, y, z) \in \mathbb{R}^3$ such that

$$x^2 + \prod_{k=1}^{g} ((z-4k)^2 + y^2 - 4)^2 = 1.$$

The *sphere with* 0 *handles* is the sphere S^2. The sphere with one handle is the torus. The spheres with two and three handles are shown in Fig. 2.1.3.

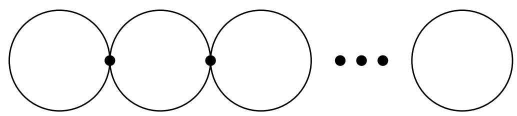


Figure 2.1.4. A 'chain of circles' in the plane

The equation $\prod_{k=1}^{g} ((z-4k)^2 + y^2 - 4) = 0$ defines a 'chain of circles' in the plane Oyz (Fig. 2.1.4). The sphere with g handles is the boundary of the 'tubular neighborhood' of this chain in the space. Hence, the sphere with g handles is obtained from the sphere by 'cutting out' $2g$ disks and then attaching g curvilinear lateral surfaces of cylinders to g pairs of boundary circles of these disks (Fig. 2.1.5).

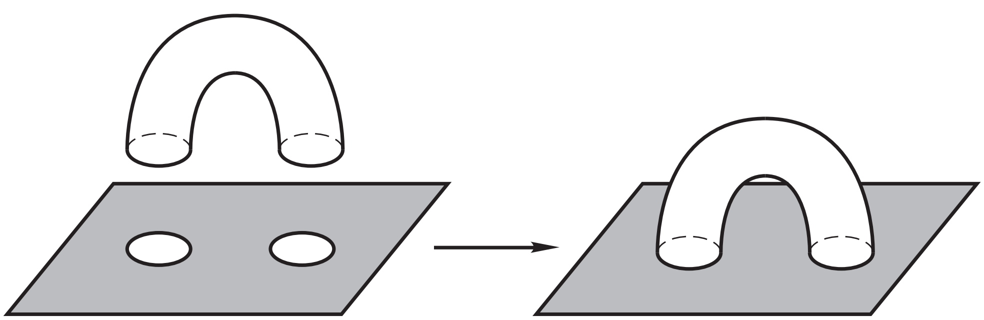


Figure 2.1.5. Attaching a handle

The **sphere with g handles and a hole** $S_{g,0}$ is the part of the sphere with g handles that lies below or on the plane situated slightly below the tangent plane at the top point (i.e., the part of S_g that lies in the domain $z \leqslant 4g + 2$). This shape is obtained from the sphere with handles by 'cutting out a hole'.

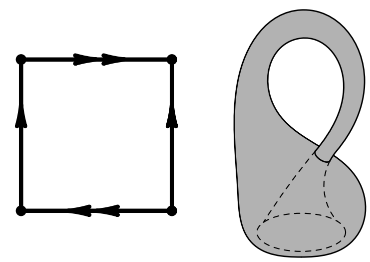


Figure 2.1.6. The Klein bottle: (a) gluing; (b) a drawing in $\mathbb{R}^3$

Informally, the *Klein bottle* is obtained from a rectangle $ABCD$ by 'gluing together' the pairs of opposite sides, the pair $\overrightarrow{AB}$, $\overrightarrow{DC}$ 'with the same direction', and the other pair $\overrightarrow{BC}$, $\overrightarrow{DA}$ 'with opposite directions' (Fig. 2.1.6 (a)).

Consider in $\mathbb{R}^4$ the circle $x^2 + y^2 = 1$, $z = t = 0$ and the family of three-dimensional normal planes to this circle. Strictly speaking, the **Klein bottle** K is the set of points in $\mathbb{R}^4$ swept by a circle ω as its center moves uniformly along the circle under consideration, while the circle ω itself rotates uniformly by angle π (in the moving three-dimensional normal plane, about its own diameter moving together with this plane).

The projection of the Klein bottle to $\mathbb{R}^3$ is shown in Fig. 2.1.6 (b).

In what follows, 'surface' is a collective term for the shapes defined above, and not a mathematical term (cf. the definition of a 2-manifold in § 4.5).

2.2. Cutting Surfaces and Cutting out of Surfaces

In the problems of this subsection, you are asked to give not rigorous proofs, but large, comprehensible, and preferably beautiful pictures.

2.2.1. (a) For every n there exist n points in $\mathbb{R}^3$ such that the segments between them have no common interior points (i.e., every graph can be drawn in $\mathbb{R}^3$ without edges crossing).

(b) Every graph can be drawn without edges crossing on a book with a certain number of sheets (Fig. 2.2.1; the definition is given after the figure) depending on the graph. More precisely, for every n there exists an integer k, as well as n points and $n(n-1)/2$ non-self-intersecting polygonal lines on a book with k sheets such that every pair of points is joined by a polygonal line and no polygonal line intersects the interior of another polygonal line.

(c) The same as in part (b) with 3 sheets instead of k.

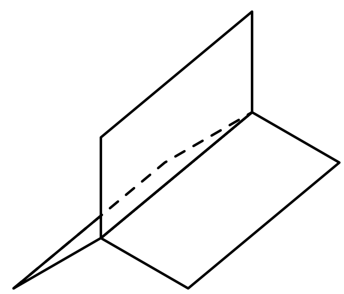
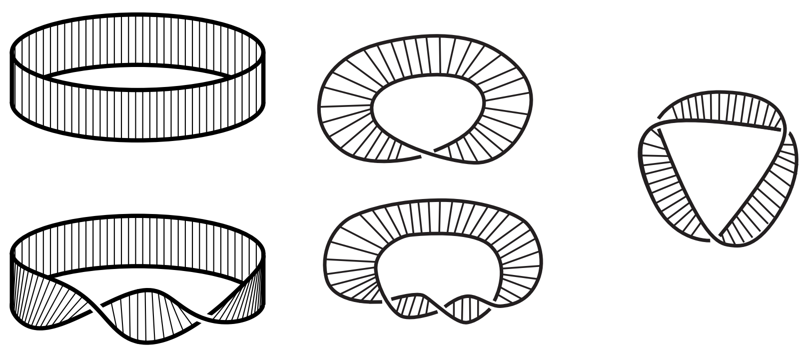


Figure 2.2.1. A book with three sheets

In $\mathbb{R}^3$ consider n rectangles XYB_kA_k, $k = 1, 2, \ldots, n$, any two of which have only the segment XY in common. The *book with n sheets* is the union of these rectangles; see Fig. 2.2.1 for the case $n = 3$.



(a) (b)

Figure 2.2.2. Nonstandard (a) annuli; (b) Möbius strips

A *nonstandard annulus* is any shape obtained from a rectangle by gluing a pair of opposite sides 'with the same direction' (Fig. 2.2.2 (a)). This informal definition can be formalized using the notions of homeomorphism and gluing (§ 2.7 and Example 5.1.1.c). In a similar way one defines a nonstandard Möbius strip (Fig. 2.2.2 (b)), torus with a hole, Klein bottle with a hole, etc. They will be used only in this subsection (one cuts nonstandard shapes out of standard ones); the word 'nonstandard' will be omitted.

2.2.2. Cut the Möbius strip so as to obtain

(a) an annulus; (b) an annulus and a Möbius strip.

2.2.3. Cut the Klein bottle (Fig. 2.1.6) so as to obtain
(a) two Möbius strips; (b) one Möbius strip.

2.2.4. Cut out the following shapes from the book with three sheets (Fig. 2.2.1):
(a) a Möbius strip; (b) a torus with a hole;
(c) a sphere with two handles and a hole;
(d) a Klein bottle with a hole.

2.2.5. Let A, B, C, D be points on the boundary circle of a torus with a hole (in this order along the circle). A rectangle $A'B'D'C'$ is attached to the torus with a hole by gluing AB to $A'B'$ and CD to $C'D'$. From the resulting shape (i.e., from a torus with a hole and a Möbius strip), cut out three pairwise disjoint Möbius strips.

2.3. Impossibility of Cutting and Separating Curves

2.3.1. (a) A torus with a hole cannot be cut out of the plane.

(b) For $k < n$, a sphere with n handles and a hole cannot be cut out of the sphere with k handles.

(c) Two disjoint Möbius strips cannot be cut out of the Möbius strip.

(d) Find all g, m, g', m' for which g' tori with a hole and m' Möbius strips (all $g' + m'$ shapes pairwise disjoint) can be cut out of a disk with g handles and m Möbius strips (see the definitions before Figs. 2.1.5 and 2.8.1).

Proof of (a). Part (a) follows from the Intersection Lemma 1.4.4 or from the (essentially equivalent) nonplanarity of the graph K_5 (Assertion 1.3.2 (a)), because the analogues of these results for the torus are false (cf. Assertion 2.4.1 (a)).

Alternatively, assume to the contrary that a torus with a hole is cut out of the plane. Take a closed non-self-intersecting curve γ on this torus with a hole such that γ does not separate it (Assertion 2.3.2.a). In the next paragraph we prove that γ does not separate the sphere, contradicting Jordan's Theorem 1.4.3 (b) (the details are necessary because e.g. the boundary circle of the disk does not separate the disk, but does separate the plane containing the disk).

Pick any two points in the plane that do not lie on γ. Join them with a polygonal line α 'in general position' with respect to γ. This polygonal line meets γ in a finite number of points. For each such point A, take a small segment α_A of α that contains A in its interior. The endpoints of α_A lie on the torus with a hole. Hence, they can be joined by a polygonal line α'_A that does not intersect γ. Replace each segment α_A with α'_A. We obtain a polygonal line that joins the given points and does not intersect γ. □

Comments on the proof of (b,c,d). Part (b) follows from Theorem 2.3.5 (c) and Assertion 2.3.3.c. Part (b) can also be deduced from Assertion 2.4.4 (c), or from Theorem 2.3.5 (a) and Assertion 2.3.3.a (observe that both Assertion 2.4.4 (c) and Theorem 2.3.5 (a) use Euler's Inequality 2.5.3 (a)). The details of deduction from Theorems 2.3.5 (c) or 2.3.5 (a) have to be checked, cf. (a).

Analogously, parts (c) can be deduced from either of Assertions 2.8.2 (a), 2.8.2 (c) or 2.8.3 (b).

To solve part (d), it is helpful to use Assertion 2.8.5 (c), see also Assertion 2.6.6 and Problem 6.7.7. □

2.3.2. (a) Draw a closed curve on the torus such that cutting along this curve does not separate the torus.

(b) The same for the Möbius strip.

(c) Draw two closed curves on the torus such that cutting along their union does not separate the torus.

(d) Draw two closed disjoint curves on the Klein bottle such that cutting along their union does not separate the Klein bottle.

Curves and graphs on the torus can be easily defined by regarding the torus as obtained from a rectangle by gluing. A (*piecewise linear*) *curve on the torus* is then a family of polygonal lines in the rectangle satisfying certain conditions (work out these conditions!). In a similar way, other surfaces can be obtained from plane polygons by gluing (for spheres with handles, see Problem 2.3.4). This allows one to define curves and graphs on other surfaces. Another formalization is given in § 5, see also § 4.

2.3.3. On the sphere with g handles S_g there are

(a) g closed pairwise disjoint curves, whose union does not separate S_g.

(b) $2g$ closed curves, of which any two intersect by a finite number of points, and whose union does not separate S_g.

(c) a non-separating wedge of $2g$ cycles.

2.3.4. For every $g > 0$, obtain S_g by gluing together sides of a $4g$-gon. (See visualization in `https://www.youtube.com/watch?v=G1yyfPShgqw` and in `https://www.youtube.com/watch?v=U5N5mg3MePM`.)

It turns out that cutting the torus along the union of any two disjoint closed curves inevitably separates the torus. This is a special case of the following generalizations of Jordan's Theorem 1.4.3 (b).

Theorem 2.3.5. (a) (Riemann) *The union of any* $g + 1$ *pairwise disjoint closed curves on* S_g *separates* S_g.

(b) (Betti) *Suppose that on* S_g *there are* $2g + 1$ *closed curves, of which any two intersect by a finite number of points. Then the union of the curves separates the sphere with g handles.*

(c) *Any wedge of* $2g + 1$ *cycles drawn without self-intersections on* S_g *separates* S_g.

Here the curves are allowed to be self-intersecting; however, the case of non-self-intersecting curves is the most interesting, and the general case can be easily reduced to it.)

These results (strictly speaking, for the *piecewise linear case*) follow from Euler's Inequality 2.5.3 (a). For part (c) the deduction is clear, for parts (a,b) see § 2.5.

2.4. Graphs on Surfaces and Map Colorings

The definition and discussion of a drawing of a graph on a surface without edges crossing is analogous to the case of the plane, see § 1.3. The formalization is outlined after Problem 2.3.2 and described in § 5.2, but can be omitted on first acquaintance.

The torus, Möbius strip, and other shapes are assumed to be *transparent*, i.e., a point (or a subset) that 'lies on one side of a surface' 'lies on the other side as well'. In a similar way, in geometry we speak about a triangle in the plane, rather than a triangle on the upper (or lower) side of the plane.

2.4.1. Draw the following graphs on the torus without edges crossing:

(a) K_5; (b) $K_{3,3}$; (c) K_6; (d) K_7; (e)* $K_{4,4}$; (f)* $K_{6,3}$.

The definition of a graph **realizable** on the torus or on a sphere with handles is analogous to that of a planar graph.

Proposition 2.4.2. *Any graph can be realized on a sphere with a certain number* (*depending on the graph*) *of handles.*

2.4.3. (a) The graph K_8; (b) the graph $K_{5,4}$; (c)* the graph $K_5 \sqcup K_5$ are not realizable on the torus.

To prove Assertions 2.4.3 and 2.4.4, we need Euler's Inequality 2.5.3 (a). Here is a version of Assertion 2.4.3 for spheres with handles.

Proposition 2.4.4. (a) *The graph K_n is not realizable on a sphere with less than $(n-3)(n-4)/12$ handles.*

(b) *The graph $K_{m,n}$ is not realizable on a sphere with less than $(m-2)(n-2)/4$ handles.*

(c)* *The disjoint union of $g+1$ copies of the graph K_5 is not realizable on the sphere with g handles S_g.*

In view of Assertions 2.4.4 (a, c), for every g there is a graph (for example, K_{g+15} or the disjoint union of $g+1$ copies of K_5) that is not realizable on S_g (the second of these graphs is realizable on S_{g+1}). The estimations in Assertion 2.4.4 are sharp [Pr14, 13.1].

Theorem 2.4.5. *For every g there is an algorithm for deciding whether a graph is realizable on S_g.*

This result is deduced from Theorem 2.6.8 (a).

2.4.6. A *map on the torus* is a partition of the torus into (curved) polygons. A coloring of a map on the torus is said to be *proper* if different polygons sharing a common boundary curve have different colors. Is it true that any map on the torus has a proper coloring with

(a) 5 colors; (b) 6 colors?

It turns out that any map on the torus has a proper coloring with 7 colors. This is a special case of the following result. A *map on S_g handles* and a *proper* coloring of such a map are defined analogously to the case of the torus.

Theorem 2.4.7 (Heawood). *If $0 < g < (n-2)(n-3)/12$, then every map on S_g has a proper coloring with n colors.*

The version of this theorem for $g=0$ is true: this is the Four Color Conjecture. In view of Ringel's results on embeddings of K_n [Pr14, 13.1] $n-1$ colors are not sufficient for $g \geqslant (n-2)(n-3)/12$.

Heawood's Theorem 2.4.7 is implied by the following result, whose proof relies on Euler's Inequality 2.5.3 (a).

2.4.8. (a) Any graph drawn on the torus without edges crossing has a vertex with at most 6 incident edges.

(b) If $0 < g < (k-1)(k-2)/12$, then any graph drawn on S_g without edges crossing has a vertex with at most k incident edges.

2.5. Euler's Inequality for Spheres with Handles

Given a graph drawn on a surface without edges crossing, a **face** is any of the connected parts into which cutting along all edges of the graph divides the surface.

On the torus there are two closed curves such that cutting along them divides the torus into different numbers of parts (Problem 2.3.2 (a)). So, the number of faces depends on the way the graph is drawn on the given surface. However, we still have a version of Euler's Formula for surfaces. These are the following inequalities 2.5.1 (d) and 2.5.3 (a).

2.5.1. (a, b, c, d) The same as in Assertions 1.4.2, with the plane replaced by a sphere with handles and a planar graph replaced by a graph drawn on the sphere with handles without edges crossing.

(d′) In a parliament consisting of n members there are several (pairwise distinct) 3-person commissions. It is known that if two persons x, y belong to a commission, then the set $\{x, y\}$ is contained in exactly two commissions. Such two commissions are said to be *adjacent*. It is also known that for any two persons A, B there is a sequence of commissions such that A is in the first commission, B is in the last commission, and any two consecutive commissions are adjacent. Show that the number of commissions is not less than $2n-4$.

Hint. There is an intuitive reduction to (d) (observe that rigorous proof of (d) requires some technicalities). For a realization of this idea in an algebraic way see [**?**, §6].

(e) If G is a subgraph of a connected graph H on a sphere with handles, then $V_G - E_G + F_G \geqslant V_H - E_H + F_H$.

Hint. Part (e) follows from part (c). Use the operations of deleting an edge, or deleting a hanging vertex.

Warning. Part (e) is not true for a disconnected graph H, but is true for a disconnected graph H if every connected component contains a vertex of G.

2.5.2. Given a connected graph with V vertices and E edges drawn on the torus without edges crossing, denote by F the number of faces.

(a) If the graph (more exactly, its drawing) contains a parallel and a meridian, then $F = E - V$.

Hint. Cut the torus along the parallel and the meridian. The result is a connected plane graph lying in a square, and containing the boundary of a square. Apply Euler's Formula to this graph.

(b) $F \geqslant E - V$.

Clarification. We assume that the graph meets the union of a parallel and a meridian in a finite number of points, and after cutting the torus along this union with subsequently unfolding the cut torus into the square we obtain from the graph a union of polygonal lines (a learned way of saying this is that the given embedding of the graph into the torus is piecewise linear, and is in general position with respect to the parallel and the meridian).

Hint. Use part (a) and Assertion 2.5.1 (e).

2.5.3. (a) **Euler's Inequality**[8]. *Given a connected graph with V vertices and E edges drawn on S_g without edges crossing, denote by F the number of faces. Then*

$$V - E + F \geqslant 2 - 2g.$$

(b) Given a graph with V vertices, E edges, and s connected components drawn on S_g without edges crossing, denote by F the number of faces. Then $V - E + F \geqslant 1 + s - 2g$.

Euler's Inequality 2.5.3 (a) can be proved analogously to the case of the torus 2.5.2 (b) using Assertion 2.3.4.

Sketch of proof of Riemann's Theorem 2.3.5 (a). Consider the case of the torus (the general case is proved analogously). Suppose that the

[8]Usually, instead of Euler's *Inequality*, which is sufficient for many applications, one considers the more complicated Euler's *Formula* 5.9.2 (cf. Assertion 2.5.2 (a)), whose statement involves the notion of a *cellular subgraph*.

union of two disjoint closed curves does not separate the torus. We may assume that the curves are simple. Similarly to the proof of Jordan's Theorem 1.4.3 (b), we use the orientability of the torus to conclude that there are a 'figure eight' and a circle that are non-self-intersecting, disjoint, and whose union does not separate the torus. Joining the figure eight and the circle by an arc on the torus, we obtain a graph with $V - E = -2$ that does not separate the torus, contradicting Euler's Inequality. □

Betti's Theorem 2.3.5 (b) follows from Euler's Inequality 2.5.3 (b) (or from Euler's Inequality 2.5.3 (a) and Riemann's Theorem 2.3.5 (a); the details are similar to the arguments in [Bi20, bottom of p. 6]).

2.6. Realizability of Hieroglyphs and Orientable Thickenings

Disks with untwisted ribbons are defined in § 1.5. We will call them hieroglyphs, cf. § 1.7. A hieroglyph is said to be **realizable** on a given surface if it can be cut out of this surface.

2.6.1. (a, b, c) The hieroglyphs corresponding to the words (*abab*), (*abcabc*), and (*abacbc*) (Fig. 1.5.2) are realizable on the torus.

A solution of (b, c) is presented in Fig. 2.6.1.

2.6.2. The hieroglyphs shown in Fig. 1.5.3
(a′, b′, c′, d′) are realizable on the sphere with two handles.
(a, b, c, d) are not realizable on the torus.

For a proof of (a′, b′, c′, d′) pick two interlacing ribbons and show that the disk with the two remaining ribbons is realizable on the torus (a proof via attaching ribbons one by one also works, but is more complicated). Parts (a,b,c,d) are proved analogously to Assertion 2.3.1 (b) (in fact, every hieroglyph with 4 ribbons that has one boundary circle cannot be realized on the torus).

Denote by $h(M)$ the number of boundary circles of a hieroglyph or a thickening M.

2.6.3. (a) If a hieroglyph M is cut out of the sphere with g handles S_g, then the number of obtained connected components of $S_g - M$ does not exceed $h(M)$.

(a') If a hieroglyph M with n ribbons is cut out of S_g, then $h(M) \geqslant n + 1 - 2g$.

(b) For every g there exists a hieroglyph not realizable on S_g.

(c) If a hieroglyph M is realizable on S_g and removing any of its ribbons results in a hieroglyph non-realizable on S_g, then M has $2g+2$ ribbons.

Here part (a') follows from part (a) and Euler's Inequality 2.5.3 (a) (cf. Assertion 2.3.1 (b)). Part (b) follows by part (a') (take e.g. hieroglyph $(a_1b_1a_1b_1 \dots a_{g+1}b_{g+1}a_{g+1}b_{g+1})$).

2.6.4. (a) Every hieroglyph with 3 ribbons is realizable on the torus.

(b) Does there exist a hieroglyph with 4 ribbons that has two boundary circles?

(c) Every hieroglyph with 4 ribbons that has three boundary circles is realizable on the torus.

(d) Every hieroglyph with n ribbons that has at least $n-1$ boundary circles is realizable on the torus.

The proof is analogous to that of Assertions 2.6.2(a$'$, b$'$, c$'$, d$'$), cf. Assertions 1.5.3 (a, b).

Theorem 2.6.5. (a) *For every g there is an algorithm for deciding whether a hieroglyph is realizable on S_g.*

(b) *Each of the following conditions on a hieroglyph M with n ribbons is equivalent to its realizability on S_g.*

(E) *The inequality $h(M) \geqslant n+1-2g$ holds.*

(I) *Among any $2g+1$ rows of the interlacement matrix* (*see the definition below*) *there are several* ($\geqslant 1$) *rows whose sum is zero modulo* 2. (*In other words, the rank of the interlacement matrix over $\mathbb{Z}_2$ does not exceed $2g$.*)

The *interlacement matrix* of a hieroglyph with n ribbons is the $n \times n$ matrix whose $a \times b$ cell contains 1 if $a \neq b$ and the letters a and b do not interlace, and 0 otherwise. Cf. § 6.7.

Here part (a) follows from (b). The condition (E) is necessary for the realizability by Assertion 2.6.3.a'. The sufficiency of (E) is proved analogously to Assertion 2.6.4, cf. Assertion 2.7.7 (b) and its proof. Criterion (I) can be proved analogously to Assertion 2.7.7 (c).

The *rank* $\operatorname{rk} M$ of a hieroglyph M is the rank of its interlacement matrix over $\mathbb{Z}_2$. The rank measures the 'complexity of intersections' on the hieroglyph.

2.6.6. A hieroglyph M can be cut out of a hieroglyph M' if and only if $\operatorname{rk} M \leqslant \operatorname{rk} M'$.

Orientable thickenings are defined in §§ 1.6 and 1.7. A thickening is said to be **realizable** on a given surface if it can be cut out of this surface.

2.6.7. Does there exist an orientable thickening of
(a) the graph K_4; (b) the graph K_5
that is not realizable on the torus?

Theorem 2.6.8. (a) *For every g there is an algorithm for deciding whether a thickening is realizable on S_g.*

(b) *Each of the following conditions on an orientable thickening M of a connected graph is equivalent to its realizability on S_g.*

(E) *The inequality $2g \geqslant 2 - V + E - h(M)$ holds, where V and E are the numbers of vertices and edges of the graph.*

(I) =2.6.5.b(I).

Given an orientable thickening of a connected graph G and a spanning tree, we construct a hieroglyph corresponding to the edges not in the tree (Fig. 1.6.4). The *interlacement matrix*, corresponding to the tree, of the orientable thickening is the interlacement matrix of the resulting hieroglyph. The *rank* of an orientable thickening is the rank of its interlacement matrix (corresponding to an arbitrary tree) over $\mathbb{Z}_2$.

Theorem 2.6.8 is reduced to Theorem 2.6.5 by contracting an edge or considering a spanning tree.

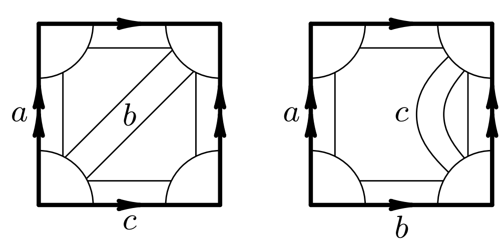


Figure 2.6.1. The disks with ribbons corresponding to the words ($abcabc$) and ($abacbc$) on the torus

2.7. Topological Equivalence (Homeomorphism)

2.7.1. Can the graph K_5 be drawn without edges crossing

(a) on the sphere; (b) on the lateral surface of a cylinder (Fig. 2.1.2)?

In this section, we do not give a rigorous definition of the notion of *homeomorphism* (topological equivalence); for a rigorous definition, see § 5.2. To 'prove' that shapes are homeomorphic, in this section you must draw a chain of pictures similar to Fig. 2.7.1.

Here it is allowed to temporarily *cut* a shape, and then *glue together* the 'edges' of the cut. For example,

• the sphere with a point removed is homeomorphic to the plane, and the lateral surface of a cylinder is homeomorphic to the annulus on the plane (here a chain of pictures can be obtained from the solution of Problem 2.7.1);

• the sphere with one handle (Fig. 2.1.5) is homeomorphic to the torus (Fig. 2.1.2);

• the disk with two ribbons (Fig. 2.7.1 (right)) is homeomorphic to the torus with a hole (Fig. 2.7.1 (left));

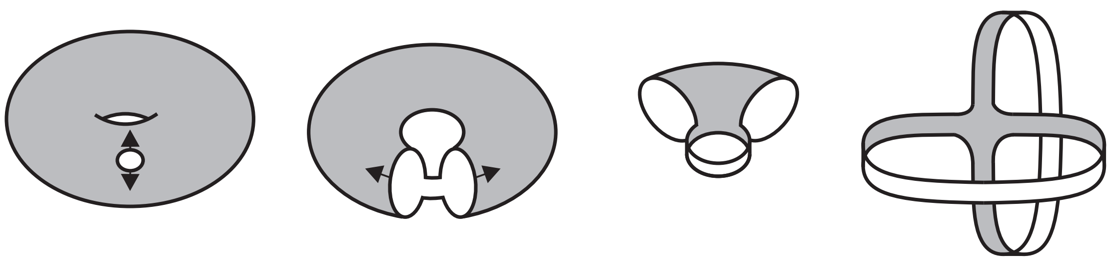


Figure 2.7.1. The torus with a hole is homeomorphic to the disk with two ribbons

• the three ribbons in Fig. 2.2.2 (b) are homeomorphic (here we can no longer do without cutting);

• the two ribbons in Fig. 2.2.2 (a) are homeomorphic (here again we cannot do without cutting).

The ribbons in Fig. 2.2.2 (a) and in Fig. 2.2.2 (b) are not homeomorphic. We will deal with *nonhomeomorphic* shapes in § 5, after introducing a rigorous definition and other notions, which allow one to turn the informal arguments of this section into rigorous proofs.

One should not confuse the notion of homeomorphism with that of *isotopy*, see Problem 6.6.1 (b) and § 15.5.

2.7.2. (a, b) The shapes in Fig. 1.5.2 (middle and right) are homeomorphic to the torus with two holes.

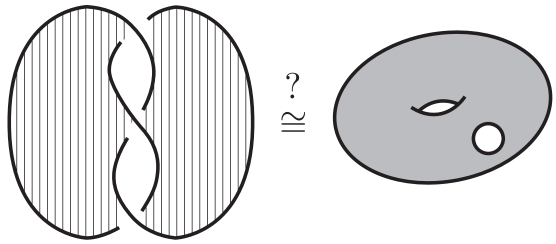


Figure 2.7.2. Are these shapes homeomorphic?

(c) The shape in Fig. 2.7.2 (left) is homeomorphic to the torus with a hole.

(d) Is the shape in Fig. 1.6.2 (right) homeomorphic to a sphere with handles and holes? If yes, with how many handles and holes?

2.7.3. (a, b, c, d) The shapes in Fig. 1.5.3 are homeomorphic to the sphere with two handles and a hole.

2.7.4. Cutting the torus

(a) along any non-separating cycle results in a shape homeomorphic to the annulus;

(b) along any non-separating 'figure eight' results in a shape homeomorphic to the disk (i.e., to a convex polygon).

2.7.5. The regular neighborhoods of different drawings of a graph in the plane without edges crossing (i.e., of isomorphic plane graphs, see Fig. 1.3.1) are homeomorphic.

Concerning hieroglyphs and thickenings, see §§ 2.6 and 1.5–1.7.

2.7.6. (a) Every hieroglyph with two ribbons is homeomorphic either to the disk with two holes or to the disk with one hole.

(b) (Riddle) To what surfaces can an orientable thickening of the graph K_4 be homeomorphic?

Proposition 2.7.7. (a) *Two hieroglyphs with the same number of ribbons are homeomorphic if and only if they have the same number of boundary circles.*

(b') Any hieroglyph no two of whose ribbons interlace is homeomorphic to the disk with holes.

(b) Euler's Formula. *Let M be a hieroglyph with n ribbons. Then $h(M) - n$ is odd, $h(M) \leqslant n + 1$, and M is homeomorphic to the sphere with $(n + 1 - h(M))/2$ handles and $h(M)$ holes.*

(c)* Mohar's Formula. *Let M be a hieroglyph of rank r with n ribbons. Then r is even and M is homeomorphic to the sphere with $r/2$ handles and $n+1-r$ holes.*

The names 'Euler's Formula' and 'Mohar's Formula' for Assertions 2.7.7, 2.7.9, and 2.8.8 (see below) are not widely used. Cf. Problems 5.9.2 and 6.7.5 (f, g).

Proposition 2.7.8. *(a) Any thickening of a tree is homeomorphic to the disk D^2.*

(b) Let M be a thickening of a connected graph with V vertices and E edges. If $V-E+h(M)=2$, then M is homeomorphic to the sphere with $h(M)$ holes.

Part (b) is proved using part (a), Proposition 2.7.7.b' and Assertions 1.5.3.a,b, 1.6.4.c.

Proposition 2.7.9. (a) *Two orientable thickenings of a connected graph are homeomorphic if and only if they have the same number of boundary circles.*

(b) Euler's Formula. *Assume that M is an orientable thickening of a connected graph with V vertices and E edges. Then $V-E+h(M)$ is even, $V-E+h(M)\leqslant 2$, and M is homeomorphic to the sphere with $(2-V+E-h(M))/2$ handles and F holes.*

(c)* Mohar's Formula. *Assume that M is an orientable thickening of rank r of a connected graph with V vertices and E edges. Then r is even, $V-E+r\leqslant 1$, and M is homeomorphic to the sphere with $r/2$ handles and $2-V+E-r$ holes.*

2.8. Non-Orientable Surfaces*

Graphs and Map Colorings on a Disk with Möbius strips

2.8.1. Draw the following graphs on the Möbius strip without edges crossing:

(a) $K_{3,3}$; (b) $K_{3,4}$; (c) K_5; (d) K_6.

2.8.2. (a) *Euler's Inequality.* Assume that a connected graph with V vertices and E edges is drawn on the Möbius strip without edges crossing so that it does not intersect the boundary circle. Denote by F the number of faces. Then $V-E+F\geqslant 1$.

(b) The graph K_7 cannot be realized on the Möbius strip.

(c) The graph $K_5 \sqcup K_5$ cannot be realized on the Möbius strip.
(d) Any map on the Möbius strip has a proper coloring with 6 colors.

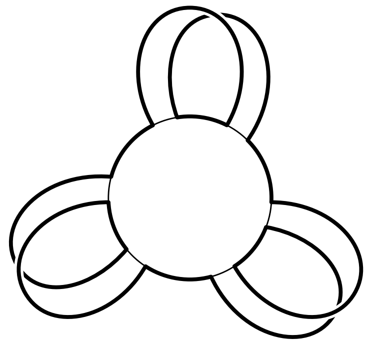


Figure 2.8.1. The disk with Möbius strips

The **disk with m Möbius strips** (Fig. 2.8.1) is the union of the disk and m ribbons such that

• each ribbon is glued along a pair of opposite sides to the boundary circle S of the disk, and the directions on these sides determined by an arbitrary direction on S 'coincide along the ribbon',

• the ribbons are 'separated', i.e., there are m pairwise disjoint arcs on S such that the endpoints of the ith ribbon are glued to two disjoint subarcs contained in the ith arc for every $i = 1, 2, \dots, m$.

2.8.3. (a) Draw m closed non-self-intersecting pairwise disjoint curves on the disk with m Möbius strips such that their union does not separate the disk with m Möbius strips.

(b) The union of any $m + 1$ pairwise disjoint closed curves on the disk with m Möbius strips separates it.

(c) Any graph can be drawn without edges crossing on a disk with a certain number (depending on the graph) of Möbius strips.

(d) For every $m > 0$, obtain the disk with m Möbius strips by gluing from a regular $4m$-gon.

2.8.4. (a) *Euler's Inequality.* Assume that a connected graph with V vertices and E edges is drawn without edges crossing on the disk with m Möbius strips, so that the graph does not intersect the boundary circle. Denote by F the number of faces. Then $V - E + F \geqslant 2 - m$.

(b) State and prove versions of Theorem 2.4.4 for the disk with m Möbius strips, where $m \neq 2$.

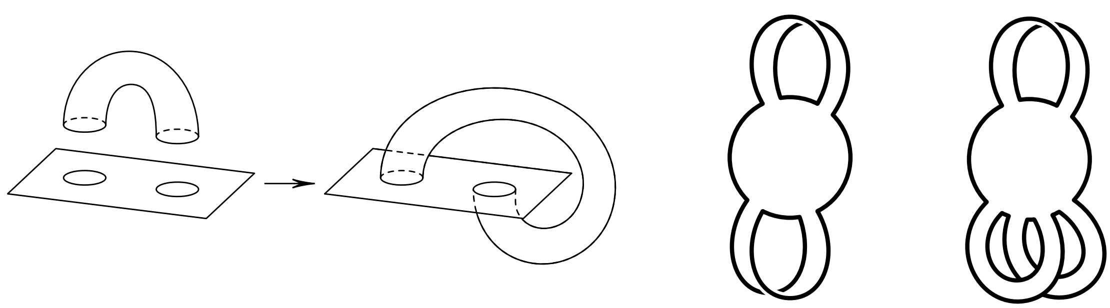
(c) State a prove a version of Heawood's Theorem 2.4.7 for the disk with m Möbius strips, where $m \neq 2$.

It turns out that the graph K_7 cannot be realized on the Klein bottle (i.e., on the disk with 2 Möbius strips), and that any map on the Klein bottle has a proper coloring with 6 colors [Fr34, SK86].

Homeomorphic Non-Orientable Surfaces

2.8.5. (a) The Möbius strip with a handle is homeomorphic to the Möbius strip with an inverted handle, see Fig. 2.1.5, 2.8.2 (a).

(b) The shape in Fig. 2.8.2 (b) (i.e., the disk with two 'twisted' 'separated' ribbons) is homeomorphic to the Klein bottle with a hole (Fig. 2.1.6).



(a) (b) (c)

Figure 2.8.2. (a) Attaching an inverted handle (cf. Fig. 2.1.5). (b) The disk with two 'twisted' 'separated' ribbons (c) The disk with ribbons corresponding to the word $(aabcbc)$ with $w(a) = 1$ and $w(b) = w(c) = 0$.

(c) The shape in Fig. 2.8.2 (c) is homeomorphic to the disk with three Möbius strips.

(d) The shapes in Fig. 2.8.3 (a) are homeomorphic.

(e) The shapes in Fig. 2.8.3 (b) (i.e., an annulus with two 'twisted' 'separated' ribbons glued to the same boundary circle and an annulus with two 'twisted' ribbons glued to different boundary circles) are homeomorphic.

Beautiful examples from Problems 2.8.5 (d, e) are of importance since they show that dissimilar shapes can still be homeomorphic.

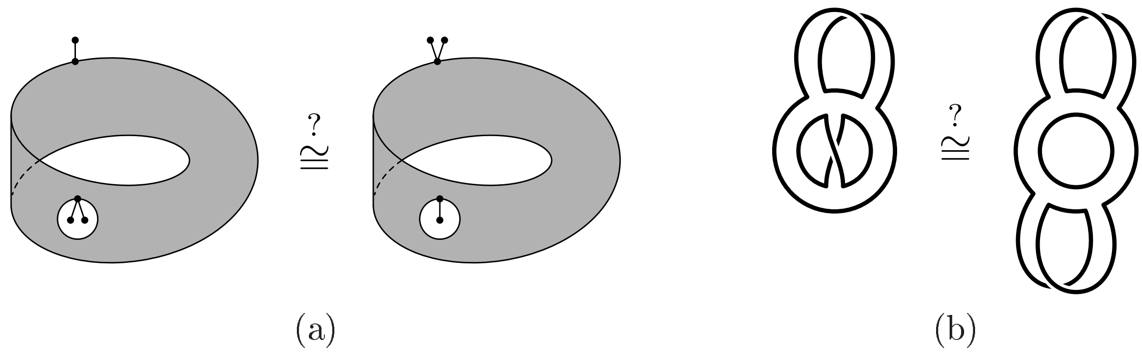


Figure 2.8.3. (a) Are the boundary circles of the Möbius strip with a hole equivalent? (b) Are these annuli with two Möbius strips homeomorphic?

Disks with Twisted Ribbons

Given a disk with ribbons and a ribbon k in it, set $w(k)=1$ if the ribbon is twisted, and $w(k)=0$ otherwise.

Figures 2.8.2 (b, c) and 1.5.1 (right), 2.8.1 show, respectively,

• the disk with ribbons corresponding to the word $(aabb)$ for which $w(a)=w(b)=1$;

• the disk with ribbons corresponding to the word $(aabcbc)$ for which $w(a)=1$ and $w(b)=w(c)=0$;

• the disk with n Möbius strips, i.e., the disk with ribbons corresponding to the word $(1122\dots nn)$ for which $w(1)=w(2)=\dots=w(n)=1$.

2.8.6. (a) How many boundary circles can a disk with two ribbons have?

(b) To what surfaces can a disk with two ribbons be homeomorphic?

(c) To one of the boundary circles of the disk with n Möbius strips and $k>0$ holes, a *twisted* (with respect to this boundary circle) ribbon is attached. The resulting shape is homeomorphic to the disk with $n+1$ Möbius strips and k holes.

2.8.7. State and prove versions of Theorems 2.6.5 (a, b) for the realizability of disks with ribbons on the disk with m Möbius strips.

Proposition 2.8.8. (a) *Two disks with the same number of ribbons are homeomorphic if and only if they have the same number of boundary circles and either both have a twisted ribbon or neither has one.*

(b) Euler's Formula. *Assume that M is a disk with n ribbons among which there is a twisted one, and M has h boundary circles. Then $h \leqslant n$, and M is homeomorphic to the disk with $n+1-h$ Möbius strips and $h-1$ holes.*

(c)* Mohar's Formula. *The* interlacement matrix *of a hieroglyph with ribbons* $1, 2, \ldots, n$ *and nonzero map* $w\colon \{1, 2, \ldots, n\} \to \{0, 1\}$ *is defined analogously to the interlacement matrix of a hieroglyph, with the difference that the diagonal cell $a \times a$ contains the number $w(a)$. Denote by r the rank of the interlacement matrix over $\mathbb{Z}_2$. Then the corresponding disk with ribbons is homeomorphic to the disk with r Möbius strips and $n-r$ holes.*

Thickenings of Graphs

2.8.9. (a) The thickening in Fig. 2.8.4 cannot be realized on the Möbius strip.

(b) Every thickening of a unicyclic graph can be realized on the Möbius strip.

(c) Which thickenings of the graph K_4 can be realized on the Möbius strip?

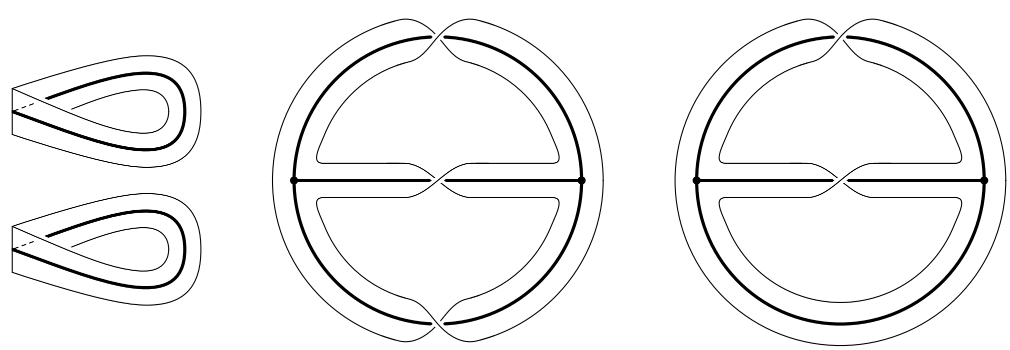


Figure 2.8.4. Thickenings that cannot be realized on the Möbius strip

2.8.10. State and prove versions of Theorems 2.6.8 (a, b) for the realizability of thickenings on the disk with m Möbius strips.

A thickening is said to be *orientable* if the boundary circles of the disks can be endowed with orientations so that every ribbon becomes untwisted, and *non-orientable* otherwise.

Proposition 2.8.11. (a) *Two thickenings of a connected graph are homeomorphic if and only if they have the same number of boundary circles and either both are orientable or both are non-orientable.*

(b) Euler's Formula. *Let M be a non-orientable thickening of a connected graph with V vertices and E edges that has h boundary circles. Then $V - E + h \leqslant 1$, and M is homeomorphic to the disk with $2 - V + E - h$ Möbius strips and $h - 1$ holes.*

(c)* Mohar's Formula. *Let M be non-orientalbe thickening of rank r of a connected graph with V vertices and E edges. Then M is homeomorphic to the disk with r Möbius strips and $1 - V + E - r$ holes.*

Answers, Hints, and Solutions to Some Problems

2.2.1. (a) See [Sk, §1, proof of the General Position Theorem 1.1.2].

(c) Draw the graph in the plane with self-intersections. We may assume that the self-intersection points are transverse (Fig. 6.6.1) and lie on the same line. Attach the third sheet along this line. Now, in a small neighborhood of each intersection point of edges, lift one of the edges 'bridgelike' over the other edge to the third sheet. In this way, eliminate all intersection points.

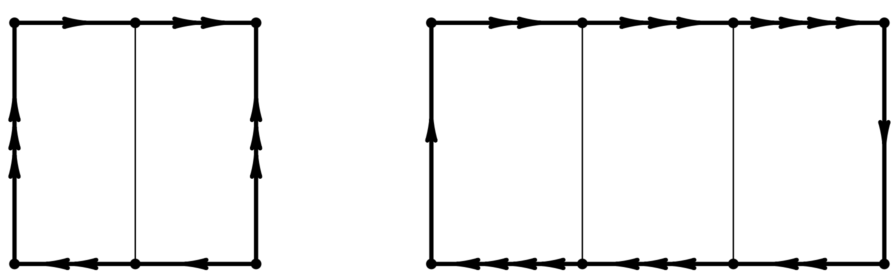


Figure 2.8.5. Cuts on the Klein bottle

2.2.3. (a) Cut Fig. 2.1.6 (right) along the plane of symmetry. Or see Fig. 2.8.5 (right).

(b) See Fig. 2.8.5 (left). It is easier to cut along the curve denoted by three arrows.

2.2.4. (a) See Fig. 2.8.6.

(b) Use Fig. 2.7.1.

2.3.2. (a) Make a cut along a *meridian*.

(b) The Möbius strip can be cut along the *midline*.

§ 5. Two-dimensional manifolds

I should say it meant something simple and obvious, but then I am no philosopher!

I. Murdoch. The Sea, the Sea.

5.1. Hypergraphs and their geometric realizations

Let us give a combinatorial definition of two-dimensional surfaces (and somewhat more general objects). This definition is convenient for theoretical purposes as well as for storing in computer memory; cf. §1.2.

Main results stated in this section (but not used later) are Theorems 5.2.4, 5.3.1, 5.3.3, and 5.6.1.

A **two-dimensional hypergraph**[14] (or **2-hypergraph**, for short) (V, F) is a collection F of three-element subsets of a finite set V. The elements of V and F are called **vertices** and **faces** (or *hyperedges*) of the 2-hypergraph. An **edge** of a 2-hypergraph is an unordered pair of vertices that is contained in some face.

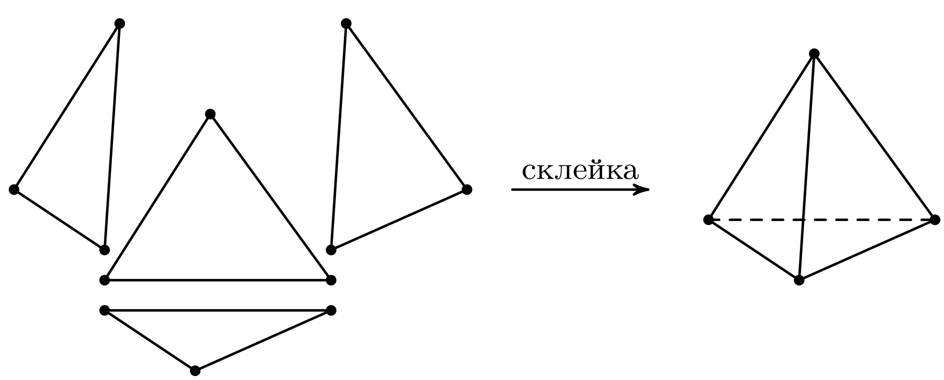


Figure 5.1.1. Building (the geometric realization of) a complete 2-hypergraph with 4 vertices

Example 5.1.1. (a) A *complete 2-hypergraph with n vertices* (or the *two-dimensional skeleton of an $(n-1)$-dimensional simplex*) is the

[14]Sometimes called a 3-*uniform* hypergraph, or a *dimensionally homogeneous* (*pure*) two-dimensional simplicial complex, see [Sk, § 6]

collection of all three-element subsets of an n-element set. See Figure 5.1.1 for $n = 4$ and Figure 5.1.2 for $n = 5$. In this section the complete 2-hypergraph on 4 vertices is called the **sphere** S^2.

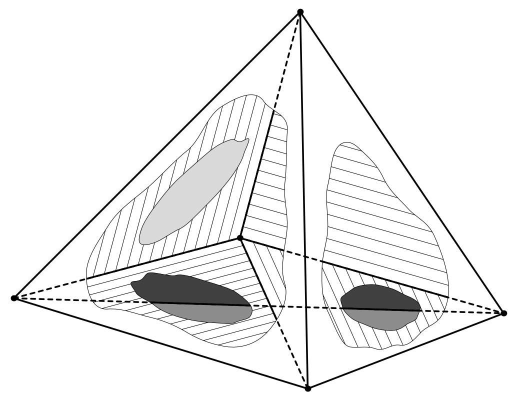


Figure 5.1.2. A complete 2-hypergraph with 5 vertices

(b) The *book with n pages* is the 2-hypergraph with vertices $a, b, 1, 2, \dots, n$ and faces $\{a, b, j\}$, $j = 1, 2, \dots, n$. See Figure 2.2.1 for $n = 3$.

(c) Suppose one has a 2-hypergraph, and a gluing diagram showing which pairs of edges should be identified, so that no two vertices of intersecting faces get identified. Such a gluing gives a new 2-hypergraph. For instance, Figure 2.1.1 shows the 2-hypergraphs obtained by gluing the sides of a square (triangulations are not shown there), and gives them names. See the remark after Assertion 5.2.3.

(d) A triangulation of 2-manifold (see §4.6) can be naturally viewed as a 2-hypergraph, which is also called a triangulation.

The definition of 2-hypergraphs being isomorphic is analogous to the one for graphs. 2-Hypergraphs (V, F) and (V', F') are called **isomorphic** if there is a 1–1 correspondence $f\colon V \to V'$ satisfying the following property: *vertices $A, B, C \in V$ lie in one face if and only if their images lie in one face.*

For $1 \leqslant i \leqslant n$, denote by $e_{n,i} \in \mathbb{R}^n$ the point whose i-th coordinate is 1 whereas the others are 0. The convex hull Δ_n of the points $e_{n+1,1}, \dots, e_{n+1,n+1} \in \mathbb{R}^{n+1}$ is called[15] the *n-dimensional simplex.* It is

[15]One could define the n-dimensional simplex as the convex hull of $(0, \dots, 0), e_{n,1}, \dots, e_{n,n} \in \mathbb{R}^n$. This might be more visually intuitive but this is less convenient for us.

a convex polyhedron with $n+1$ vertices; the union of its edges 'forms' the complete graph K_{n+1}. The **geometric realization** (or *body*) of a 2-hypergraph (V, F) is the union of those two-dimensional faces of the simplex with vertex set V that correspond to the faces from F.

Remark 5.1.2 (on geometric realization of hypergraphs)**.** Similarly to the case of graphs, one builds a geometric shape from a 2-hypergraph, and calls it the *geometric realization* (cf. the above rigorous definition). Informally speaking, the shape is obtained by gluing several triangles corresponding to the faces. The gluing procedure does not have to happen in three-dimensional space; the procedure is either done in higher dimensions, or even abstractly, without any reference to an ambient space.

For example, Figure 5.1.1 shows how to build the geometric realization of the complete 2-hypergraph with 4 vertices. The geometric realization of the 2-hypergraph that is obtained as a surface triangulation is homeomorphic to that surface. More generally, 2-hypergraphs, just like graphs, can be specified by *geometric shapes*, including 'smooth' or self-intersecting ones. See the last two rows of Figure 2.1.1. One shape specifies multiple 2-hypergraphs.

Usually all these 2-hypergraphs are homeomorphic (see §5.2, Theorem 5.2.4 and the example before Problem 10.3.3). Then a 2-hypergraph bears the name of the shape. In this case non-isomorphic but homeomorphic 2-hypergraphs have the same name.

Despite having a geometric realization, a 2-hypergraph is a combinatorial object. It is impossible, say, to take a point on its face. However, 'taking a point on a face of the geometric realization of a 2-hypergraph' can be formalized as 'taking the newly added vertex of the new 2-hypergraph obtained by the subdivision of that face'; see Figure 5.2.2 on the right. We will not follow such a level of formality.

5.2. Homeomorphic 2-hypergraphs

Remark 5.2.1 (homeomorphism of graphs)**.** (a) The operation of *edge subdivision* is shown in Figure 5.2.1. Two graphs are called *homeomorphic* if one of them can be obtained from the other (more precisely, from a graph isomorphic to the other) using edge subdivisions and the inverse operations. Equivalently, two graphs are homeomorphic if there is a graph that can be obtained from either of the two using edge subdivisions.

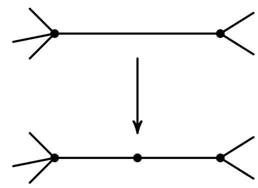


Figure 5.2.1. Edge subdivision

(b) The definition of a homeomorphism for subsets of Euclidean space is given in §3.1. It turns out that *graphs* G_1 *and* G_2 *are homeomorphic if and only if the realizations* $|G_1|$ *and* $|G_2|$ *are homeomorphic.* This criterion *motivates* the definition of a graph homeomorphism, which allows us to study certain shapes using combinatorial language.

(c) A *one-dimensional polyhedron* is a homeomorphism class of graphs. A topologist is usually interested in polyhedra even if calling them graphs. On the other hand, graphs and their realizations are convenient tools for studying polyhedra and storing them in computer memory. A combinatorialist or discrete geometer are mostly interested in graphs, though they might find polyhedra useful as well.

The definition of homeomorphic (combinatorial topology equivalent) 2-hypergraphs is analogous to the one for graphs.

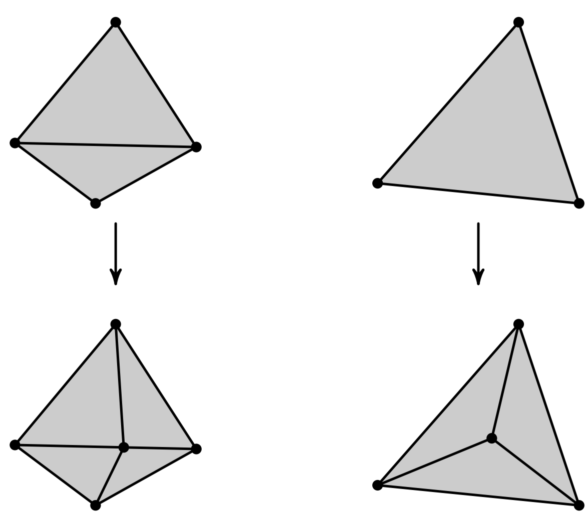


Figure 5.2.2. Subdivision of an edge and a face

The operation of an **edge subdivision** of a 2-hypergraph is shown in Figure 5.2.2, on the left.

5.2.2. The operation of a *face subdivision* in Figure 5.2.2, on the right, can be expressed using edge subdivision and its inverse.

Two 2-hypergraphs are said to be **homeomorphic**, if one of them can be obtained from the other (more precisely, from a 2-hypergraph isomorphic to the other) using the operations of edge subdivision and its inverse.

A *two-dimensional polyhedron* is a homeomorphism class of 2-hypergraphs. An analogue of Remark 5.2.1.c is valid for 2-hypergraphs.

A graph is said to be **embeddable** (or *realizable*) in a 2-hypergraph if a certain 2-hypergraph homeomorphic to the given one contains a graph homeomorphic to the given one.

5.2.3. (a) The 2-hypergraph with vertices $0, 1, \ldots, n$ and faces $\{0, 1, 2\}, \{0, 2, 3\}, \ldots, \{0, n-1, n\}$ is homeomorphic to complete 2-hypergraph with three vertices.

(b) The same for the set of faces $\{0, 1, 2\}, \{0, 2, 3\}, \ldots, \{0, n-1, n\}, \{0, n, 1\}$.

(c) The 2-hypergraphs in each separate column of Figure 2.1.1 are homeomorphic to each other (for some triangulation of square), while the 2-hypergraphs from different columns are not.

Hint: the material of the following sections can be used in order to prove that certain 2-hypergraphs are not homeomorphic.

(d) Any two triangulations of a triangle are homeomorphic.

(e) The spheres S^2 defined in Example 5.1.1.a,c are homeomorphic.

Both (d,e) are non-trivial. Part (d) can be proved in a direct geometric way (check that your proof does not work for the Möbius band), or follows from Theorem 5.4.3. Part (e) follows from Theorem 5.3.3 (or from a more complicated Theorem 5.2.4.a).

Theorem 5.2.4. *(a) Two-dimensional hypergraphs are homeomorphic if and only if their geometric realizations are homeomorphic.*

(b) The 2-hypergraphs corresponding to different triangulations of the same 2-manifold in $\mathbb{R}^m$ (see §4.5) are homeomorphic.

This is an important statement (‘Hauptvermutung’). It illustrates the connection between the notions of ‘combinatorial’ homeomorphism of 2-hypergraphs and 'topological' homeomorphism of their geometric realizations.

Theorem 5.2.4 is neither proved nor used in this book. This result is nontrivial even when one of the 2-hypergraphs is a triangle (Assertion 5.2.3 (d)) or a sphere with handles (§2.1).[16]

5.3. Recognition of 2-hypergraphs being homeomorphic

Theorem 5.3.1. *There exists an algorithm deciding whether*

(a) *a 2-hypergraph is homeomorphic to the sphere* S^2*;*

(b) *two arbitrary 2-hypergraphs are homeomorphic.*

Theorem 5.3.1.b is neither proved nor used in this book. Theorem 5.3.1 (a) follows from Theorem 5.3.3 on sphere recognition. The latter and Theorem 5.6.1 on classification of surfaces can be regarded as important special cases of Theorem 5.3.1 (b), which suggest how to prove the general case (see Problem 5.4.4 (b) and the notion of *attaching word* before Problem 10.5.10). Let us introduce the notions required to state these special cases.

A 2-hypergraph is called **connected**, if any two vertices can be joined by a path along the edges.

A 2-hypergraph is called **locally Euclidean**, if for every its vertex v, the faces containing v form a chain

$$\{v, a_1, a_2\},\ \{v, a_2, a_3\},\ \ldots,\ \{v, a_{n-1}, a_n\} \quad \text{or}$$
$$\{v, a_1, a_2\},\ \{v, a_2, a_3\},\ \ldots,\ \{v, a_{n-1}, a_n\},\ \{v, a_n, a_1\}$$

for some pairwise distinct vertices $a_1, \ldots, a_n$.

E.g. 2-hypergraphs that are triangulations of surfaces in Figure 2.1.1, or of a disk with ribbons (§ 1.5), are locally Euclidean.

[16] Be careful: visually intuitive explanations of this and analogous results might not be proofs! For example, in [Pr14, proof of Theorem 11.5] the following things are not defined: 'surface edges', 'piecewise linear graph on the surface', and 'transverse intersection of edges'. To overcome this, one needs a version of Triangulation Theorem 4.6.4. An easier way is to prove the equality of the Euler characteristics not for arbitrary closed two-dimensional surfaces, but for the examples in question, and take in place of G_2 the specific triangulation that we constructed (this suffices for Theorem 11.5). Even after this, the phrase 'Graph G_1 can be modified in order to...' in not obvious; it seems that this fact is as difficult as Theorem 5.2.4.b.

5.3.2. (a) For which n is the complete 2-hypergraph on n vertices locally Euclidean?

(b) There is a 2-hypergraph that is not locally Euclidean but with each edge incident to two faces.

(c) A 2-hypergraph homeomorphic to a locally Euclidean one is locally Euclidean itself.

The **Euler characteristic** of a 2-hypergraph K with V vertices, E edges and F faces is the number

$$\chi(K) := V - E + F.$$

Methods for computing the Euler characteristics are presented in §5.5.

Theorem 5.3.3 (Sphere recognition). *A 2-hypergraph is homeomorphic to the sphere S^2 if and only if it is connected, locally Euclidean, and its Euler characteristic equals* 2.

A sketch of the proof is presented in §5.4. For higher dimensional analogues see §10.1.

5.4. Proof of Sphere Recognition Theorem 5.3.3

5.4.1. (a) The Euler characteristic of the sphere S^2 equals 2.

(b) The Euler characteristics of homeomorphic 2-hypergraphs are equal.

The 'only if' part of Theorem 5.3.3 follows from Assertion 5.3.2 (c) and 5.4.1 (a, b). (Being closed and orientable, see §§5.6, 5.7, is also required for being homeomorphic to S^2, but is implied by the other hypothesis in Theorem 5.3.3.)

Proof of the 'if' part of Theorem 5.3.3. This part is reduced to its version for thickenings (Proposition 2.7.8.b). Denote by

- K the given hypergraph;
- V, E, F, n the number of its vertices, edges, faces, and boundary circles;
- M of the union *caps* and *ribbons* corresponding to its vertices and edges (see an informal explanation near Fig. 1.6.3 (left), and a rigorous definition below in this subsection).

By Assertions 5.2.3.a,b any patch, any ribbon, and any cap is homeomorphic to D^2. Hence M is a thickening of the union of edges. Clearly, M has $F + n$ boundary circles. Since $V - E + F = 2$, by the connectivity and Assertion 1.6.4.c we have $n = 0$. Then by Proposition 2.7.8.b M is homeomorphic to the sphere with F holes. The thickening M is K with F holes. Hence by Assertion 5.4.2.d K homeomorphic to the sphere. □

The **boundary** ∂N of a locally Euclidean 2-hypergraph N is the union of all its edges each of which is contained in a single face.

5.4.2. (a) The boundary is a disjoint union of cycles, i.e., graphs homeomorphic to a triangle.

(b) The number of boundary circles is the same for homeomorphic locally Euclidean 2-hypergraphs.

(c) 2-Hypergraphs 'representing' annulus and Möbius band are not homeomorphic.

(d) Let K and L be homeomorphic locally Euclidean hypergraphs. Denote by K_+ and L_+ the hypergraphs obtained from them by attaching disks to all the boundary components (i.e. attaching cones over all the boundary components). Then K_+ and L_+ are homeomorphic.

The *barycentric subdivision* G' of a graph G is obtained by subdividing all its edges. The *barycentric subdivision of a face* of a 2-hypergraph is the result of the replacement of the face by six new faces that are obtained by drawing the 'medians' in the triangle representing the face (Figure 5.4.1). The **barycentric subdivision** K' of a 2-hypergraph K is the result of the barycentric subdivision of all its faces.

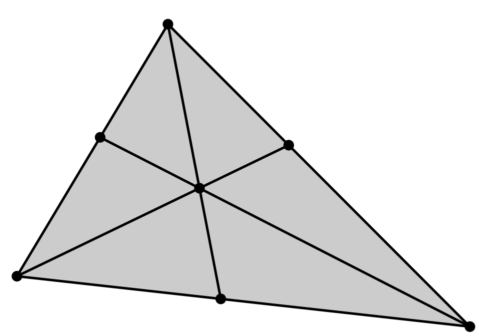


Figure 5.4.1. Barycentric subdivision

Since the barycentric subdivision can be obtained via edge subdivisions, K' is homeomorphic to K.

Denote by K'' the 2-hypergraph obtained from a 2-hypergraph K by barycentrically subdividing it twice. We will use the following notation (see Figure 1.6.3 on the left, where a triangulation of 2-manifold K is shown):

• a **cap** is the union of the faces of the triangulation K'' that contain a certain vertex of the triangulation K;

• a **ribbon** is the union of the faces of the triangulation K'' that intersect a certain edge of the triangulation K but avoid the vertices of the triangulation K;

• a **patch** is a connected component of the union of the remaining faces of the triangulation K'', i.e., the union of all faces of K'' belonging neither to caps nor to ribbons.

Theorem 5.4.3. *A 2-hypergraph is homeomorphic to the disk D^2 if and only if it is connected, locally Euclidean, has one boundary circle, and its Euler characteristic equals* 1.

5.4.4. (a) There exists an algorithm that takes a 2-hypergraph homeomorphic to S^2 and outputs a sequence of edge subdivisions and inverse operations that transform the 2-hypergraph to S^2.

(b) There exists an algorithm recognizing whether a 2-hypergraph is homeomorphic to the book with 3 pages.

5.5. Euler characteristic of a 2-hypergraph

5.5.1. Придумайте связный локально евклидов 2-гиперграф, имеющий

(a) эйлерову характеристику -99.

(b) пустой край и эйлерову характеристику -10.

(c) пустой край и эйлерову характеристику 1.

For a solution the following transformations are useful. From a locally Euclidean 2-hypergraph one can obtain other locally Euclidean 2-hypergraphs by

• *cutting a hole*, i.e. removing a face disjoint from the boundary,

• *attaching a handle*, i.e. cutting a hole and attaching to its boundary some torus with hole, see Remark 5.1.1.c), and

• *attaching a Möbius film*, or a *cross-cap*, i.e. cutting a hole and attaching to its boundary some Möbius band.

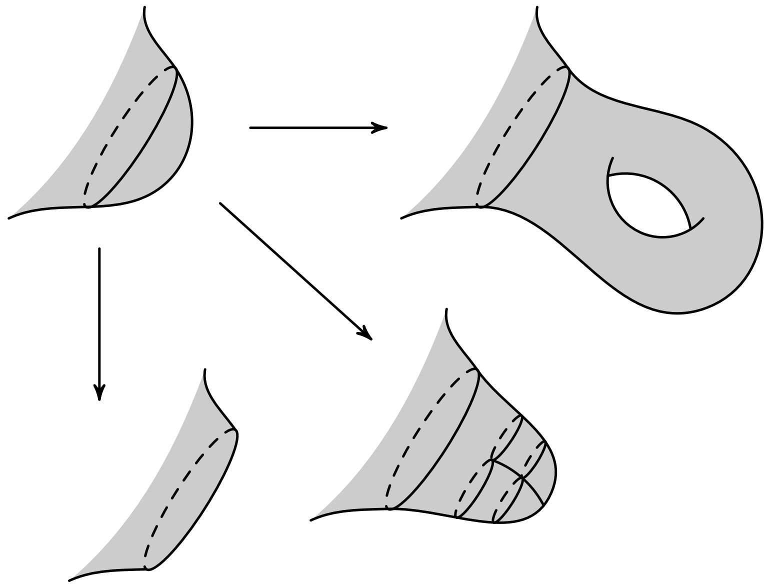


Figure 5.5.1. Attaching a handle and a Möbius film; cutting a hole

See Figure 5.5.1 and Remark 5.1.1.c. Before we prove in §5.8 that these operations are well-defined (up to a homeomorphism), we do not assume that.

5.5.2. (a) Define *careless attaching a handle* to be cutting two holes and attaching to their boundary some annulus (cylinder, disk with a hole) see Figure 2.1.5. Prove that this operation is not-well defined.

Hint: see Figure 2.8.2 (a) and use §5.7.

(b) Define *careful attaching a handle* and prove that this is the same as attaching a handle, up to a homeomorphism.

(c) The projective plane (cf. Example 4.5.3) with a hole is homeomorphic to the Möbius band. (Rigorously: *any* projective plane with a hole is homeomorphic to *some* Möbius band.)

(d) Сфера с m пленками Мёбиуса и дыркой гомеоморфна диску с m лентами Мёбиуса (см. рисунок 2.8.1 и определение после него).

(e) The Klein bottle is homeomorphic to the sphere with two Möbius films.

(f) The torus with a Möbius film is homeomorphic to the Klein bottle with a Möbius film.

(g) The result of attaching a Möbius film is homeomorphic to the result of cutting a hole and identifying the antipodal points of its boundary circle.

(h) The result of attaching a handle is homeomorphic to the result of cutting out square $ABCD$ and gluing directed edges AB and DC, AD and BC.

5.5.3. Find the Euler characteristic of
(a) сферы; (b) кольца; (c) тора; (d) ленты Мёбиуса;
(e) сферы с g ручками; (f) сферы с g ручками и h дырками;
(g) бутылки Клейна; (h) проективной плоскости.

We recommend to compute the Euler characteristic (for example, in Problem 5.5.3) not by definition but using its properties. They are presented in Problems 5.4.1.b and 5.5.4.

5.5.4. (a) (Riddle) Guess and prove the formula for the Euler characteristic of a union.

(b) Cutting a hole decreases the Euler characteristic by 1.

(c) (Riddle) How the Euler characteristic is changed under attaching a handle or a Möbius film?

5.5.5. The triangulations of spheres with distinct numbers of handles, which you constructed in Problem 4.6.3 (e), are not homeomorphic. (This fact is not obvious since seemingly different shapes might happen to be homeomorphic, see §2.7 and especially §2.8.)

5.5.6. Find the Euler characteristic of

(a) the disk with m Möbius bands (see Figure 2.8.1 and definition thereafter);

(b) the Klein bottle with g handles;

(c) the projective plane with g handles;

(d) the sphere with m Möbius films;

(e) the sphere with m Möbius films and h holes.

5.5.7. Which 2-hypergraphs from Problem 5.5.6 are homeomorphic?

5.5.8. Denote by K a triangulation of 2-manifold.

(a) *The Riemann Theorem.* Suppose $g+m$ pairwise disjoint loops are chosen in K so that cutting along any of the first g of them gives two boundary circles, and cutting along any of the last m of them gives one boundary circle. If $2g+m>2-\chi(K)$ then the union of these loops splits the triangulation. (This generalizes the Riemann Theorem 2.3.5 (a) and is implied by (d) cf. [Pr14, § 11.4].)

(b) *The Euler inequality.* A connected subgraph G of K with V vertices and E edges splits the triangulation into at least $E - V + \chi(K)$ parts. In other words, $\chi(G) \geqslant \chi(K)$.

(c)* What is the minimum number of parts in a splitting of K by a subgraph with V vertices, E edges and s connected components?

(d) Cut a locally Euclidean hypergraph along a non-splitting cycle (formed by some edges). The resulting hypergraph has the same Euler characteristic as the original one.

5.6. Classification of surfaces

Theorem 5.6.1 (Classification of surfaces). *Every connected locally Euclidean 2-hypergraph is homeomorphic either to a sphere with handles and holes, or to a sphere with Möbius films and holes.*

These triangulations are not homeomorphic for different triples (ε, g, h)*, set to* $(0, g, h)$ *for a sphere with* g *handles and* h *holes, and to* $(1, g, h)$ *for a sphere with* g *Möbius bands and* h *holes.*

A proof is sketched in 5.7. It gives an algorithm detecting homeomorphism between a 2-hypergraph and the aforementioned classes (ε, g, h) of 2-hypergraphs, as well as an algorithm detecting homeomorphism between locally Euclidean 2-hypergraphs. Compare to Theorem 6.7.6.

A *piecewise linear (PL) two-dimensional manifold* is a homeomorphism class of locally Euclidean 2-hypergraphs. If there is no ambiguity with the notion of 2-manifolds from §4.5, we say '**2-manifold**' as a shorthand for 'PL two-dimensional manifold'.

From now on, instead of the term 'locally Euclidean 2-hypergraph' we use a common term '**triangulation of 2-manifold**'. Earlier it would not be convenient for a beginner, since in the study of 2-manifolds from the piecewise linear viewpoint, the primary object is a 2-hypergraph, and not a 2-manifold.

A locally Euclidean 2-hypergraph is called **closed**, if each its edge belongs to two faces (as opposed to one; that is, for each vertex the second option from the definition of being locally Euclidean takes place). For instance, in Figure 2.1.1 only the four last 'hypergraphs' are closed. By 'sealing' (capping with a disk) each boundary circle of a disk with ribbons one obtains a closed locally Euclidean 2-hypergraph.

5.7. Orientable triangulations of 2-manifolds

An *orientation* of a two-dimensional triangle is an ordering of its vertices up to an even permutation. An orientation is conveniently pictured by a closed curve with an arrow inside the triangle (or by an ordered pair of non-collinear vectors).

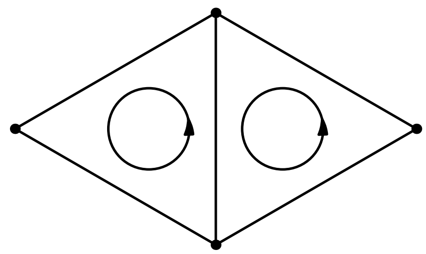


Figure 5.7.1. Agreeing orientations

An *orientation* of a triangulation of 2-manifold is a choice of face orientations *agreeing* with one another along every edge contained in two faces, so that the orientations of adjacent faces induce the *opposite* directions on their common edge (Figure 5.7.1). A triangulation of 2-manifold is called *orientable* if it has an orientation[17].

It is not difficult to see that *a smooth* 2*-manifold is orientable in the sense of* §4.10 *if and only if it has an orientable triangulation.*

5.7.1. (a) Homeomorphic triangulations of 2-manifold are simultaneously orientable or non-orientable.

(b) The sphere, the torus, a sphere with handles are orientable.

(c) The Möbius band, the Klein bottle, the projective plane (Figure 2.1.1) are non-orientable.

(d) The torus is not homeomorphic to the Klein bottle.

5.7.2. (a) The orientability is preserved when cutting a hole.

(b) A disk with ribbons (see §1.5) is orientable if and only if no ribbon is twisted.

(c) The Euler characteristic of a closed orientable triangulation of 2-manifold is even. (This follows by Theorem 5.6.1 or by Assertion 6.7.3 (b).)

[17] The notion of orientability is 'impossible' to introduce for arbitrary 2-hypergraphs (think why), but is could be introduced for 2-hypergraphs each of whose edges is contained in at most two faces.

Is it correct that a surface is orientable iff it does not contain Möbius band? Different formalization of this informal question have different answers.

5.7.3. (a) A triangulation of 2-manifold is orientable if and only if no homeomorphic triangulation contains (as a subhypergraph) a triangulation of Möbius band.

(b)* There exists a non-orientable triangulation of 2-manifold that does not contain a triangulation of Möbius band.

The criterion from part (a) does not give an *algorithm* recognizing orientability. (Such an algorithm is obtained from the following strengthening of the criterion: replace the words 'no homeomorphic triangulation contains' by the words 'its second barycentric subdivision does not contain'. However, the corresponding algorithm is slow, i.e. has 'exponential complexity'.) A polynomial algorithm is presented in §6.1 (or can be obtained from Assertion 5.7.4.a).

5.7.4. (a) A closed triangulation of 2-manifold is orientable if and only if there exists a collection of faces of its barycentric subdivision such that every edge of the subdivision is incident to exactly one face of the collection.

(b) For any closed triangulation of 2-manifold, there exists a set of orientations on all faces of its barycentric subdivision such that the orientations of any two adjacent faces disagree.

Sketch of the proof of Surface Classification Theorem 5.6.1. The lack of homeomorphism (i.e. the second assertion of the theorem) is proved using orientability, the number of connected boundary components, and the Euler characteristic. That is, this part follows from Assertions 5.7.1 (a), 5.4.2 (b), 5.4.1 (b) and the results of Problems 5.5.6 (e), 5.5.3 (g).

The proof of homeomorphism (i.e. the first assertion of the theorem) is analogous to that of Theorem 5.3.3. That is, this part follows from Assertions 2.7.9 (b), 2.8.11 (b), and Assertions 5.7.2 (a, b). □

In Theorem 5.6.1, the number g of handles is called the *orientable genus* of a triangulation of 2-manifold. It can be found from the equation $2-2g-h=\chi$. The number m of Möbius bands is called the *non-orientable genus* and can be found from the equation $2-m-h=\chi$. See Problems 5.5.3 (g) and 5.5.6 (a).

5.8. Attaching a handle or a Möbius band is well-defined

The 2-hypergraphs obtained from a given locally Euclidean one by attaching a handle or a Möbius band, are unique up to a homeomorphism. For cutting a hole, this is Homogeneity Lemma 5.8.1.

Lemma 5.8.1 (homogeneity)**.** *Let p and q be any two faces of a locally Euclidean 2-hypergraph K. If both p and q are disjoint from ∂K, then $K - p$ and $K - q$ are homeomorphic.*

The fact that the result of attaching a handle or a Möbius band does not depend on the disks to which the handle is attached, also follows from Homogeneity Lemma 5.8.1. However, the independence from the attaching map is a priori not obvious (though it is usually not discussed in textbooks). Indeed, the result of gluing two quadrilaterals $ABCD$ and $A'B'C'D'$ to one another along the edges AB and $A'B'$, CD and $C'D'$, depends on the choice of attaching map (i.e., on the choice of directions along the edges used for gluing). Moreover, in the following paragraph we define a analogous operation of 'attaching a candle', which is not well-defined up to a homeomorphism.

A *candle* is the union of a quadrilateral $ABCD$ with segments CC_1, DD_1, DD_2. Given a surface M and an arc XY in its boundary, *attaching a candle* is taking the union of M and the candle, and identifying the arcs AB and XY. This can be done in two ways: identify A with X, and B with Y, or vice versa. The two thus obtained shapes are homeomorphic when M is a disk, but any homeomorphism between them reverses the orientation on the disk. The two thus obtained shapes are not homeomorphic when M is a disk with candle.

For higher-dimensional manifolds, the result of the attaching an analogue of a handle may depend on the choice of gluing (a remark for experts: $\mathbb{C}P^2\#\mathbb{C}P^2$ and $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ are not homeomorphic).

In order to have the independence of the way of gluing one needs the attached object has to be 'symmetric'. For attaching a handle, the independence follows from Assertion 5.8.2 (b) (or 5.8.2 (c) or 5.5.2.h), while for attaching a Möbius film this follows from Assertion 5.8.3 (or 5.5.2.g).

5.8.2. (a) The quadrilateral whose antipodal sides are endowed with 'agreeing' directions is homeomorphic to the quadrilateral whose antipodal sides are endowed with the opposite 'agreeing' directions. Formally, there exists a refinement K of the 2-hypergraph with vertices 1, 2, 3, 4 and faces $\{1,2,3\},\{1,3,4\}$, and an isomorphism $K \to K$, sending 1, 2, 3, 4 to 2, 1, 4, 3, respectively.

(b) The annulus whose boundary circles are endowed with 'agreeing' directions is homeomorphic to the annulus whose boundary circles are endowed with the opposite 'agreeing' directions.

(c) The torus with a hole and with a choice of direction along the boundary circle is homeomorphic to the torus with a hole and with the opposite choice of direction along the boundary circle.

5.8.3. The Möbius band with a direction on its boundary circle is homeomorphic to the Möbius band with the opposite direction along the boundary circle.

5.9. Regular neighborhoods and cellular subgraphs

The notion of a regular neighborhood is informally explained near Fig. 1.6.3 (left). An example of a regular neighborhood of a subgraph in a hypergraph one can take the union U of caps and ribbons corresponding to the vertices and the edges of the subgraph; that is, the union of those faces of the second barycentric subdivision that intersect the subgraph. Let us give the general definition.

A hypergraph L is obtained from a complex K by an *elementary collapse* if $K = L \cup \sigma$ and $L \cap \sigma = \partial\sigma - \mathrm{Int}\,\tau$ for some faces σ, τ of K such that $\tau \subset \partial\sigma$. A hypergraph K *collapses to* L (notation: $K \searrow L$) if there exists a sequence of elementary collapses $K = K_0 \searrow K_1 \searrow \ldots \searrow K_n = L$. A hypergraph K is *collapsible* if it collapses to a point.

A **regular neighborhood** of a subhypergraph A in a hypergraph K is a subhypergraph of some subdivision of K which contains A and collapses to A.

5.9.1. (a) The cone of any graph is collapsible.

(b) Construct three hypergraphs none of which collapses to a hypergraph homeomorphic to any other.

(c) The Euler characteristic is preserved under collapses.

(d) The Euler characteristic of a subgraph and of its regular neighborhood in a 2-hypergraph are equal.

(e) The union U is indeed a regular neighborhood.

The *complement* $G - H$ *in a graph* G *to a vertex set* H is formed by the vertices of the graph G that do not lie in H, and the edges of the graph G without endpoints in H.

Let G be a subgraph of a hypergraph K (i.e., a subgraph of the graph formed by the vertices and the edges of the hypergraph K). The *complement* $K - G$ is formed by the faces of the hypergraph K that do not intersect G.

The following definition formalize the construction of gluing a hypergraph out of a square (Figure 2.1.1) or a polygon.

Denote by $|K|$ the geometric realization of a graph K or a hypergraph K.

A vertex set A in a graph K is called (topologically) *cellular* if each connected component of $|K| - |A|$ is homeomorphic (topologically) to the open interval. We will be using the following (equivalent) combinatorial definition. A vertex set H in a graph G is called **cellular** if each connected component of the complement $G'' - H$ is homeomorphic to a segment each of whose endpoints belongs to an edge of the graph G'' incident to a vertex from H.

A subgraph A in a hypergraph K is called (topologically) *cellular* if each connected component of $|K| - |A|$ is homeomorphic (topologically) to the open disk. We will be using the following (equivalent) combinatorial definition. A subgraph G in a hypergraph K is called **cellular** if each connected component C of the complement $K'' - G''$ is homeomorphic to a disk[18] each of whose boundary edges lies in a face of the hypergraph K'' intersecting G. For example,

• a point in the sphere is cellular whereas a point in the torus is not;

• the union of the edges of a hypergraph is cellular.

5.9.2. *The Euler formula.* If K is a 2-hypergraph, and $G \subset K$ is a connected cellular subgraph with V vertices and E edges, then $V - E + F = \chi(K)$, where F is the number of connected components of the complement $K' - G'$.

[18]In many applications of the notion 'cellular', the condition 'homeomorphic to a disk' could be replaced by a weaker condition $\chi(C) = 1$, which is easier to verify. If the component C is locally Euclidean, then the cellularity condition is equivalent to this weaker condition as well as to the following one: the component C is split by any polygonal line with the endpoints on the boundary of C.

Hint. The formula follows from the inclusion-exclusion principle (Problem 5.5.4.a), since $\chi(D^2) = 1$.

5.9.3. (a) If a connected graph can be embedded to the sphere with g handles, then it is homeomorphic to a cellular subgraph of a sphere with at most g handles.

(b) The same for spheres with Möbius films.

§ 6. Homology of two-dimensional manifolds

> And the leap is not — is not what I think you sometimes see it as — as breaking, as acting. It's something much more like a quiet transition after a lot of patience and — tension of thought, yes — but with that [enlightenment] as its discipline, its orientation, its truth. Not confusion and chaos and immolation and pulling the house down, not something experienced as a great significant moment.
>
> I. Murdoch, The Message to the Planet.

6.1. Orientability criterion

The definitions of a piecewise linear (PL) 2-manifold and its triangulation are presented in §5.6. The definitions of a smooth 2-manifold and its triangulation are presented in §4.5. Either of these two approaches can be used for this section. However, a careful treatment is only presented in the PL language in some places.

The definition of orientability of a triangulation is given in §5.7. There is a nice and simple criterion of orientability: 'does not contain a Möbius band' (a precise formulation is given in Problem 5.7.3 (a)). There is a simple algorithm recognizing orientability as follows. It suffices to check the orientability of each connected component. First, orient a face of the component arbitrarily. Then at each step orient a face adjacent to any of the faces already oriented, until all faces are oriented, or two adjacent faces with disagreeing orientations are found.

In this section we will give an algebraic criterion of orientability, which, basically, is merely a reformulation of the definition of orientability in algebraic language. However, this criterion is important not on its own but rather as an illustration of obstruction theory. Moreover, similar considerations lead to Assertion 6.1.2 (b), and are applied in the *classification of thickenings* [Sk]. Cf. §6.8, §4.11.

Theorem 6.1.1 (Orientability). *A 2-manifold N is orientable if and only if its first Stiefel—Whitney class $w_1(N) \in H_1(N, \partial)$ is zero.*

The group $H_1(N, \partial)$ and the class $w_1(N)$ are defined later. They arise naturally and can be defined rigorously in the process of *inventing* the Orientability Theorem, which we will start in a moment. The computation of the group $H_1(N)$ is given in §6.4.

In this section the word 'group' can be regarded synonymous with the word 'set' (with the exception of Problems 6.2.5, 6.5.2, and §6.7). The constructions will remain interesting.

6.1.2. (a) Draw a closed non-self-intersecting curve on the disk with three Möbius bands, so that the complement to the curve is orientable.

(b) Any closed connected 2-manifold contains a closed non-self-intersecting curve whose complement is orientable. (More formally: for any closed connected triangulation of 2-manifold there is a subgraph of a homeomorphic triangulation T, such that the subgraph is homeomorphic to the circle, and the complement to the image of this subgraph in the second barycentric subdivision of T, see §5.9, is orientable.)

6.2. Cycles

The notion of a cellular decomposition of a hypergraph formalizes the examples 'glued of polygons' from Example 5.1.1.c. A **cellular decomposition** of a hypergraph K is a pair $K_0 \subset K_1 \subset K$ of its subhypergraphs in which K_1 is a cellular subgraph in K and K_0 is a cellular set of vertices in K_1 (see §5.9 for definitions). The graph K_1 is called the *one-dimensional skeleton* of the cellular decomposition. *Edges* and *faces* of a cellular decomposition $K_0 \subset K_1 \subset K$ are the connected components of the complement $K_1'' - K_0$ and connected components of the complement $K'' - K_1''$, respectively.

Many constructions are done more conveniently for cellular decompositions rather than for hypergraphs, since many 'interesting' hypergraphs have 'many' faces, but admit 'economical' cellular decompositions. For computations, it is more convenient to draw cellular decompositions rather than more cumbersome polygonal decompositions. Triangulations are special cases of cellular decompositions. Other examples are shown in Figure 2.1.1. In the following considerations, except the examples, the reader may substitute cellular decompositions with triangulations.

In this section T is a cellular decomposition of a 2-manifold N, while o is a choice of orientations on the faces of T.

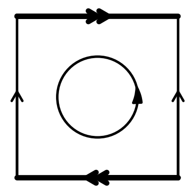


Figure 6.2.1. Collection o of orientations, and the obstruction cycle $\omega(o)$

Color an edge of a cellular decomposition T in red if the orientations of the incident faces *do not agree* along this edge, i.e., induce the same direction on the edge. The collection of the red edges is called the **obstruction cycle** $\omega(o)$.

For instance, in Figure 6.2.1 the Klein bottle is represented as a square with glued sides, i.e., it is decomposed into a single polygon. The faces incident to the horizontal edge from the two sides, coincide. But their (or rather its) orientations do not agree along the edge. Besides, the orientation of the only face agrees with itself along the vertical edge. Hence, in Figure 6.2.1 the obstruction cycle consists of a single horizontal edge (shown in bold).

So, if a decomposition is not a triangulation, then the orientation of a face incident to an edge from two sides does not have to agree with itself along this edge. Moreover, a pair of faces (coinciding or not) might have orientations that agree along one edge but disagree along another edge.

6.2.1. (a) For each edge of the single-face cellular decomposition of the Möbius band (i.e., of the representation of the Möbius band as a square with glued sides, see the third column in Figure 2.1.1), find out if the orientation of the only face agrees with itself along this edge.

(b) The same question for the projective plane (Figure 2.1.1).

6.2.2. (a) Draw the obstruction cycle for the single-face cellular decomposition of the Möbius band.

(b) The same for the projective plane.

Many of the following facts (for example, Problems 6.2.3 (a, b)) can be first proved for triangulations and then for cellular decompositions.

6.2.3. (a) A collection o of face orientations determines an orientation of a cellular decomposition if and only if $\omega(o) = \varnothing$.

(b) If a 2-manifold is closed, then each vertex has an even number of incident edges of the obstruction cycle (by convention, a loop counts with multiplicity two).

(c) The complement to the obstruction cycle $\omega(o)$ (formally, the union of the faces of the second barycentric subdivision that do not intersect $\omega(o)$) is orientable.

A **cycle** (homological, one-dimensional, mod 2) in a graph (or in a hypergraph) is an unordered collection of its edges such that any vertex has an even number of incident edges from the collection. The words 'homological', 'one-dimensional' and 'mod 2' will be omitted. Cycles in the sense of graph theory will be called 'closed curves'.

For instance, the graphs in Figure 1.2.1 have 2, 8, and 8 cycles, respectively. The union of edges in the single-face cellular decomposition of the Klein bottle (Figure 6.2.1) is the 'figure eight', so this graph has four cycles.

6.2.4. How many cycles are there in a connected graph with V vertices and E edges?

On the set of all cycles in a given graph (or a hypergraph) consider the operation of the (mod 2) **sum** (i.e., the symmetric difference).

6.2.5. The *homology group* $H_1(G)$ of a graph G (one-dimensional, with coefficients mod 2) is the group of all cycles in the graph G.

(a) The sum of cycles is a cycle.

(b) Homeomorphic graphs have isomorphic homology groups.

(c) For a connected graph G with V vertices and E edges, one has $H_1(G) \cong \mathbb{Z}_2^{E-V+1}$.

(d) Non-self-intersecting closed curves in a graph G generate $H_1(G)$.

6.3. Homologous cycles

If $\omega(o) \neq \varnothing$, then o does not determine an orientation of a cellular decomposition T. All is not lost though: one can try to modify o in order to make the obstruction cycle empty. For this, let us find out how $\omega(o)$ depends on o. The answer is formulated conveniently using the mod 2 sum (i.e., the symmetric difference) of edge sets in an arbitrary graph.

The (homological) **boundary** ∂a of a face a in a hypergraph is the set of edges of the geometric boundary of this face.

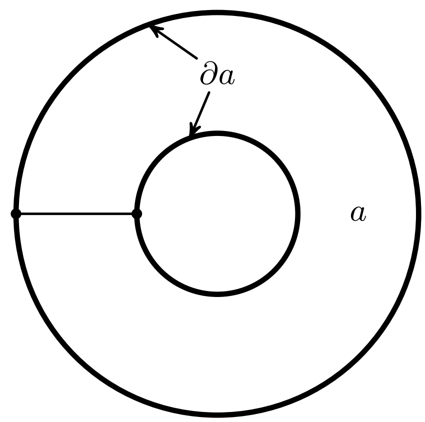


Figure 6.3.1. Homological (algebraic) boundary of a complicated face

For a face of a cellular decomposition, the definition is more involved. The (homological) **boundary** ∂a of a face a is the set of all those edges of the geometric boundary of the face that are adjacent to the face just from one side (Figure 6.3.1).

As for cycles, the word 'homological' will be omitted. For the single-face cellular decomposition of the Klein bottle (Figure 6.2.1) the boundary of the only face is empty.

6.3.1. (a) What is the boundary of the only face in the single-face cellular decomposition of the projective plane (see Figure 2.1.1)?

(b) The boundary of a face is a cycle.

(c) When the orientation of single face a is reverted, the cycle $\omega(o)$ changes to the sum with the boundary of that face: for the resulting collection o' of orientations one has $\omega(o') - \omega(o) = \partial a$.

(d) When the orientations of *several faces* $a_1, \dots, a_k$ are reverted, the cycle $\omega(o)$ changes to the sum with the boundaries of these faces: for the resulting collection o' of orientations one has

$$\omega(o') - \omega(o) = \partial a_1 + \ldots + \partial a_k.$$

Two cycles are called **homologous** (or *congruent modulo boundaries*), if their difference is the sum of the boundaries of several faces.

6.3.2. (a) When the collection o of orientations is changed, the obstruction cycle $\omega(o)$ is replaced by a homologous cycle.

(b) If $\omega(o)$ is a boundary, then it is possible to change o to o' so that $\omega(o') = \varnothing$.

Proposition 6.3.3. *A closed triangulation of* 2*-manifold is orientable if and only if some* (*or, equivalently, any*) *obstruction cycle is homologous to the empty cycle.*

Sketch of the proof. It is clear that this condition is necessary for orientability. Conversely, suppose that some obstruction cycle is homologous to the empty cycle. Then there exists a collection o of face orientations of which $\omega(o)$ is the boundary. Then by Assertion 6.3.2 (b) it is possible to change o to o' so that $\omega(o') = 0$. Therefore, the triangulation is orientable. ☐

6.3.4. (a) Any two cycles in the single-face cellular decomposition of the sphere (see Figure 2.1.1) are homologous.

(b) The boundary circles on the torus with two holes are homologous (for any cellular decomposition).

(c) The boundary circle of the Möbius band is homologous to the empty cycle (for any cellular decomposition).

6.3.5. For the single-face cellular decomposition of the torus (Figure 2.1.1)

(a) the 'meridian' cycle is not homologous to the empty cycle;

(b) different cycles are not homologous.

6.3.6. (a) In the single-face cellular decomposition of the projective plane (Figure 2.1.1) different cycles are not homologous.

(b) In the complete hypergraph on 9 vertices any two cycles are homologous.

(c) Any two cycles are homologous in the single-face cellular decomposition of the Zeeman dunce hat.

(The *Zeeman dunce hat* is obtained from a triangle ABC by gluing all three its sides directed so that $\overrightarrow{AB} = \overrightarrow{AC} = \overrightarrow{BC}$.)

6.3.7. (a) Homology is an equivalence relation on the set of cycles.

(b) Any cycle in a connected triangulation T of 2-manifold is homologous to a closed non-self-intersecting polygonal line in some subdivision of T.

(c) Is the same true for an arbitrary connected hypergraph T?

6.3.8. (a) The sum of the boundaries of all faces of a closed triangulation of 2-manifold is empty.

(b) The sum of the boundaries of all faces of a triangulation of 2-manifold equals to the boundary.

(c) The sum of the boundaries of any proper subset of faces of a connected closed triangulation of 2-manifold is non-empty.

6.3.9. (a) Any cycle in a hypergraph is homologous to some cycle in any cellular graph in this hypergraph.

(b) If two cycles in a cellular decomposition of a hypergraph are homologous in the hypergraph, then they are homologous in the cellular decomposition as well.

6.4. Homology and the first Stiefel—Whitney class

Recall the definitions, motivated and introduced in the previous sections. A **cycle** in a hypergraph is an unordered collection of edges such that every vertex is incident to an even number of them. The **boundary** ∂a of a face a in a hypergraph is the collection of all edges of the geometric boundary of this face. Two cycles are called **homologous** if their difference is the sum of several boundaries.

The **homology group** $H_1(K)$ (one-dimensional, with coefficients mod 2) of a hypergraph K is the group of cycles up to homology.

The homology group appears in solutions of specific problems (e.g. in checking orientability, see §6.2-§6.3). It is important that the homology group is defined in a short way regardless of the problems, and for arbitrary hypergraphs.

6.4.1. (a) On the set $H_1(K)$ the sum operation is well-defined by the formula $[\alpha]+[\beta]=[\alpha+\beta]$.

(b) The set $H_1(K)$ with this operation is a group.

(c) The homology groups of homeomorphic hypergraphs are isomorphic. More precisely, if a hypergraph K is obtained from a hypergraph L by edge subdivision, then the naturally defined homomorphism $H_1(L)\to H_1(K)$ is an isomorphism.

The *homology group* $H_1(T)$ (one-dimensional, with coefficients mod 2) of a cellular decomposition T of a hypergraph is defined analogously. By definition, the **boundary** ∂a of a face a of a cellular decomposition of a hypergraph is the collection of those edges of the geometric boundary of a that are adjacent to a from an odd number of sides (Figure 6.3.1).

6.4.2. (a) For the aforementioned single-face cellular decompositions of the sphere, the torus, the projective plane, the Klein bottle (Figures 2.1.1 and 6.2.1) the number of elements in $H_1(T)$ equals 1, 4, 2, 4, respectively.

(b) For a cellular decomposition T of a hypergraph K we have $H_1(T) \cong H_1(K)$.

The *homology group* $H_1(N)$ (one-dimensional, with coefficients mod 2) of a 2-manifold N is the group $H_1(T)$ for any triangulation T of the manifold (or even for any cellular decomposition T of a triangulation). The homology group is well-defined by Assertion 6.4.1 (c) (and 6.4.2 (b)).

The **first Stiefel—Whitney class** of a cellular decomposition T of a closed triangulation of 2-manifold is the homology class of an obstruction cycle:

$$w_1(T) := [\omega(o)] \in H_1(T).$$

This is well-defined by Assertion 6.3.2 (a).

The **first Stiefel—Whitney class** of a closed 2-manifold N is the first Stiefel—Whitney class of any triangulation T of 2-manifold N (or even of any cellular decomposition T of a triangulation): $w_1(N) := w_1(T)$. This is well-defined in the following sense (see also Assertion 6.4.2 (b)).

6.4.3. The map from Assertion 6.4.1 (c) sends $w_1(L)$ to $w_1(K)$.

Orientability Theorem 6.1.1 is a reformulation of Assertion 6.3.3.

6.5. Computations and properties of homology groups

In the arguments involving homology classes of cycles, it is convenient first to work with representing cycles, and then prove that the actual choice of the representatives does not play a role.

6.5.1. Find the homology group and draw the curves representing its basis for (any triangulation of)

(a) the sphere with g handles;

(b) the sphere with g handles and h holes;

(c) the sphere with m Möbius bands;

(d) the sphere with m Möbius bands and h holes.

6.5.2. If T is a cellular decomposition of a connected closed 2-manifold, then $H_1(T) \cong \mathbb{Z}_2^{2-\chi(T)}$.

6.5.3. (a) If M and N are closed 2-manifolds, then $H_1(M\#N) \cong$ $\cong H_1(M) \oplus H_1(N)$ (the operation $\#$ of connected sum is defined analogously to Figure 5.5.1).

(b) Does that formula hold for non-closed 2-manifolds M and N?

6.5.4. (a) For any hypergraphs K and L sharing at most one point, $H_1(K \cup L) \cong H_1(K) \oplus H_1(L)$.

(b) Does that formula hold if there are two common points?

6.5.5. (a) For any connected graph K one has

$$H_1(K \times I) \cong H_1(K) \quad \text{and} \quad H_1(K \times S^1) \cong H_1(K) \oplus \mathbb{Z}_2.$$

(Come up with your own definitions of the product of a graph with the interval/the circle, or find the definitions in [Sk, §6.16 'Cartesian products'].)

(b) The group $H_1(K)$ is not changed under collapsing. (Hence the group $H_1(K)$ is not changed by passing to the regular neighborhood.)

Let T be a cellular decomposition of a triangulation of 2-manifold N (perhaps, with a non-empty boundary). A *cycle relative to the boundary* (or a *relative cycle*, for brevity) in T is a collection of edges of T such that every non-boundary vertex is incident to an even number of the edges from the collection. Two relative cycles are said to be *homologous relative to the boundary*, if their difference is a sum of the boundaries of several faces and of some boundary edges. The homology groups $H_1(T, \partial)$, $H_1(N, \partial)$ relative to the boundary, and the classes $w_1(T) \in H_1(T, \partial)$, $w_1(N) \in H_1(N, \partial)$ are defined analogously to above.

6.5.6. (a, b) Formulate and solve the analogues of Problems 6.5.1 (b, d) for the homology groups relative to the boundary.

6.6. Intersection form: motivation

The intersection form is among the most important tools and research objects in topology and its applications. See [DZ93]. The intersection form arises naturally, for instance, when proving Assertions 6.6.1 (b) and 6.6.2. See also the Mohar formulas 2.7.7 (c) and 2.8.8 (c).

6.6.1. (a) Regular neighborhoods (see Figure 1.6.3, on the left, and §5.9) of isomorphic graphs in the same surface are not necessarily homeomorphic.

(b) Regular neighborhoods of the images of homotopic embeddings of a given graph into a 2-manifold are homeomorphic. (The definitions of homotopy are analogous to the ones given in §3.2, 3.4, 3.7.)

Two embeddings $f_0, f_1 \colon G \to N$ are called *isotopic* if there exists a family $U_t \colon N \to N$ of *homeomorphisms* depending continuously on the parameter $t \in [0,1]$, such that $U_0 = \mathrm{id}$ and $U_1 \circ f_0 = f_1$. It is clear that regular neighborhoods of the images of homotopic embeddings of a given graph into a surface are homeomorphic. In contrast, Assertion 6.6.1 (b) is not obvious.

6.6.2. On Topologist's planet, shaped as a solid torus, there are rivers Meridian and Parallel. The Little Prince and Topologist traveled around the planet along two different closed routes. The prince crossed the Meridian 9 times and the Parallel 6 times, while Topologist crossed the rivers 8 and 7 times, respectively. Then their routes had to intersect. (When crossing a river a character ends up on the other bank of the river. More rigorously, the intersection of the river and character's path are *transverse*, see the definition below.)

An heuristic argument, leading to the notion of the intersection number. Let N be a 2-manifold and let a, b be closed curves on N. Let us assume that a and b

- are subgraphs of a certain hypergraph representing N;
- are in *general position*; that is, they intersect transversely (Figure 6.6.1) in finitely many points, none of which is a self-intersection point of either a or b.

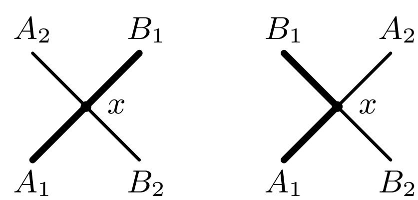


Figure 6.6.1. A transverse intersection and a non-transverse intersection

An intersection point x of two curves on a 2-manifold is called *transverse* if the curves are non-self-intersecting in a neighborhood of the point, and every sufficiently small closed curve S_x winding around x intersects the two curves in two pairs of points that *alternate* along S_x (that is, if A_1, B_1 are the intersection points of the first curve with S_x, and A_2, B_2 are the intersection points of the second curve with S_x, then these points are situated along S_x in the order $A_1A_2B_1B_2$). In other words, in order for the point x to be transverse, two short 'segments' of the first curve that are incident to x need to be on the different sides of the second curve in a small neighborhood of x, see Figure 6.6.1.

In this situation $|a \cap b|$ mod 2 does not change if a and b are replaced by homologous curves satisfying the same condition (the subgraphs, corresponding the curves, are homologous cycles; this is what is meant by 'homologous' curves).

6.7. Intersection form: definition and properties

The construction of the preceding section can be reworked in order to define the intersection form via transversality. We will present a different definition. Instead of transversality it will use the more convenient notion of the dual decomposition into polygons, see §4.8.

Take a triangulation T of a 2-manifold N (in other words, a hypergraph representing N). Take the dual decomposition T^* into polygons. Then 1-cycles in T^* are defined analogously. For 1-cycles x in T, and y in T^*, set

$$[x] \cap [y] := |x \cap y|_2$$

to be the parity of the number of their intersection points.

6.7.1. (a) The intersection product of 1-cycles is bilinear:

$$x \cap (z+t) = x \cap z + x \cap t \quad \text{and} \quad (x+y) \cap z = x \cap z + y \cap z.$$

(b) The intersection of a cycle and a boundary equals zero.

(c) The product $\cap\colon H_1(T) \times H_1(T^*) \to \mathbb{Z}_2$ is well-defined.

(d) Let $T, \overline{T}$ be triangulations of a 2-manifold N, where $\overline{T}$ is obtained from T by a single edge subdivision. Define 'natural' maps $f\colon H_1(T) \to H_1(\overline{T})$ and $f^*\colon H_1(T^*) \to H_1(\overline{T}^*)$ (cf. Assertion 6.4.1 (c)). Prove that $x \cap y = f(x) \cap f^*(y)$ for any 1-cycles x in T, and y in T^*.

A solution of (c) is presented in §10.7.

By Assertion 6.7.1 (d) one obtains the symmetric bilinear **intersection form**

$$\cap\colon H_1(N) \times H_1(N) \to \mathbb{Z}_2.$$

6.7.2. (a) Find the intersection form of the sphere with g handles (that is, find the matrix of this form in some basis of the homology group).

(b) Find the intersection form of the sphere with m Möbius bands.

(c) The rank of the intersection form of a disk with ribbons is equal to the rank defined in the Mohar formula 2.8.8 (c).

(d) The intersection form is symmetric: $\alpha \cap \beta = \beta \cap \alpha$.

6.7.3. Let N be a closed 2-manifold. The definition of the first Stiefel—Whitney class $w_1(N) \in H_1(N)$ is presented in §6.4.

(a) For any $a \in H_1(N)$, one has $w_1(N) \cap a = a \cap a$.

(b) $w_1(N) \cap w_1(N) = \rho_2\chi(N)$.

6.7.4. *Poincaré duality.* The intersection form of any closed 2-manifold N is non-degenerate; that is, for any $\alpha \in H_1(N) - \{0\}$ there exists $\beta \in H_1(N)$ such that $\alpha \cap \beta = 1$.

6.7.5. (e) The intersection form can be degenerate for a 2-manifold with non-empty boundary.

(f) Find the intersection form and its rank for the sphere with g handles and h holes.

(g) Find the intersection form and its rank for the sphere with m Möbius bands and h holes.

(h) Can every bilinear symmetric form $\mathbb{Z}_2^k \times \mathbb{Z}_2^k \to \mathbb{Z}_2$ be represented as the intersection form of some 2-manifold?

Theorem 6.7.6. *2-manifolds are homeomorphic if and only if their intersection forms are isomorphic, and the manifolds either both are closed or both have non-empty boundary.*

6.7.7. A 2-manifold M with boundary can be cut from a 2-manifold N if and only if $\mathrm{ori} M \leqslant \mathrm{ori} N$ and $\mathrm{rk}\, M - \mathrm{ori} M \leqslant \mathrm{rk}\, N - \mathrm{ori} N$. Here rk is the rank of the intersection form, and $\mathrm{ori} \in \{0, 1\}$ is the orientability.

6.7.8. (a) There are 2-manifolds with boundary intersecting by the 2-disk, having the same rank $r > 0$ of the intersection form, and whose union has the same rank r of the intersection form. (Then $\mathrm{rk}(M_1 \cup M_2) < \mathrm{rk}\, M_1 + \mathrm{rk}\, M_2$.)

(b) If two 2-manifolds with boundary intersect by the 2-disk, then $\mathrm{rk}(M_1 \cup M_2) \leqslant \mathrm{rk}\, M_1 + \mathrm{rk}\, M_2$.

§ 8. Vector fields on higher-dimensional manifolds

The main results of this section are stated in § 8.1 and § 8.7. In § 8.7 we use definitions introduced at the beginning of § 8.6. Let

$$D^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1^2 + \ldots + x_n^2 \leqslant 1\} \quad \text{and}$$
$$S^{n-1} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1^2 + \ldots + x_n^2 = 1\}.$$

If you find the case $n > 3$ difficult, you can read this section assuming that $n = 3$, since already this case is interesting.

8.1. Vector fields on the Euclidean space

Definitions of general, non-vanishing and unit vector fields on a subset $N \subset \mathbb{R}^n$ and of their homotopies are straightforward generalizations of the case $n = 2$ (§ 3.3 and § 3.4). Homotopy of maps is defined in § 3.7.

8.1.1. (a) Any non-vanishing vector field v on $\mathbb{R}^{2k}$ is homotopic to the vector field $-v$.

(b) The radial vector field on S^{2k-1} is homotopic to the central vector field.

8.1.2. State and prove versions of Problems 3.4.4 (a—e), 3.4.5 (a, c), 3.4.6 (b), 3.4.7 (a, b) and 3.7.2 (b, c, d, e) for vector fields on $\mathbb{R}^n$ and maps to S^{n-1}.

8.1.3. The following statements are equivalent. (You do not need to prove the statements, only their equivalence.)

(1) **The Brouwer Fixed Point Theorem.** *Any map $f\colon D^n \to D^n$ from the ball to itself has a fixed point, i. e. a point $x \in D^n$ such that $f(x) = x$.*

(2) **Non-retractability of the ball onto the boundary sphere.** *There does not exist a map from the ball to its boundary sphere that is equal to the identity on the sphere, i. e. a map $f\colon D^n \to S^{n-1}$ such that $f(x) = x$ for every $x \in S^{n-1}$.*

(3) *The identity map of the sphere S^{n-1} is not homotopic to the constant map (i. e. to the map to a point).*

These results can be proved using a higher-dimensional version of the Sperner Lemma 3.6.3 (Sp) and piecewise-linear approximation (see Problem 8.2.2). We discuss a similar (but more complicated) proof using the *degree modulo 2* of a map, an important notion introduced in 1911 by Luitzen Egbertus Jan Brouwer, which will be used later in the book. More precisely, statement (3) follows from Problems 8.3.5 (a, b), 8.3.6 (c) and 8.3.7 (c, d).

Theorem 8.1.4 (Borsuk-Ulam). *Для любого отображения $f : S^d \to \mathbb{R}^d$ существует такое $x \in S^d$, что $f(x) = f(-x)$.*

This theorem has many equivalent formulations, see Theorem 8.1.5 and [Ma03]. The equivalence of the following assertions to each other and to Theorem 8.1.4 is simple.

A map $f : S^n \to \mathbb{R}^m$ is called *odd*, or *equivariant*, or *antipodal* if $f(-x) = -f(x)$ for any $x \in S^n$.

Theorem 8.1.5 (Borsuk-Ulam). (a) *For any equivariant maps $f : S^d \to \mathbb{R}^d$ there exists $x \in S^d$ such that $f(x) = 0$.*

(b) *There are no equivariant maps $S^d \to S^{d-1}$.*

(c) *No equivariant map $S^{d-1} \to S^{d-1}$ extends to D^d.*

(d) *If S^d is the union of $d+1$ closed sets (or $d+1$ open sets), then one of the sets contains opposite points.*

Part (c) follows by Assertion 8.3.8.f whose simple proof is sketched in Problems 8.3.8.a-e.[19]

Theorem 8.1.6. *The balls D^n and D^k are not homeomorphic if $n \neq k$.*

This is deduced from Theorems 8.1.3 (3) and 8.1.7 (a).

Theorem 8.1.7. (a) *For $k < n$, any map $S^k \to S^n$ is homotopic to the map to a point.*

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[19] This slightly simplifies the proof from [BSS] and [Ma03, pp. 153-154]. For other proofs of Theorems 8.1.4, 8.1.5 and Assertion 8.3.8.f see §3 (for $d = 2$), [Ma03, §2], and the references therein. E.g. Theorem 8.1.5.a can be deduced from its following 'quantitative version': *If $0 \in \mathbb{R}^d$ is a regular point of a (PL or smooth) equivariant map $f : S^d \to \mathbb{R}^d$, then $|f^{-1}(0)| \equiv 2 \mod 4$.*
See the definition of a regular point e.g. in §8.3. This quantitative version is proved analogously to [Sk, Lemmas 1.4.3 and 2.2.3]: calculate $|f^{-1}(0)|$ for a specific f and prove that $|f^{-1}(0)|$ modulo 4 is independent of f. Realization of this simple idea is technical, see [Ma03, §2.2].

(b) *For $n \geqslant 2$, any map $S^n \to S^1$ is homotopic to the map to a point.*

The proof of Theorem 8.1.7 (b) is a straightforward generalization of the proof of Theorem 3.1.9 (b) (§ 3.11). The proof of Theorem 8.1.7 (a) is based on piecewise-linear (or smooth) approximation similar to the proof of Theorem 3.1.9 (a) in § 3.11. More precisely, Theorem 8.1.7 (a) follows from the result of Problem 8.2.2.

Definitions of a tangent vector field on the n-dimensional sphere is a straightforward generalizations of the case $n = 2$ (§ 4.1).

Theorem 8.1.8 (Hopf). (a) *The sphere S^n admits a non-vanishing tangent vector filed if and only if n is odd.*

(b) *The identity map of S^n is homotopic to the antipodal map if and only if n is odd.*

For odd n this follows by giving an explicit formula for a field or a homotopy. For even n part (a) follows by (b), and part (b) follows by the results Problems 8.4.3.c and 8.4.5.d (i.e., using the *degree*; here the degree modulo 2 is not sufficient!). Alternatively, one show that $\chi(S^{2k}) = 2$ (by constructing the vector field of the velocities of water flowing from the North Pole to the South Pole), and use the Hopf Theorem 8.7.4.

Solutions to many problems are similar to the solutions in the low-dimensional cases (§ 3 and § 4). This hint will not be repeated for each such problem.

Hint to 8.1.6. The following statements (A) and (B) follow from Theorems 8.1.7 (a) and 8.1.3 (3) respectively.

(A) For any $k < n$ and any point $x \in D^n$, any map $S^{k-1} \to D^n - \{x\}$ is homotopic to the map to a point;

(B) For any k the inclusion $i : S^{k-1} \to D^k - \{0\}$ is not homotopic to the map to a point.

Proof of the Theorem 8.1.6 using (A) and (B). Suppose that there exists a homeomorphism $h : D^k \to D^n$. By (A) for $x = h(0)$, there exists a homotopy $H : S^{k-1} \times [0,1] \to D^n - \{x\}$ between $h \circ i$ and the map to a point $a \in D^n - \{x\}$. The map

$$\widehat{H} := h^{-1} \circ H : S^{k-1} \times [0,1] \to D^k - \{0\}$$

is continuous as a composition of continuous maps. We have

$$\widehat{H}(y,0) = h^{-1}hiy = iy \quad \text{and} \quad \widehat{H}(y,1) = h^{-1}H(y,1) = h^{-1}a.$$

Hence $\widehat{H}$ is a homotopy between i and the map to the point $h^{-1}a$. This contradicts (B).[20]

8.2. Piecewise-linear approximation

Let

$$S^n_{PL} := \partial I^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\colon\ \max(|x_1|, \ldots, |x_{n+1}|) = 1\}$$

be the surface of the $(n+1)$-dimensional cube (also called the standard piecewise-linear sphere). Let $\pi\colon S^n \to S^n_{PL}$ be the central projection whose center is at the origin.

A *triangulation* of the sphere S^n_{PL} is a decomposition of S^n_{PL} into finitely many n-dimensional simplices such that the intersection of each two of the simplices is a simplex of dimension less than n (this includes the case of disjoint simplices). A map $S^k_{PL} \to S^n_{PL}$ is called **piecewise-linear** if it is linear on every simplex of some triangulation of the sphere[21] S^k_{PL}.

8.2.1. Which of the following maps $S^n_{PL} \to S^n_{PL}$ are piecewise-linear:
(a) constant map;
(b) identity map;
(c) antipodal map (i. e. central symmetry with the center at the origin);
(d) restriction of an isometry $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to S^n_{PL};
(e) central projection from the point $(1/2, \ldots, 1/2)$;
(f) map $S^2_{PL} \to S^2_{PL}$, given by the formula $(x, y, z) \mapsto (z^2, x, y)$;
(g) $\pi(\Sigma w_k)\pi^{-1}$ for the k-fold winding $w_k\colon S^1 \to S^1$ of S^1 (see the definition of suspension Σg below)?

The suspension Σg of a map $g\colon S^1 \to S^1$ is the map $f\colon S^2 \to S^2$ given by

$$f(\cos\alpha\cos\theta, \cos\alpha\sin\theta, \sin\alpha) := (g(\cos\theta, \sin\theta)\cos\alpha, \sin\alpha).$$

8.2.2. (a—e) State and prove versions of Assertions 3.11.3 for maps $f, g\colon S^k \to S^n$, $k < n$.

[20] It suffices to use (A) for points $x \in \operatorname{Int} D^n$ only. For this, take any interior point x_0 in a sufficiently small neighbourhood of the point $h^{-1}(0)$, so that $x_0 \in \operatorname{Int} D^k$ and $h(x_0) \in \operatorname{Int} D^n$, and set $x = h(x_0)$.

[21] Instead of introducing the sphere S^n_{PL}, we could define triangulations and piecewise-linear maps for S^n.

Hint to 8.2.1. (g) The map is PL only for $k\in\{0,1,-1\}$. It suffices to prove that for every $k\notin\{0,1,-1\}$ there exists an arbitrarily small $x>0$ such that $\tan(kx)\neq k\tan(x)$. Here your school trigonometry is more effective than the Taylor formula.

8.3. Modulo 2 degree of a map

The following constructions are already interesting in the case $n=1$ (try to solve the problems below for $n=1$ first if this case makes sense). Note that for $n=1$ they are different from those discussed in § 3.8; see Assertion 8.4.6.

In this and the next subsections we assume that $g\colon S^n_{PL}\to S^n_{PL}$ is a piecewise-linear map, and we use a triangulation of S^n_{PL} such that g is linear on every simplex of the triangulation.

A **regular value** of the map g is any point $y\in S^n_{PL}$ outside the union of g-images of the boundaries of the simplices of the triangulation. It is clear that such a point exists.

8.3.1. (a—d) Find a regular value for each of the maps $S^n_{PL}\to S^n_{PL}$ defined in Problem 8.2.1 (a—d).

8.3.2. For every regular value y, the set $g^{-1}(y)$ is finite.

Let the **mod 2 degree** $\deg_2 g\in\mathbb{Z}_2$ of g be the parity of the number of g-preimages of a regular value y.

8.3.3. The mod 2 degree of a PL map is well-defined, i.e., is independent of y.

This follows from Assertion 8.3.6.

By the result of Problem 8.2.2 (e) for every map $f\colon S^n\to S^n$ there exists a piecewise-linear map $g\colon S^n_{PL}\to S^n_{PL}$ homotopic to the map $\pi f\pi^{-1}$. Define the **mod 2 degree** of f by $\deg_2 f:=\deg g$.

8.3.4. The mod 2 degree of a map is well-defined, i.e., is independent of g.

This follows from Assertions 8.3.7.abc.

8.3.5. Assuming Assertions 8.3.3 and 8.3.4, find the mod 2 degrees of the following maps:

(a—g) the maps $S^n\to S^n$ analogous to the maps $S^n_{PL}\to S^n_{PL}$ defined in Problem 8.2.1.

(2, 3, 4) 'taking d-th power' $S^n \to S^n$, for $S^2 = \mathbb{C}P^1$, $S^3 \subset \mathbb{H}$ and $S^4 = \mathbb{H}P^1$. (Here the smooth version of the definition presented in footnote 22 works better than the piecewise-linear version.)

8.3.6. Let y_0, y_1 be regular values of the map g. Join y_0 and y_1 by a polygonal line $l \subset S^n_{PL}$ such that

• l has no self-intersections,

• $l \cap g\sigma = \varnothing$ for any $(n-2)$-simplex σ of the triangulation,

• for any $(n-1)$-simplex τ of the triangulation, $l \cap g\tau$ consists of at most one point, and if $l \cap g\tau$ is one point then this point splits a small part of l near this point into two polygonal lines that are contained in the g-images of different n-simplices.

(A polygonal line with these properties is called a *regular path* for the map g.)

Then $g^{-1}(l)$ is a union of finitely many pairwise disjoint (closed and non-closed) polygonal lines whose end points form the set $g^{-1}\{y_0, y_1\}$.

Triangulations of the cylinder $S^n_{PL} \times I$ (and other subsets of $\mathbb{R}^d$) are defined in the same way as triangulations of the sphere S^n. (For some subsets a triangulation may not exist.) A homotopy $S^n_{PL} \times I \to S^n_{PL}$ is called *piecewise-linear* if it is linear on every simplex of some triangulation of the cylinder $S^n_{PL} \times I$.

8.3.7. (a) Let $G\colon S^n_{PL} \times I \to S^n_{PL}$ be a homotopy linear on every simplex of some triangulation of the cylinder $S^n_{PL} \times I$. Take a point $y \in S^n_{PL}$ outside the union of G-images of $(n-1)$-dimensional simplices of the triangulation. (Such a point is called a *regular value* of the homotopy G.) Then $G^{-1}(y)$ is a union of finitely many pairwise disjoint (closed and non-closed) polygonal lines whose end points form the set $G^{-1}(y) \cap S^n_{PL} \times \{0, 1\}$.

(b) For any two piecewise-linearly homotopic piecewise-linear maps $g, g'\colon S^n_{PL} \to S^n_{PL}$ there exists a common regular value $y \in S^n_{PL}$ such that $|g^{-1}(y)| \equiv |(g')^{-1}(y)| \mod 2$.

(c) If two piecewise-linear maps $S^n_{PL} \to S^n_{PL}$ are homotopic, then they are piecewise-linearly homotopic.

(d) The mod 2 degrees of homotopic maps are equal.

8.3.8. Take an equivariant PL map $f : S^k \to S^k$ such that $f|_{S^{k-1}} = \mathrm{id}$. Let

$$D^k_\pm := \{(x_1, \dots, x_{k+1}) \in S^k \ : \ \pm x_{k+1} \geqslant 0\}.$$

Let $f^+ : S^k \to S^k$ be the 'union' of f on D^k_+ and the identity on D^k_-. Let $f^- : S^k \to S^k$ be the 'union' of f on D^k_- and the identity on D^k_+. Denote by $\deg_2$ the degree modulo 2.

(a) Find $\deg_2 f^+$ and $\deg_2 f^-$ for the standard n-winding $f : S^1 \to S^1$, $n = 3, 5$.

(b) $f^-(x) = -f^+(-x)$.

(c) $\deg_2 f^+ = \deg_2 f^-$.

(d) $\deg_2 f = \deg_2 f^+ + \deg_2 f^- + 1$.

(e) $\deg_2 f = 1$.

(f) Any equivariant map $S^k \to S^k$ has an odd degree.

Hint to 8.3.2. It suffices to prove that $|g^{-1}(y) \cap \Delta| \leqslant 1$ for any simplex Δ of the triangulation in question of the set S^n_{PL}. Suppose that for some Δ there exist two distinct points $x_1, x_2 \in \Delta$ such that $g(x_1) = g(x_2) = y$. Denote by x the intersection of $\partial\Delta$ with the line through the points x_1, x_2. Then there exists $t \in \mathbb{R}$ such that $x = tx_1 + (1-t)x_2$. The map g is linear on the simplex Δ, hence

$$g(x) = g(tx_1 + (1-t)x_2) = tg(x_1) + (1-t)g(x_2) = ty + (1-t)y = y.$$

Thus $g^{-1}(y) \cap \partial\Delta \neq \varnothing$. This contradicts the assumption that y is a regular value of the map g.

8.4. Degree of a map

The *sign* of a g-preimage x of a regular value y is defined as $+1$, if the restriction of g to the simplex of the triangulation that contains x preserves the orientation, and as -1 if the restriction reverses the orientation. Let the **degree** $\deg g$ be the sum $d_y(g)$ of the signs of g-preimages of a regular value y.

8.4.1. (a) The degree of a PL map is well-defined, i.e., is independent of y.

(b) For any d there exists a PL map $g : S^n \to S^n$ of degree d.

Part (a) follows from Assertion 8.4.4. Part (b) is proved using the sum and the inverse element constructions (§14.4) or follows from Assertion 8.5.2 (b).

By the result of Problem 8.2.2 (e) for every map $f\colon S^n \to S^n$ there exists a piecewise-linear map $g\colon S^n_{PL} \to S^n_{PL}$ that is homotopic to the map $\pi f \pi^{-1}$. Define **degree** $\deg f := \deg g$.[22]

8.4.2. The degree of a map is well-defined, i.e., is independent of g.

This follows from Assertions 8.3.7.c and 8.4.5.ab.

8.4.3. (a—g), $(2, 3, 4)$ Solve the analogue of Problem 8.3.5 for the degree.

8.4.4. Under the assumptions of Problem 8.3.6 every non-closed polygonal line joins either two points of the same sign in $g^{-1}(y_0)$ and in $g^{-1}(y_1)$, or two points of different signs in $g^{-1}(y_0)$, or two points of different signs in $g^{-1}(y_1)$.

Hint: state and prove the analogous assertion for a linear map from an $(n+1)$-simplex onto the n-simplex.

Alternatively, denote by $y_0 = z_1, z_2, \ldots, z_s = y_1$ consecutive vertices of l, and by $x_1 \ldots x_t$ consecutive vertices of a non-closed polygonal line which is a connected component of $g^{-1}(l)$. Define a_j by $g(x_j) = y_{a_j}$. Then $a_j = a_{j-1} + \operatorname{sgn}_g \Delta_j$, where Δ_j is the simplex containing $x_{j-1}x_j$.

8.4.5. (a) Under the assumptions of Assertion 8.3.7.a any non-closed polygonal line joins either two points of the same sign in $S^n_{PL} \times 0$ and in $S^n_{PL} \times 1$, or two points of different signs in $S^n_{PL} \times 0$, or two points of different signs in $S^n_{PL} \times 1$.

(b,d) State and prove analogues of Assertions 8.3.7.b,d for the degree.

8.4.6. In the case $n = 1$, the definition of the degree of a map given in this subsection is equivalent to the one given in § 3.8.

For generalizations of the notion of a degree, see for example § 8.10, § 14 and [Sk, § 9 'Homotopy classification of maps'].

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[22] Here is the definition using *smooth* approximation. (Proofs of statements omitted here can be found, e.g. in [Pr14, § 18.1].) Every map $f\colon S^n \to S^n$ is homotopic to some *smooth* map h. A point $y \in S^n$ is called a *regular value* of h if $\operatorname{rk} dh(x) = n$ for any point $x \in h^{-1}(y)$ (here $dh(x)$ is the derivative of h at x). There exists a regular value $y \in S^n$. The set $h^{-1}(y)$ is finite. Let $\operatorname{sgn}\det dh(x)$ be the *sign* of a preimage x of y. Let *degree* $\deg f$ be the sum of the signs of h-preimages of y.

8.5. Homotopy classification of maps to the sphere

For a subset $N \subset \mathbb{R}^m$ denote by $\pi^n(N)$ the set of all maps $N \to S^n$ up to homotopy. Note the difference between this set and the group $\pi_n(N)$ whose definition is more complicated, see §§ 14.1,14.4.

Theorem 8.5.1 (Hopf). *The degree* $\deg\colon \pi^n(S^n) \to \mathbb{Z}$ *is a 1–1 correspondence.*

A proof is sketched in parts (a—d) of the next problem. Cf. [Pr14, §18.3, §18.5].

8.5.2. We call a map $S^n \to S^n$ a *Pontryagin* map if there exist disjoint n-dimensional closed balls $D_1, \dots, D_{k+l} \subset S^n$ such that the set $S^n - D_1 - \dots - D_{k+l}$ is mapped to the point $(0, \dots, 0, -1)$, the centres of the balls are mapped to the point $(0, \dots, 0, 1)$, and the radii of each ball are mapped bijectively to the meridians of the sphere.

A Pontryagin map is called a (k, l)-*Pontryagin map* if the map is orientation-preserving on k balls and orientation-reversing on l balls.

(a) For every d there exists a Pontryagin map of degree d.

(b) Any map $S^n \to S^n$ is homotopic to a Pontryagin map.

(c) For any k, l, any two (k, l)-Pontryagin maps are homotopic.

(d) Any (k, l)-Pontryagin map is homotopic to some $(k + 1, l + 1)$-Pontryagin map. (The proof of part (b) shows that it is sufficient to prove that any (k, l)-Pontryagin map is homotopic to a map that has a regular value with $(k + 1)$ preimages of sign $+1$ and $(l + 1)$ preimages of sign -1.)

8.5.3. * A *framed point set* in S^n is an unordered set of points in S^n, with a *framing*, i.e. with an n-tuple of linearly independent vectors tangent to S^n at each point of the set.

Two framed point sets in S^n are called *framed cobordant* if there exist

• a compact one-dimensional submanifold L in $S^n \times [0, 1]$ with boundary (the definition is similar to § 4.5), and such that $L \cap S^n \times 0$ and $L \cap S^n \times 1$ coincide with the framings of the first and the second sets respectively,

• an ordered set ξ of n linearly independent vector fields on L that are tangent to $S^n \times [0, 1]$, are normal to L, and whose restrictions to

$L \cap S^n \times 0$ and to $L \cap S^n \times 1$ coincide with the framings of the first and of the second set respectively.

Prove that the set $\pi^n(S^n)$ is in 1–1 correspondence with the set of framed point sets in S^n up to framed cobordism.

This correspondence and its generalizations are called the *Pontryagin correspondence.*

8.6. Higher-dimensional manifolds

Informally, an n-dimensional manifold is a shape whose every point has a small neighborhood homeomorphic to the n-dimensional ball. Rigorous definitions of n-dimensional *smooth* manifolds, their boundary, their being closed, and connected, are straightforward generalizations of the case $n = 2$ (§ 4.5). In this book manifolds are allowed to have non-empty boundary. We abbreviate 'smooth manifolds' to 'manifolds'. Examples of manifolds are spheres, balls, and their Cartesian products.

8.6.1. If $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are smooth submanifolds, then $M \times N \subset \mathbb{R}^m \times \mathbb{R}^n$ is a smooth submanifold.

Further examples appear naturally later in the book. The methods we study are so strong that they give beautiful non-trivial results on manifolds while requiring barely any knowledge of examples of manifolds (see e.g. § 8.7, § 9.1).

We assume that all manifolds are compact unless specified otherwise.

8.6.2. State and prove higher-dimensional versions of Problems 4.5.1 (a, b, c, d, e, f), 4.5.2 (a), 4.5.4 and 4.5.5.

Example 8.6.3 (Constructing manifolds by gluing; cf. § 2.1, Example 5.1.1.c**)**.
(a) *Projective space* $\mathbb{R}P^n$

• is obtained from the sphere S^n by gluing antipodal pairs of points, equivalently,

• is obtained from the disk D^n by gluing antipodal pairs of points on its boundary sphere, equivalently,

• $\mathbb{R}P^n := (\mathbb{R}^{n+1} - \{0\})/\!\sim$, where $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R} - \{0\}$.

We can think of this set as an n-submanifold in $\mathbb{R}^{(n+1)(n+2)/2}$ that is the image of the sphere S^n under the map

$$(x_1, \dots, x_{n+1}) \mapsto (x_k x_l)_{1 \leqslant k \leqslant l \leqslant n+1}.$$

Similar descriptions as submanifolds will be omitted in the following parts.

(b) '3-dimensional Möbius band' $D^2 \widetilde{\times} S^1$ is obtained from the 3-dimensional cylinder $D^2 \times I$ by gluing together the points $(x, 0)$ and $(\sigma(x), 1)$ for each $x \in D^2$. Here $\sigma\colon D^2 \to D^2$ is a reflection in a line.

(c) Generalizing the construction of $\mathbb{R}P^3$, define the *lens space* as

$$L(p, q) := S^3/(z_1, z_2) \sim (z_1 e^{2\pi i/p}, z_2 e^{2\pi iq/p})_{z_1, z_2 \in \mathbb{C}, |z_1|^2+|z_2|^2=1},$$

where p, q are two coprime positive integers. This space is obtained by gluing together faces of the union of two p-gonal pyramids that share the base. Each top face A is glued to the bottom face that is obtained from A via the composition of

• the rotation through $2\pi q/p$ around the line containing the vertices of the pyramids, and

• the reflection in the plane containing the base of the pyramids.

It is clear that $L(1, 1) = S^3$ and $L(2, 1) = \mathbb{R}P^3$.

8.6.4. The following sets of matrices are submanifolds of the set $\mathbb{R}^{n^2}$ of all matrices of size $n \times n$ for (a–c), of $\mathbb{C}^4$ for (d) and of $\mathbb{R}^4$ for (e):

(a) $GL(n, \mathbb{R}) = \{\text{real } n \times n\text{-matrices } A\colon\ \det A \neq 0\}$;

(b) $SL(n, \mathbb{R}) = \{\text{real } n \times n\text{-matrices } A\colon\ \det A = 1\}$;

(c) $SO(n) = \{\text{real } n \times n\text{-matrices } A\colon AA^T = E, \det A = 1\}$, where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

(d) $SU(2) = \{\text{complex } 2 \times 2\text{-matrices } A\colon A\overline{A}^T = E\}$;

(e) $SO(1, 1) = \{\text{real } 2 \times 2\text{-matrices } A\colon AIA^T = I\}$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Theorem 8.6.5 (Hopf). *For any closed connected n-manifold N there exists a 1–1 correspondence* $\deg\colon \pi^n(N) \to \mathbb{Z}$ *if N is orientable and* $\deg_2\colon \pi^n(N) \to \mathbb{Z}_2$ *otherwise.*

This is proved analogously to the Hopf Theorem 8.5.1.

8.6.6. Let V and W be smooth k- and l-submanifolds of $\mathbb{R}^n$ (or of a smooth n-manifold). They are (more precisely, the pair V, W is) called *transversal* if for any $z \in V \cap W$ there exists a closed neighborhood Oz of z in $\mathbb{R}^n$, and a diffeomorphism $\varphi : Oz \to [-1, 1]^n$ such that

$$\varphi(V \cap Oz) = [-1, 1]^k \times 0^{n-k} \quad \text{and} \quad \varphi(W \cap Oz) = 0^{n-l} \times [-1, 1]^l.$$

(a) If V and W are transversal, then $V \cap W$ is a smooth submanifold.

(b) Immersions $v : V \to \mathbb{R}^d$ and $w : W \to \mathbb{R}^d$ are (more precisely, the pair v, w is) called *transversal* if for any $x \in V$ and $y \in W$ such that $v(x) = w(y)$ there exist closed neighborhoods O of $v(x) = w(y)$ in $\mathbb{R}^d$, Ox of x in V, and Oy of y in W, and a diffeomorphism $\varphi : O \to [-1, 1]^d$ such that $v|_{Ox}$ and $w|_{Oy}$ are injective, and

$$\varphi(O \cap v(Ox)) = [-1, 1]^k \times 0^{d-k} \quad \text{and} \quad \varphi(O \cap w(Oy)) = 0^{d-l} \times [-1, 1]^l.$$

Is it correct that if $v, w : S^2 \to \mathbb{R}^3$ are transversal immersions, then $u^{-1}(v(S^2))$ is a 1-submanifold of S^2?

(c) If at every point of $V \cap W$ the sum of the tangent spaces to V and to W is $\mathbb{R}^n$, then $V \cap W$ is a smooth submanifold.

(d) Under the assumption of (c) V and W are transversal.

(e) Given three pairwise *tangent-transversal* submanifolds (in the sense of (c)), the sum of their normal spaces at any point of their triple intersection is the normal space to the triple intersection.

8.7. Vector fields on higher-dimensional manifolds

Definitions of tangent and normal vector fields on n-manifolds, as well as homotopy of vector fields, are straightforward generalizations of the case $n = 2$ (§§ 4.1, 4.10, 3.4).

8.7.1. Each of the following manifolds admits a non-vanishing tangent vector field:

(a) $S^1 \times S^1 \times S^1$; (b) $S^2 \times S^1$;

(c) Cartesian product of a sphere with handles with S^1; (d) $S^{2k-1} \times S^q$.

8.7.2 (cf. Assertion 4.6.1)**.** Any connected manifold with non-empty boundary admits a non-vanishing tangent vector field.

Theorem 8.7.3 (Hopf)**.** *(a) Any odd-dimensional manifold admits a non-vanishing tangent vector field.*

(b) No product of even-dimensional spheres admits a non-vanishing tangent vector field.

Parts (a) and (b) follow from Theorem 8.7.4 together with Assertions 8.8.2.df and 8.8.3.b, respectively.

Definitions of a *triangulation* and a *polyhedral decomposition* for n-manifolds are analogous to those for 2-manifolds given in § 4.5. A version of the Triangulation Theorem 4.6.4 holds for n-manifolds.

The *Euler characteristic of a polyhedral decomposition of a manifold* is defined as the alternating sum over k of the numbers of k-dimensional faces. The *Euler characteristic* $\chi(N)$ *of a manifold* N is defined as the Euler characteristic of some polyhedral decomposition of this manifold. As for 2-manifolds, a higher-dimensional version of Theorem 5.2.4.b and Assertion 10.4.3 (c) imply that the Euler characteristic is well-defined. It is important to note that there are effective methods to calculate the Euler characteristic (Assertions 8.8.3, 10.4.3, 10.4.5 and 10.6.10).

Theorem 8.7.4 (Hopf). *A closed connected manifold admits a non-vanishing tangent vector field if and only if its Euler characteristic is zero.*

Problems 8.8.1 and 8.8.2 guide you towards the proof of Theorem 8.7.4.

The products of the torus and of the Klein bottle respectively with the arc (or the circle) are not homeomorphic. This is proved using the following notion of *orientability*. An *orientation* of an n-dimensional vector space V over $\mathbb{R}$ can be defined as a non-degenerate multilinear antisymmetric form $V^n \to \mathbb{R}$. A manifold N is called *orientable* if we can choose orientations on all tangent spaces of N in such a way that the orientation on the tangent space at the point $x \in N$ depends continuously on x. Cf. § 9.4.

For example, any triangulation of the '3-dimensional Möbius band' $D^2 \widetilde{\times} S^1$ is non-orientable, but every triangulation of the manifolds D^3, S^3 and $\mathbb{R}P^3$ is orientable.

8.7.5. (a) The product of a triangulation of a 2-manifold with a segment (or with a circle) is orientable if and only if the triangulation of the 2-manifold is orientable.

(b) A triangulation of a 3-manifold is orientable if and only if it is not homeomorphic to any triangulation that contains some triangulation of the 3-manifold $D^2 \widetilde{\times} S^1$.

(c) For which p, q is the lens space $L(p, q)$ orientable?

8.7.6. (a) The manifold $\mathbb{R}P^n$ is orientable if and only if n is odd.

(b) The manifold $\mathbb{C}P^n$ is orientable for any n.

8.7.7. (a) An n-dimensional manifold in $\mathbb{R}^{n+1}$ admits a non-vanishing normal vector field if and only if the manifold is orientable. (This implies that, similar to the beginning of § 6.8, there are no closed non-orientable n-manifolds in $\mathbb{R}^{n+1}$.)

(b) For $m \geqslant 2n$, any n-manifold with non-empty boundary in $\mathbb{R}^m$ admits a non-vanishing normal vector field.

(c) For $m \geqslant 2n+1$, any n-manifold in $\mathbb{R}^m$ admits a non-vanishing normal vector field.

Theorem 8.7.8 (Normal Fields)**.** *For $m \geqslant 2n$ and for $m \leqslant n+2$, any closed orientable n-manifold in $\mathbb{R}^m$ admits a non-vanishing normal vector field.*

The cases $m = n+1$ and $m \geqslant 2n+1$ correspond to Assertions 8.7.7 (a, c). The cases $m = n+2$ and $m = 2n$ are similar to Theorem 4.10.3 (a) on normal fields, see Problem 8.9.1.

A non-vanishing normal vector field might not exist for $(m, n) = (4, 2)$ on closed non-orientable manifolds, or for $n+2 < m < 2n$, or for $n+2 = m$ for manifolds with boundary (see examples 4.10.3 (b, c, d) and 8.9.2 (a)). A complete answer to the following question of M. Hirsch is not known: *for which (m, n) does every n-manifold in $\mathbb{R}^m$ admit a non-vanishing normal vector field*?

In 1931 Hopf found a map $S^3 \to S^2$ that is not homotopic to the map to a point (see Assertions 8.10.6 (a, b); according to Assertion 8.10.6 (c) there exist infinitely many pairwise non-homotopic maps $S^3 \to S^2$). The existence of such a map is a surprise in view of Theorem 8.1.7 (a, b). This is one of the most important examples in topology.

For a manifold N, the set $V(N)$ is defined in the same way as in § 4.2.

Theorem 8.7.9 (Hopf—Pontryagin—Freudenthal, 1938)**.** *There exist 1–1 correspondences $V(S^3) \to \mathbb{Z}$ and $\pi^2(S^3) \to \mathbb{Z}$.*

These correspondences (Hopf invariants) will be constructed explicitly in § 8.10. They are group isomorphisms (with respect to the group operation on $V(S^3)$ given by multiplication of quaternions and the group operation on $\pi^2(S^3)$ defined in § 14.4). Explanations and proof are presented in § 8.10.

Theorem 8.7.10 (Vector Fields Classification)**.** *If a closed orientable n-manifold N satisfies the condition $V(N) \neq \varnothing$ (i. e. if $\chi(N) = 0$), then there exists a surjective map $D\colon V(N) \to H_1(N; \mathbb{Z})$.*

For $n = 2$, the map D is bijective.

For $n = 3$, for any $a \in H_1(N; \mathbb{Z})$ the number $|D^{-1}(a)|$ is a largest divisor of the class $[2a] \in H_1(N; \mathbb{Z})/T$, where T is the torsion subgroup.

For $n \geqslant 4$, every class has exactly two preimages under the map D.

For $n = 2$, this theorem is a folklore result from the early 20th century, see § 4.11. For $n = 3$, a simple proof of this theorem is presented, for example, in [CRS07]; for introductory problems see [Sk20, §8.9]. Theorem 8.7.10 is equivalent to the Pontryagin Theorem [Sk20, Theorem 8.9.5] by the Stiefel Theorem 9.1.3. For $n \geqslant 4$, Theorem 8.7.10 is apparently a folklore result from the middle of the 20th century [Ko81, Theorem 18.2]. Here we describe the foundations of the theory that is used to prove this result. For applications in Physics, see [MM95].

8.8. Existence of tangent vector fields

8.8.1. (a) A non-vanishing tangent vector field, defined on the vertices of a sufficiently fine triangulation of a 3-manifold, can be extended to the union of the edges.

(b) A non-vanishing tangent vector field, defined on the union of the edges of a sufficiently fine triangulation of a 3-manifold, can be extended to the union of the 2-dimensional faces.

(c) Any two non-vanishing vector fields on $S^1 \subset S^3$, tangent to S^3, are homotopic.

(d) Given a non-vanishing tangent vector field on S^3 and a homotopy of its restriction to $S^1 \subset S^3$, the homotopy can be extended to a homotopy of the whole vector field on S^3.

(c′) Any two non-vanishing vector fields, tangent to a 3-manifold, defined on the union of the edges of a sufficiently fine triangulation of the 3-manifold, are homotopic.

(d′) Suppose we have a non-vanishing tangent vector field on a 3-manifold and a homotopy of the restriction of the field to the union of the edges of a sufficiently fine triangulation of the 3-manifold. Then the homotopy can be extended to a homotopy of the whole vector field on the 3-manifold.

8.8.2. Let N be a closed n-manifold and let v be a non-vanishing tangent vector field, defined on the union of $(n-1)$-dimensional faces of a sufficiently fine triangulation of N.

(a) Construct an assignment $\varepsilon(v)$ of integers to the n-simplices of the triangulation that obstructs the extension of the field v to the whole manifold N.

(Hint: the construction is analogous to § 4.8. Alternatively, one can construct an assignment of integers to the vertices of the dual decomposition, see definition in § 9.7.)

(b) How is the assignment $\varepsilon(-v)$ obtained from the assignment $\varepsilon(v)$?

(c) The sum $e(N)$ of the numbers in the assignment $\varepsilon(v)$ does not depend on v.

(d) If n is odd then $e(N)=0$.

(e) For any k, map $f:D^k\to S^k$ and number $d\in\{+1,-1\}$ there exists a map $g:D^k\to S^k$ such that $g=f$ on ∂D^k and

$$\deg(f\cup(-g):S^k\to S^k)=d.$$

(f) If N is connected and $e(N)=0$ then N admits a non-vanishing tangent vector field.

(g) We have $e(N)=\chi(N)$.

8.8.3. (a) (Additivity) If M, N and $M\cup N$ are manifolds of dimension n and $M\cap N$ is a manifold of dimension $n-1$, then

$$\chi(M\cup N)=\chi(M)+\chi(N)-\chi(M\cap N).$$

(b) (Multiplicativity) If M and N are manifolds then

$$\chi(M\times N)=\chi(M)\chi(N).$$

Hint to 8.8.2. (d) The obstruction $e(-v)$ to the extension of the field $-v$ has the opposite sign to the obstruction $e(v)$. On the other hand, $e(-v)=e(v)$.

Hint to 8.8.3. (b) For closed manifolds M,N we sketch a proof using the general position similar to § 4.7. A tangent vector field on $M\times N$ is in general position if both ‘projections’ onto the factors are in general position. Let u and v be tangent fields in general position on M and on N respectively. Then $u+v$ is a tangent vector field in general position on $M\times N$. Define the subsets $e(u)\subset M$, $e(v)\subset N$ and $e(u+v)\subset M\times N$ consisting of finitely many points with signs as in § 4.7. Then $e(u+v)=e(u)\times e(v)$. Adding up the signs of the points we obtain $\chi(M\times N)=\chi(M)\chi(N)$.

8.9. Existence of normal vector fields

8.9.1. (a) Given a closed orientable n-manifold $N \subset \mathbb{R}^m$, construct a group $H_{2n-m}(N;\mathbb{Z})$ and an obstruction $\overline{e}(N) \in H_{2n-m}(N;\mathbb{Z})$ (*normal Euler class*) for the existence of a non-vanishing normal vector field on N.

(b, c, d, e) State and prove higher-dimensional versions of Assertions 4.10.4 (b, c, d, e).

Part (a) is similar to Problem 4.10.4.a. In (b) for $m \geqslant 2n$ the proof is similar to Assertion 4.10.4 (b). In (b) for $m = n+2$ use Assertion 8.1.7 (b).

8.9.2. (a) There exists an orientable 3-manifold with boundary in $\mathbb{R}^5$ that does not admit a non-vanishing normal vector field.

(b)* There exists a closed orientable 4-manifold in $\mathbb{R}^7$ that does not admit a non-vanishing normal vector field.

An example to (a) is the composition $S^2 \times D^1 \to S^2 \times D^3 \to \mathbb{R}^5$ of the embedding given by the formula $(x,t) \to (x,tx)$ and the standard embedding. A normal field to such an embedding can be used to construct a tangent field on S^2.

A proof of (b) is better postponed till after studying § 16.4.

8.9.3. (a; cf. Problem 8.9.1 (a)) Given an orientable n-manifold $N \subset \mathbb{R}^m$ with non-empty boundary, construct group $H_{2n-m}(N,\partial;\mathbb{Z})$ and an obstruction $\overline{e}(N) \in H_{2n-m}(N,\partial;\mathbb{Z})$ (*normal Euler class*) to the existence of a non-vanishing normal vector field on N.

(b) Prove the completeness of this obstruction for $m = 2n-1$.

In the following parts we assume that $H_{2n-m}(N,\partial;\mathbb{Z})$ does not contain any elements of order 2.

(c) If $m-n$ is odd then $\overline{e}(N) = 0$.

(d) If $m = 2n-1$ and n is even, then N admits a non-vanishing normal vector field.

(e)* If $m = 2n-1 = 5$, then $\overline{e}(N)$ is even. (The proof is better postponed till after studying § 12.)

8.9.4. Any two normal fields on a 2-manifold in $\mathbb{R}^m$ are homotopic if $m \geqslant 6$.

This is proved using Assertion 8.1.7 (a) for $k = 1, 2$.

8.9.5. For a submanifold $N \subset \mathbb{R}^m$ we denote by $V(N \subset \mathbb{R}^m)$ the set of non-vanishing normal vector fields on N up to homotopy within the class of non-vanishing normal vector fields. Describe $V(N \subset \mathbb{R}^4)$ and $V(N \subset \mathbb{R}^5)$ for

(a) $N = S^2$; (b) Möbius band N; (c) Klein bottle N.

The answer in part (c) depends on the embedding into $\mathbb{R}^4$; in other parts of this problem the answer does not depend on the choice of the embedding into $\mathbb{R}^4$ or into $\mathbb{R}^5$, though this is not obvious. For (a) the descriptions are equivalent to Theorem 3.1.9 (b) the Hopf Theorem 8.5.1. For (b, c) use Theorem 3.1.9 (a, b) and the Hopf Theorem 8.5.1. *Answers:* (a) 0 and $\mathbb{Z}$; (b) 0 for $\mathbb{R}^5$.

Hint to 8.9.3. (e) Any element of the group $H_2(N;\mathbb{Z})$ can be realized by some closed oriented 2-submanifold $F \subset N$, cf. § 14.9. This fact and the Poincaré Duality 10.8.1 (b) imply that it suffices to prove that $\overline{e}(f) \cap [F] \in \mathbb{Z}$ is even. This number is an obstruction to the construction of a field on F that is normal to $f(N)$. For the residue modulo 2 we have $\rho_2(\overline{e}(f) \cap [F]) + w_2(F) = 0$, analogously to the Whitney—Wu formula 12.6.3. This equation also follows from the Whitney—Wu formula 13.4.3 (b) and the relation $5\varepsilon_F = \tau_F \oplus \nu_{F\subset N} \oplus \nu_{N\subset\mathbb{R}^5}|_F$.

8.10. Vector fields on the 3-dimensional sphere

8.10.1. (a) Construct three linearly independent tangent vector fields on S^3.

(b) Construct a 1–1 correspondence $V(S^3) \to \pi^2(S^3)$.

For (a) you can give an explicit formula for the fields (for example, using the fact that S^3 is the group of unit quaternions). Part (b) follows by (a).

The definition of the linking number lk can be found, for example, in [Sk, § 4.3 'Linking number of closed polygonal lines in 3-space'].

8.10.2. (a) Split the complement to a line in the 3-dimensional space into a disjoint union of closed oriented curves such that the linking number of any two curves is $+1$.

(b) Construct a map $S^3 \to S^2$ such that the preimages of any two distinct points under this map are closed curves whose linking number is ± 1.

See visualization in [Ho] and the construction after 8.10.3.

Let

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{C} - \{0\}$.

8.10.3. (a) We have $\mathbb{C}P^1 \cong S^2$.

(This means that there exists a continuous map $f\colon \mathbb{C}^2 - \{0\} \to S^2$ such that $f(x) = f(y)$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C} - \{0\}$. The definition of being homeomorphic given before Problem 3.1.7 does not apply here since $\mathbb{C}P^1$ is not given as a subset of a Euclidean space.)

(b) We have $\mathbb{C}P^n \cong S^{2n+1}/\sim$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ and $x \sim y$ if $x = e^{i\varphi}y$ for some $\varphi \in \mathbb{R}$.

(This means that there exists a continuous map $f\colon \mathbb{C}^{n+1} - \{0\} \to S^{2n+1}$ such that $f(x) = f(y)$ if and only if $x = \mu y$ for some $\mu > 0$.)

(c) Represent $\mathbb{C}P^n$ as a subset of a Euclidean space.

Identify S^2 with $\mathbb{C}P^1$ (see Assertion 8.10.3.a). Represent S^3 as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2\colon |z_1|^2 + |z_2|^2 = 1\}.$$

Define the **Hopf map** $\eta\colon S^3 \to S^2$ by the formula

$$\eta(z_1, z_2) = (z_1 : z_2).$$

Cf. Assertion 8.10.3.b.

8.10.4. (a) For any $x \in S^2$ we have $\eta^{-1}x \cong S^1$.

(b) The preimages of the Hopf map are intersections of S^3 with complex lines $a_1z_1 + a_2z_2 = 0$, where $a_1, a_2 \in \mathbb{C}$.

(c) We have $\mathbb{C}P^2 \cong D^4/\sim$, where $x \sim y$ if $x, y \in S^3$ and $\eta(x) = \eta(y)$.

(This means that there exists a continuous map $f\colon S^5 \to D^4$ such that $x \sim y$ in the sense of Assertion 8.10.3.b if and only either $f(x) \notin S^3$ and $f(x) = f(y)$, or $f(x), f(y) \in S^3$ and $\eta(f(x)) = \eta(f(y))$.)

A subset $A \subset X \subset \mathbb{R}^m$ is called a **retract** of the set X if there exists a map $X \to A$ whose restriction to A is the identity map.

8.10.5. (a) The subset $\mathbb{R}P^1$ is not a retract of the set $\mathbb{R}P^2$.

(b) The subset $\mathbb{C}P^1$ is not a retract of the set $\mathbb{C}P^2$.

The proof of part (b) is based on the fact that the Hopf map is not homotopic to the map to a point. To prove this fact we need the following notion.

The **Hopf invariant** of a map $S^3 \to S^2$ is the linking number of the preimages of two general position points under a smooth (or piecewise-linear) approximation of this map. Let us give more details of this definition. Any map $f\colon S^3 \to S^2$ is homotopic to a PL map g, i.e. to a map simplicial for some triangulations of S^3 and S^2. Take two points $y_1, y_2 \in S^2$ in the interiors of 2-faces of the triangulation of S^2 (*regular values* of g). Then $g^{-1}y_i = S^1_{i1} \sqcup S^1_{i2} \sqcup \ldots \sqcup S^1_{ik_i}$ is a PL *link* (i. e. a set of closed pairwise disjoint closed polygonal lines without self-intersections) for $i = 1, 2$. [23] The orientations of S^2 and S^3 define orientations on these curves. Define the *Hopf invariant* to be

$$H(f) := \sum_{i=1,j=1}^{k_1,k_2} \mathrm{lk}(S^1_{1i}, S^1_{2j}).$$

8.10.6. (a) The Hopf invariant is well-defined, i.e. is independent of y_1, of y_2, and of g.

(b) We have $H(\eta) = 1$.

(c) For any n there exists a map $S^3 \to S^2$ whose Hopf invariant is n.

(d) The Hopf invariant of f does not change under homotopy of f.

Parts (a,d) are proved similarly to Assertions 8.3.3, 8.3.4, 8.4.1, 8.4.2. Parts (b) and (c) are easily proved assuming (a).

To prove Hopf– Pontryagin– Freudenthal Theorem (Theorem 8.7.9) it remains to show that the Hopf invariant is injective. Problem 8.10.7 sketches a proof of Theorem 8.7.9 that generalizes the method of coverings used in § 3.9 (although this does not explicitly mentioned the Hopf invariant, this proves its injectivity). A different proof of injectivity is sketched in Problems 8.11.1 and 8.11.2.

8.10.7. For a subset $X \subset \mathbb{R}^m$, a map $\widetilde{f}\colon X \to S^3$ is called *a lift* of a map $f\colon X \to S^2$ if $f = \eta \circ \widetilde{f}$.

(a) The map $\eta_*\colon \pi^3(S^3) \to \pi^2(S^3)$, defined by taking the composition with the Hopf map, is well-defined.

[23] Here are some details for a smooth approximation. The proofs can be found, for example, in [Pr14, §18.4]. Any map $f\colon S^3 \to S^2$ is homotopic to a *smooth* map g. A point $y \in S^2$ is a *regular value* of g if $\mathrm{rk}\, dg(x) = 2$ for any point $x \in g^{-1}y$. There are regular values $y_1, y_2 \in S^2$. Then $g^{-1}y_i = S^1_{i1} \sqcup S^1_{i2} \sqcup \ldots \sqcup S^1_{ik_i}$ is a smooth *link*.

(b) **Local Triviality Lemma.** *For any point $x \in S^2$ there exists a homeomorphism*

$$h\colon \eta^{-1}(S^2 - \{x\}) \to (S^2 - \{x\}) \times S^1 \quad \text{such that} \quad \mathrm{pr}_1 \circ h = \eta.$$

(c) **Path Lifting Property.** *Any path $s\colon [0,1] \to S^2$ has a lift $\widetilde{s}\colon [0,1] \to S^3$.*

(d) Any map $D^3 \to S^2$ has a lift $D^3 \to S^3$.

(e) Any map $S^3 \to S^2$ is homotopic to a map that has a lift (i. e. the map η_* is surjective).

(f) **Homotopy Lifting Property.** *For any map $F_0\colon D^3 \to S^3$ and any homotopy $f_t\colon D^3 \to S^2$ of the map $f_0 = \eta \circ F_0$ there exists a homotopy $F_t\colon D^3 \to S^3$ of the map F_0 such that $f_t = \eta \circ F_t$.*

(g) If the compositions $S^3 \to S^2$ of maps $S^3 \to S^3$ with the Hopf map are homotopic then the maps $S^3 \to S^3$ themselves are homotopic (i. e. the map η_* is injective).

8.10.8. Any map $S^3 \to \mathbb{C}P^2$ is homotopic to the map to a point.

This is proved using the analogue $S^5 \to \mathbb{C}P^2$ of the Hopf map (see the details in Problem 14.5.4).

Hint to 8.10.7. (c, f) The statements follow from part (b) in the same way as in Problems 3.9.2 (a, a′, b), cf. § 14.2.

(e) Regard this map $\varphi\colon S^3 \to S^2$ as a map $D^3_+ \to S^2$ taking ∂D^3_+ to 1. Part (d) implies that there exists a lift $\widetilde{\varphi}_+\colon D^3_+ \to S^3$ of the latter map. We have $\widetilde{\varphi}_+(\partial D^3_+) \subset \eta^{-1}(1) = S^1$, hence the restriction $\widetilde{\varphi}_+|_{\partial D^3_+}$ can be extended to a map $\widetilde{\varphi}_-\colon D^3_- \to \eta^{-1}(1)$. The maps $\widetilde{\varphi}_+$ and $\widetilde{\varphi}_-$ combine to a map $\widetilde{\varphi}\colon S^3 \to S^3$. The map $\eta \circ \widetilde{\varphi}$ has a lift and is homotopic to φ.

(g) Similar to part (e). Prove and use the following statement: *Any homotopy $S^3 \times I \to S^2$ between maps that have lifts is homotopic, while keeping them unchanged on $S^3 \times \{0,1\}$, to a homotopy that has a lift $S^3 \times I \to S^2$.*

8.11. Framed links

8.11.1. (a) There exists an oriented 2-submanifold with boundary in $\mathbb{R}^3$ whose boundary is the *trefoil knot* [Sk20u, § 1].

(b) The same as (a) but for the *Hopf link* instead of the trefoil knot [Sk20u, § 2].

(c) The same as (a) but for an arbitrary oriented link (i. e. a set of closed pairwise disjoint smooth curves without self-intersections) instead of the trefoil knot, and oriented boundary instead of boundary.

(d) For any two oriented knots (i. e. closed smooth curves without self-intersections) in $\mathbb{R}^3$ there exists a (compact) 2-submanifold $L\subset\mathbb{R}^3\times I$ with boundary, meeting $\mathbb{R}^3\times\{0,1\}\supset\partial L$ orthogonally and such that $L\cap\mathbb{R}^3\times 0$ and $L\cap\mathbb{R}^3\times 1$ coincide with the first and the second knot respectively.

(e) The same for any two oriented links in $\mathbb{R}^3$ (which might have different numbers of components).

8.11.2. A *framed link in* S^3 is an oriented link in S^3 equipped with a normal field. Two framed links in S^3 are *framed cobordant* if there exist

• a (compact) 2-submanifold $L\subset S^3\times I$ with boundary, and such that $L\cap S^3\times 0$ and $L\cap S^3\times 1$ coincide with the first and the second link respectively,

• a normal to L field ξ whose restrictions to $L\cap S^3\times 0$ and to $L\cap S^3\times 1$ coincide with the vector field on the first and on the second link respectively.

Denote by $\Omega^1_{fr}(3)$ the set of all framed links in S^3 up to framed cobordism.

(a) Define the map $H\colon \Omega^1_{fr}(3)\to\mathbb{Z}$ by setting $H(l)$ to be the linking number of a framed link l and the image of this link under the translation along the normal field. This map is also called the Hopf invariant. Prove that this map is well-defined.

(In this part, state and use without proof the smooth version of [Sk, the Triviality Lemma 4.7.1].)

(b) Define the map $J\colon\mathbb{Z}\to\Omega^1_{fr}(3)$ by setting $J(n)$ to be the class of the standard circle equipped with a normal field 'winding around it' n times. Prove that $H(J(n))=n$.

(c) The Hopf invariant is injective. (Hence the maps H and J are mutually inverse bijections.)

(d) The set $\pi^2(S^3)$ is in 1–1 correspondence with the set $\Omega^1_{fr}(3)$. (Cf. Problem 8.5.2.)

8.11.3. (a) Define the Hopf invariant $H\colon \pi_5(S^3)\to\mathbb{Z}$.

(b) This invariant is zero.

§ 9. Collections of vector fields

'You mean...' he would say, and then he would rephrase what I had said in some completely simple and concrete way, which sometimes illuminated it enormously, and sometimes made nonsense of it completely.

I. Murdoch. Under the Net

9.1. Introduction and Main Results

Definitions of (smooth) manifolds, their being closed, orientability, boundary, triangulation, tangent vector fields on them are analogous to the two-dimensional case (§§4.5, 4.10, 8.6, 8.7).

Eduard Stiefel, a student of Heinz Hopf, considered the problem of existence of a pair, a triple, etc. of linearly independent tangent vector fields on a manifold. Through developing Hopf's ideas, around 1934 Stiefel came up with the definition of characteristic classes. It is interesting that Stiefel started with the 3-dimensional case, and tried to construct an orientable 3-manifold with no triple of linearly independent tangent vector fields. Formalization was completed by Norman Steenrod in 1940-s. This new theory was invented to prove Theorems 9.1.3, 9.1.4, 9.1.5 below, Whitney non-embedding theorems and Pontryagin—Thom non-cobordance theorems (stated in § 12, § 16), and many other results (see e.g. Propositions in this section, and §§ 9, 12, 13, 16).

An n-manifold N is said to be **parallelizable** if there is a family of n tangent vector fields on N linearly independent at every point of N. Например, окружность, тор, S^3 и $\mathbb{R}P^3$ параллелизуемы (Assertion 8.10.1.a), а любое неориентируемое многообразие, S_g при $g \neq 1$ и S^{2k} не параллелизуемы (по теоремам Эйлера-Пуанкаре 4.6.2 и Хопфа 8.7.3.b). Recall that S_g denotes the sphere with g handles.

9.1.1. The products $S_g \times I$ and $S_g \times S^1$ are parallelizable for any g.

For $S_g \times I$ this follows because $S_g \times I$ embeds into $\mathbb{R}^3$.

9.1.2. (a) Any collection of $n-1$ linearly independent tangent vector fields on an orientable n-manifold can be extended to a collection of n such fields.

(b) If there exists a collection of k linearly independent tangent vector fields on a manifold, then there exists a collection of k orthonormal tangent vector fields.

Hint: this follows since the Gram—Schmidt orthogonalization process is 'canonical'.

(c) Any n-submanifold of a parallelizable n-manifold (e.g. of $\mathbb{R}^n$) is parallelizable.

Theorem 9.1.3 (Stiefel)**.** *Every orientable* 3*-manifold is parallelizable.*

For generalizations see Theorems 9.1.9, 9.8.3 (a) and 12.6.1.

A manifold N is said to be k**-parallelizable** if there is a family of k tangent vector fields on N linearly independent at every point of N.

Theorem 9.1.4. *If $n+1=2^r m$ for some odd m, then $\mathbb{R}P^n$ is not 2^r-parallelizable.*

Theorem 9.1.5 (division algebras)**.** *If $\mathbb{R}^n$ has a structure of division algebra, then n is a power of 2.*

More precisely, $\mathbb{R}^n$ *has a structure of division algebra only for* $n=1, 2, 4, 8$. Moreover, S^n *is parallelizable only for* $n=0, 1, 3, 7$. These famous theorems of Bott—Milnor—Kervaire (see references in [MS74, § 4]) are also proved using topological methods (but more advanced) [Hi95].

Theorems 9.1.4 and 9.1.5 can easily be obtained from the Obstruction Theorem 9.9.1.a and Assertions 9.9.6 (a, b). (The Hopf proof [Hi95], which did not use characteristic classes, was obtained at the same time as the Stiefel proof, which did use them).

Proposition 9.1.6. *For any closed connected* 2*-manifold F the following conditions are equivalent*:

- *$F\times S^1$ is 2-parallelizable*;
- *$F\times I$ is 2-parallelizable*;
- *F has even Euler characteristics.*

This could be proved directly analogously to Proposition 9.1.10 below. Alternatively, this follows from the Obstruction Lemma 9.5.1 (completeness) and Assertions 9.7.4 (a), 9.3.5.bc.

Some implications of this and the following propositions are trivial without characteristic classes.

Proposition 9.1.7. *Let M be a closed 3-manifold.*

(a) *The manifold $M \times S^1$ is 2-parallelizable.*

(b) *The manifold $M \times S^1$ is 3-parallelizable if and only if M is 2-parallelizable.*

(c) *The manifold $M \times S^1$ is parallelizable if and only if M is orientable.*

The 'only if' part of (b) and the 'if' part of (c) follow from Proposition 9.8.5 (a) and the Stiefel Theorem 9.1.3, respectively.

A manifold is **almost parallelizable (almost k-parallelizable)** if its complement to a point is parallelizable (k-parallelizable).

9.1.8. (a) The connected sum of almost parallelizable manifolds is almost parallelizable.

(b) The manifold $\mathbb{R}P^4$ is almost 3-parallelizable.

(b') The manifold $\mathbb{C}P^2$ is not almost 3-parallelizable.

(b") The manifold $\mathbb{C}P^2$ is almost 2-parallelizable.

(c) The product of two Moebius bands is not 3-parallelizable.

(d) The product of the Moebius band and a closed 2-manifold of odd Euler characteristics is not 2-parallelizable.

Proposition 9.1.9. *Any orientable 4-manifold is almost 2-parallelizable.*

This follows by Assertions 9.8.7 (d) and 9.8.10 (b, c).

Proposition 9.1.10. *Let F and F' be closed connected 2-manifolds.*

(a) The manifold $F \times F'$ is almost 3-parallelizable if and only if one of F, F' is orientable, and the other has even Euler characteristics.

(b) If $F \times F'$ is almost 2-parallelizable, then either some of F, F' is orientable, or both have even Euler characteristics.

The 'only if' part of (a) follows by Assertion 9.1.8.c, Proposition 9.1.6 and Assertions 9.8.4. The 'if' part of (a) follows by Assertion 9.8.4.a because $(F \times I) \times (F' \times I)$ is 5-parallelizable by Assertion 9.1.1 and Proposition 9.1.6. Part (b) follows by Assertion 9.1.8.d.

It would be interesting to know if the converse to (b) holds.

For an n-manifold N denote by N_0 the complement (in N) to the interior of some n-dimensional ball in N. We abbreviate 'a k-tuple of tangent to A (normal to B) vector fields' to 'a k-tuple tangent to A

(normal to B)'. Unless explicitly written otherwise, we assume that a k-tuple is orthonormal.

Hint to 9.1.1 for $S_g \times S^1$. *First proof.* Since S_g is orientable, there exists a (non-vanishing) vector field $n = (n_1, n_2, n_3)$ on $S_g \subset \mathbb{R}^3$ normal to S_g. Then the following three vector fields on S_g are tangent to S_g (but, possibly, vanishing):

$$u_1 = (0, n_3, -n_2), \quad u_2 = (-n_3, 0, n_1), \quad u_3 = (n_2, -n_1, 0).$$

Let $v = (v_1, v_2)$ be a (non-vanishing) tangent vector field on $S^1 \subset \mathbb{R}^2$. We define three vector fields on $S_g \times S^1 \subset \mathbb{R}^5$ by the formula $w_j = (u_j, n_j v)$. It is clear that each of these fields is continuous and tangent to $S_g \times S^1$.

Let W be the (3×5)-matrix whose rows are the vectors w_1, w_2, w_3. Then the first three columns, i. e. the vectors u_1, u_2, u_3, span the orthogonal complement of n in $\mathbb{R}^3$. At least one out of the fourth and the fifth columns, i. e. one of the vectors $v_j n$, $v_2 n$, is a non-zero multiple of the vector n. Hence the columns of the matrix W span $\mathbb{R}^3$. Therefore $\operatorname{rk} W = 3$. It follows that the rows of the matrix W, i. e. the vectors w_1, w_2, w_3, are linearly independent.

Second proof. The manifold $S_{g,0}$ admits a pair tangent to $S_{g,0}$. Hence $S_{g,0} \times S^1$ admits a triple tangent to $S_{g,0} \times S^1$. Assertion 9.3.4.b implies that this triple extends to $S_{g,0} \times S^1 \cup S_g \times D^1_+$. Then by Assertion 9.3.3 the triple extends to $S_g \times S^1$.

Hint to 9.1.6. The homology class $[* \times S^1] \in H_1(F \times S^1)$ is non-zero since its *intersection* with the class $[F \times *] \in H_2(F \times S^1)$ is non-zero. (So it is not necessary to compute the group $H_1(F \times S^1)$!)

Hint to 9.1.8. (b') Define the obstructions (using Assertion 9.8.1.d)

- $w_2 \in \mathbb{Z}_2$ to the existence of a triple on $\mathbb{C}P^1$ tangent to $\mathbb{C}P^2$;
- $w_2^S \in \mathbb{Z}_2$ to the existence of a quadruple on $\mathbb{C}P^1 \times I$ tangent to $\mathbb{C}P^2 \times I$;
- $e \in \mathbb{Z}$ to the existence of a field on $\mathbb{C}P^1$ normal to $\mathbb{C}P^1$, and tangent to $\mathbb{C}P^2$;
- $e^S \in \mathbb{Z}$ to the existence of a field on $\mathbb{C}P^1 \times I$ normal to $\mathbb{C}P^1 \times I$, and tangent to $\mathbb{C}P^2 \times I$.

Then $w_2 = w_2^S \overset{(*)}{=} \rho_2 e^S = \rho_2 e = [\mathbb{C}P^1] \cap [\mathbb{C}P^1] = 1 \neq 0$. Here (*) holds because $\mathbb{C}P^1 \times I$ is parallelizable.

(b”) This holds by Assertion 9.8.6.b.

(c) Denote by S, S' the middle circles of the Moebius bands F, F'. Let us prove that no neighborhood of $S \times S'$ in $F \times F'$ is 3-parallelizable. Analogously to Problem 9.3.5.a define an obstruction $w_2(F \times F')|_{S\times S'} \in \mathbb{Z}_2$ to such 3-parallelizability. In the following two paragraphs we show that this obstruction is non-zero.

Take a non-zero vector field u on S tangent to S. Take an arc $I \subset F$ such that cutting F along I gives a square. Take a vector field v on S tangent to F normal to S, and such that $v = 0$ only at the point $S \cap I$. Take analogous pair (u', v') on F'. Take the triple $(u, u', v + v')$ on $S \times S'$ (linearly dependent at some points and) tangent to $F \times F'$.

This triple is linearly dependent only at those points where $v = v' = 0$, i.e. only at the point $(S \cap I) \times (S' \cap I')$. If we go around this point on the torus $S \times S'$, the triple makes a homotopy non-trivial loop in SO_4 (because the pair (u, u') ‘does not change’, while the vector $v + v'$ makes one turn in $SO_2 \cong S^1$; see Assertions 9.3.2.de and 9.8.1.e).

(d) Denote by F the 2-manifold. Denote by S' the middle circle of the Moebius band F'. Let us prove that no neighborhood of $F \times S'$ in $F \times F'$ is 2-parallelizable. Analogously to Problem 9.3.5.a define an obstruction $w_3(F \times F')|_{F\times S'} \in \mathbb{Z}_2$ to such 2-parallelizability. In the following two paragraphs we show that $w_3(F \times F')|_{F\times S'} = \rho_2\chi(F) \neq 0$.

Take an arc I', and a pair u', v' as in the hint to (c). Take a point $p \in F$ and a pair u, v on F as in the sketch of the proof of Assertion 9.3.6.c (in §9.3). Take the pair $(u + v', v + u')$ (linearly dependent at some points and) tangent to $F \times F'$.

The pair u, v is linearly independent outside $p \cup \omega$. The pair u', v' is linearly independent outside the point $\omega' := S' \cap I'$. The pair $(u + v', v + u')$ is linearly independent on $\omega \times \omega'$. Hence the pair $(u + v', v + u')$ is linearly independent outside the point $p \times \omega'$. This point adds to the obstruction $w_3(F \times F')|_{F\times S'}$ the residue $\rho_2\chi(F)$.

9.2. Parallelizability on a two-dimensional submanifold

A *PL k-submanifold* of a smooth manifold N is the collection of some faces of some triangulation of N, which collection is PL homeomorphic to some PL k-manifold.

Lemma 9.2.1 (Submanifold). *Any closed PL 2-submanifold of an orientable 3-manifold N has a parallelizable neighborhood in N.*

This lemma follows from the Stiefel Theorem but is used in its proof. We present two independent proofs of this lemma: a geometric one in this section (using the idea of [Ki89] and a suggestion of I. Zhiltsov) and an algebraic one in § 9.3.

In general, the more complicated the situation, the more pronounced is the advantage of algebraic methods over geometric ones. So sometimes it is easier to invent a geometric idea, but translate it into algebraic language instead of turning it directly into a proof.

Homological ideas are used both in the algebraic proof of the Submanifold Lemma and in reduction of the Stiefel Theorem 9.1.3 to this lemma. The reduction is based on the exhaustion of a 3-manifold with neighborhoods of 2-manifolds in it, see § 9.7.

9.2.2. (a) There exists an orientable 3-manifold with boundary that contains a PL submanifold PL homeomorphic to the Klein bottle (or, equivalently, contains a closed connected non-orientable PL 2-submanifold of Euler characteristic zero).

(b) One of such 3-manifolds admits a triple of fields.

(c) Same as (a) for $\mathbb{R}P^2$ instead of the Klein bottle.

9.2.3. (a) Any orientable 2-manifold is PL homeomorphic to a PL submanifold of $\mathbb{R}^3$.

(b) Any non-orientable 2-manifold is PL homeomorphic to a PL submanifold of the connected sum of several $\mathbb{R}P^3$'s.

9.2.4. (a) There are a 3-manifold M, and PL homeomorphic closed PL 2-submanifolds of M that have no homeomorphic neighborhoods (one neighborhood is orientable, the other is not).

(b) If a PL 2-submanifold F of a 3-manifold is PL homeomorphic to S^2, then some neighborhood of F is PL homeomorphic to $F \times [0, 1]$.

(c) Every closed orientable PL 2-submanifold F of an orientable 3-manifold has a neighborhood PL homeomorphic to $F \times [0, 1]$.

The definition of 3-manifolds being diffeomorphic is analogous to the one introduced at the end of § 4.5.

Lemma 9.2.5. *Each PL homeomorphic closed PL 2-submanifolds F_1, F_2 of orientable 3-manifolds M_1, M_2 have diffeomorphic neighborhoods.*

Proof of Submanifold Lemma 9.2.1. By Assertions 9.2.3 (a, b) and Lemma 9.2.5 some neighborhood of F in N is diffeomorphic to some neighborhood of some PL submanifold PL homeomorphic to F

- in $\mathbb{R}^3$, if F is orientable;
- in the connected sum of several $\mathbb{R}P^3$, if F is non-orientable.

Now the lemma follows from Assertions 9.1.2.c and 9.1.8.a. □

Let us generalize the definition of the product of a 2-manifold with a segment. Cut a closed 2-manifold F along a union S of disjoint closed curves on F. We obtain a 2-manifold F' with boundary and with a fixed point free involution $\sigma\colon \partial F' \to \partial F'$. The *thickening* $F \mathbin{\widetilde{\times}_S} D^1$ of the 2-manifold F is the 3-manifold obtained from $F' \times D^1$ by gluing together points (x, t) and $(\sigma(x), -t)$ for every $x \in \partial F'$ and $t \in D^1$:

$$F \mathbin{\widetilde{\times}_S} D^1 := F' \times D^1/(x,t) \sim (\sigma(x), -t)_{x\in\partial F', t\in D^1}.$$

Here we use the construction of 3-manifolds by gluing (see Remark 8.6.3; cf. § 13.2). E.g. some neighborhood of F in an (orientable) 3-manifold is diffeomorphic to some (orientable) thickening of F.

9.2.6. (a) If $S = \varnothing$ then $F \mathbin{\widetilde{\times}_S} D^1 = F \times D^1$.

(b) The thickening $F \mathbin{\widetilde{\times}_S} D^1$ is orientable if and only if there is an orientation on $F - S$ that changes every time we cross a curve from S (i. e. if $[S] = w_1(F)$, see § 6.4).

(c) Two thickenings of a 2-manifold for *homologous* unions of curves (see § 6.3) are diffeomorphic.

Hint to 9.2.2. (a) Here is an orientable 3-manifold containing a copy of the Klein bottle:

$$\frac{S^1 \times [-1,1] \times [0,1]}{(x,y,t,0) \sim (x,-y,-t,1)} \supset \frac{S^1 \times 0 \times [0,1]}{(x,y,0,0) \sim (x,-y,0,1)}.$$

Another description of this construction. Take an embedding of the Klein bottle into $\mathbb{R}^4$ (see Fig. 2.1.6 (b)). Take the projection $\mathbb{R}^4 \to \mathbb{R}^3 \times 0$. On the image under projection take a normal vector field parallel to the fourth coordinate. Take a normal field of (undirected) segments that are perpendicular to the above field, and that intersect the Klein bottle at their interior points. These segments form the required 3-manifold.

(b) The projection of the 3-manifold constructed in part (a) to $\mathbb{R}^3 \times 0$ is locally 1–1. Hence a triple of fields with the required properties is obtained from the triple of orthonormal fields on $\mathbb{R}^3$. Cf. Problem 9.1.2.c.

(c) Take a neighborhood of $\mathbb{R}P^2$ in $\mathbb{R}P^3$.

9.3. Another proof of Submanifold Lemma 9.2.1

Recall that $SO_3 \subset \mathbb{R}^9$ denotes the space of positively oriented orthonormal frames in $\mathbb{R}^3$.

9.3.1. The space SO_3 is connected.

9.3.2. (a) Any orientation-preserving isometry of $\mathbb{R}^3$ that fixes the origin is a rotation around a line passing through the origin.

(b) The space SO_3 is homeomorphic to (see definitions in § 3.1)

- the space of rotations of $\mathbb{R}^3$ around the lines containing the origin;
- the closed 3-dimensional ball with identified antipodal points on its boundary (cf. Remark 8.6.3.a).

(c) There are exactly two homotopy classes of maps from S^1 to $\mathbb{R}P^2$. The non-trivial class is presented by the diameter of a disk from which $\mathbb{R}P^2$ is obtained via gluing its boundary to itself by the antipodal map.

(d) There are exactly two homotopy classes of maps from S^1 to SO_3. The non-trivial class is presented by the diameter of a 3-dimensional ball from which SO_3 is obtained via gluing.

(e) Consider the composition $S^1 \xrightarrow{f} S^1 \xrightarrow{h} SO_2 \xrightarrow{\text{in}} SO_3$ of an arbitrary map f, homeomorphism $h(e^{i\varphi}) := \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ and the standard embedding in. This composition is homotopic to the constant map if and only if $\deg f$ is even.

(f) Any map $S^2 \to SO_3$ extends to D^3.

Part (c) is reduced to Theorem 3.1.9 (a) analogously to Problems 3.9.2 (a, a$'$, b) (cf. Problem 14.2.1). Part (d) is analogous to parts (c, f). Part (f) is reduced to Problem 8.1.7 (a) for $k = n - 1 = 2$ analogously to Problems 3.9.2 (a, a$'$, b), constructing a 2:1 map $S^3 \to SO_3$ using the last of the 'models' for the space SO_3 listed in (b).

9.3.3. If a 3-manifold is almost parallelizable, then it is parallelizable. (The converse is trivial.)

This follows from Assertion 9.3.2 (f).

9.3.4. (a) On any closed orientable 2-submanifold of a 3-manifold N there is a pair tangent to N.

(b) Let F be a closed orientable 2-manifold. Any triple on $F_0 \times 0 \subset F \times I$ tangent to $F \times I$ extends to a triple on $F \times 0$ tangent to $F \times I$.

Hint to (b). The obstruction to the required extension equals $w_2(F \times I)$ (see Assertion 9.3.5). The obstruction is complete. By Assertion 9.1.1 for $S_g \times I$, the product $F \times I$ is parallelizable. Hence $w_2(F \times I) = 0$.

9.3.5. Let F be a closed connected 2-manifold.

(a) Construct an obstruction $w_2(F \times I) \in \mathbb{Z}_2$ to the existence of a pair on $F \times 0$ tangent to $F \times I$.

(b) If $w_2(F \times I) = 0$, then such a pair exists (such an obstruction is said to be *complete*).

(c) We have $w_2(F \times I) = \rho_2 \chi(F)$.

In (a) the construction is analogous to § 4.7—4.11 (as well as to the Obstruction Lemma 9.5.1). Use either general position or a triangulation. For (a,b,c) apply Assertions 9.3.2 (d, e).

Sketch of the construction of (a). (Этот текст получен редактированием текста А. Мирошникова.) Возьмем триангуляцию T многообразия $F \times 0$ настолько мелкую, что

($*$) для любой ее грани касательные к $F \times I$ пространства в любых двух точках этой грани не ортогональны.

На ее 1-остове возьмем произвольную пару (u, v), касательную к $F \times I$. Возьмем грань f триангуляции T, ее ориентацию и точку p в ней. Спроецируем на касательное пространство в точке p грани f сужение пары (u, v) на границу ∂f. Ввиду свойства ($*$) при такой проекции линейно независимые пары векторов переходят в линейно независимые. Поэтому при обходе границы по направлению ориентации пара u, v дает отображение $S^1 \to SO_3$. Его гомотопический класс $\varepsilon_{(u,v)}(f) \in \pi_1(SO_3) \cong \mathbb{Z}_2$ не зависит от выборов точки p и ориентации грани f.

Определим $w_2(F \times I) := \sum_{f \in T} \varepsilon_{(u,v)}(f) \in \pi_1(SO_3) \cong \mathbb{Z}_2$. Можно проверить, что препятствие $w_2(F \times I)$ не зависит от выборов пары (u, v) и триангуляции T.

9.3.6. Let F be a closed connected 2-submanifold of a 3-manifold N.

(a) Construct an obstruction $w_2(N)|_F \in \mathbb{Z}_2$ to the existence of a pair on F tangent to N (not to F!).

(b) If $w_2(N)|_F = 0$, then such a pair exists.
(c) If N is orientable, then $w_2(N)|_F = \rho_2\chi(F) + w_1(F)^2$.

The Submanifold Lemma 9.2.1 follows by Assertion 9.3.6.b because $w_2(N)|_F = 0$ by Assertion 9.3.6.c and by the equality $\rho_2\chi(F) = w_1(F)^2$ of Assertion 6.7.3 (b).

Parts (b,c) show that for a closed connected 2-submanifold F of an orientable 3-manifold N, some neighborhood of F is parallelizable if and only if $\rho_2\chi(F) = w_1(F)^2$. So the latter equality follows from (and is an algebraic version of) embeddability of F into a parallelizable 3-manifold (§9.2). This clarifies the relation between the proofs in this and the previous sections.

Sketch of the proof of Assertion 9.3.6.c. Take a field u on F tangent to F, and such that $u \neq 0$ on the complement to some point $p \in F$, and if we go around p on F, the vector makes $\chi(F)$ turns in $SO_2 \cong S^1$. Take a 1-cycle $\omega \not\ni p$ in a triangulation of F representing the class $w_1(F)$. Take an orientation on $F - \omega$. Using this orientation we construct a vector field v on F tangent to F normal to u, and such that $v = 0$ only on $\{p\} \cup \omega$. Take analogous vector field v' on F tangent to F normal to u, and such that $v' = 0$ only on $\{p\} \cup \omega'$ for some 1-cycle $\omega' \not\ni p$ in the dual cell subdivision of F representing the class $w_1(F)$.

Since N is orientable, the cross product $u \times v'$ is defined, is tangent to N, and is normal to F. On F take the pair $(u, v + u \times v')$ tangent to N.

This pair is linearly dependent at exactly those points where either $u = 0$ or $v = v' = 0$. So the subset on which this pair is linearly dependent is $\{p\} \cup (\omega \cap \omega')$. This is a finite set. If we go on F around p, this pair makes the loop $\rho_2\chi(F)$ in SO_3; see Assertion 9.3.2.e. If we go on F around any point of $\omega \cap \omega'$, this pair makes a homotopy non-trivial loop in SO_3 (because u 'does not change', while the vector $v + u \times v'$ makes one turn in $SO_2 \cong S^1$; see Assertion 9.3.2.e). Hence $w_2(N)|_F = \rho_2\chi(F) + |\omega \cap \omega'|_2 = \rho_2\chi(F) + w_1(F)^2$. □

Sketch of the proof of Assertion 9.3.4.a. Analogously to Assertion 9.3.6.a using Assertion 9.8.1.d we construct an obstruction $w_2(N \times I)|_{F \times I} \in \mathbb{Z}_2$ to the existence of a triple on $F \times I$ tangent to $N \times I$. Then

$$w_2(N)|_F = w_2(N \times I)|_{F \times I} = 0, \quad \text{where}$$

• the first equality follows from Assertion 9.8.1.e, and

• the second equality holds since F is orientable, so $F \times I$ is parallelizable (Assertion 9.1.1 for $S_g \times I$).

So we are done by Assertion 9.3.6.b. □

These sketches sketch have further important generalizations, see Assertions 9.4.8 (c, d), 9.8.5, 9.8.8 and 9.9.4, as well as § 12.3.

9.4. Orientability of 3-dimensional manifolds

An *orientation* of a tetrahedron or a triangle is an ordering of its vertices up to an even permutation. It is clear that for a triangle this definition is equivalent to the one given in § 5.7. The orientation (1234) of a tetrahedron *induces* the orientations (123), (243), (134), (142) of its (2-dimensional) faces. (These orientations of the faces agree along their common edges.) A face in a triangulation of a 3-manifold is called *interior* if it is contained in at least two tetrahedra. A triangulation of a 3-manifold is called *orientable* if there are orientations on all tetrahedra of the triangulation such that the orientations induced from both sides of every interior face are opposite to each other (cf. Fig. 5.7.1). Such a collection of face orientations is called an *orientation* of the triangulation. Analogously one defines orientability of triangulation of a manifold having arbitrary dimension.

A smooth manifold is orientable in the sense of the definition before Assertion 8.7.5 if and only if it has a triangulation that is orientable in the above sense.

See Assertions 8.7.5, 8.7.6, and examples before them. In Assertion 9.2.3 we proved (in a different language) that any 2-manifold admits an embedding into an orientable triangulation of some 3-manifold (i. e. is homeomorphic to some subtriangulation).

9.4.1. Every closed connected 3-manifold has an orientable 2-submanifold whose complement is orientable.

This follows by Assertions 9.4.3.b and 9.4.7.b.

In this subsection T is any triangulation of a closed 3-manifold N.

9.4.2 (cf. Assertion 5.7.4 (b))**.** There are orientations on all 3-faces of the barycentric subdivision of T such that the orientations of any two neighboring faces disagree.

The following definitions appear naturally when we attempt to determine whether a 3-manifold is orientable (analogously to § 6). A set of 2-faces of T is called a **2-cycle** if every edge is contained in the boundaries of an even number of 2-faces in the set. The **boundary** ∂a of a tetrahedron $a \in T$ is the set of all boundary faces of this tetrahedron. We call the sum of boundaries of several tetrahedra a **2-boundary**. Two 2-cycles are called **homologous** if their difference is a sum of boundaries of several tetrahedra. The 2-**dimensional homology group** $H_2(T)$ (with coefficients in $\mathbb{Z}_2$) is the group of all homology classes of 2-cycles. For computations of the group $H_2(T)$, see Theorem 10.8.1 (a) and § 11.5.

9.4.3 (Riddle)**.** Define the **first Stiefel—Whitney class** $w_1(T) \in H_2(T)$ so that the following hold.

(a) The triangulation T is orientable if and only if $w_1(T) = 0$.

(b) The complement to any 2-cycle representing $w_1(T)$ is orientable.

(c) For any closed 2-manifold F, we have $w_1(F \times S^1) = w_1(F) \times S^1$.

(Take an arbitrary triangulation of F, and a convenient cellular subdivision of $F \times S^1$. Define an appropriate meaning of $\times$.)

An *edge subdivision* operation is shown in Fig. 9.4.1 on the left.

9.4.4. *Face subdivision* and *tetrahedron subdivision* operations in Fig. 9.4.1 can be expressed in terms of edge subdivision operations.

Two triangulations are called **homeomorphic** if one can be obtained from the other by edge subdivision operations and inverses of edge subdivision operations.

9.4.5. (a) Two homeomorphic triangulations of a 3-manifold are either both orientable or both non-orientable.

(b,c,d) Find $H_2(N)$ for $N = S^1 \times S^2$, $(S^1)^3$, $\mathbb{R}P^3$.

Hint. This is analogous to Assertion 6.4.2.b, cf. the definition of a cellular decomposition in §10.4.

The 2-dimensional homology groups (and analogous groups, see below) of homeomorphic triangulations are isomorphic. Moreover, their first Stiefel—Whitney classes are 'the same'. This is formalized by the following assertion.

9.4.6. (a) For a fixed N, the group $H_2(T)$ does not depend on the choice of T. More precisely, if a triangulation U is obtained from T

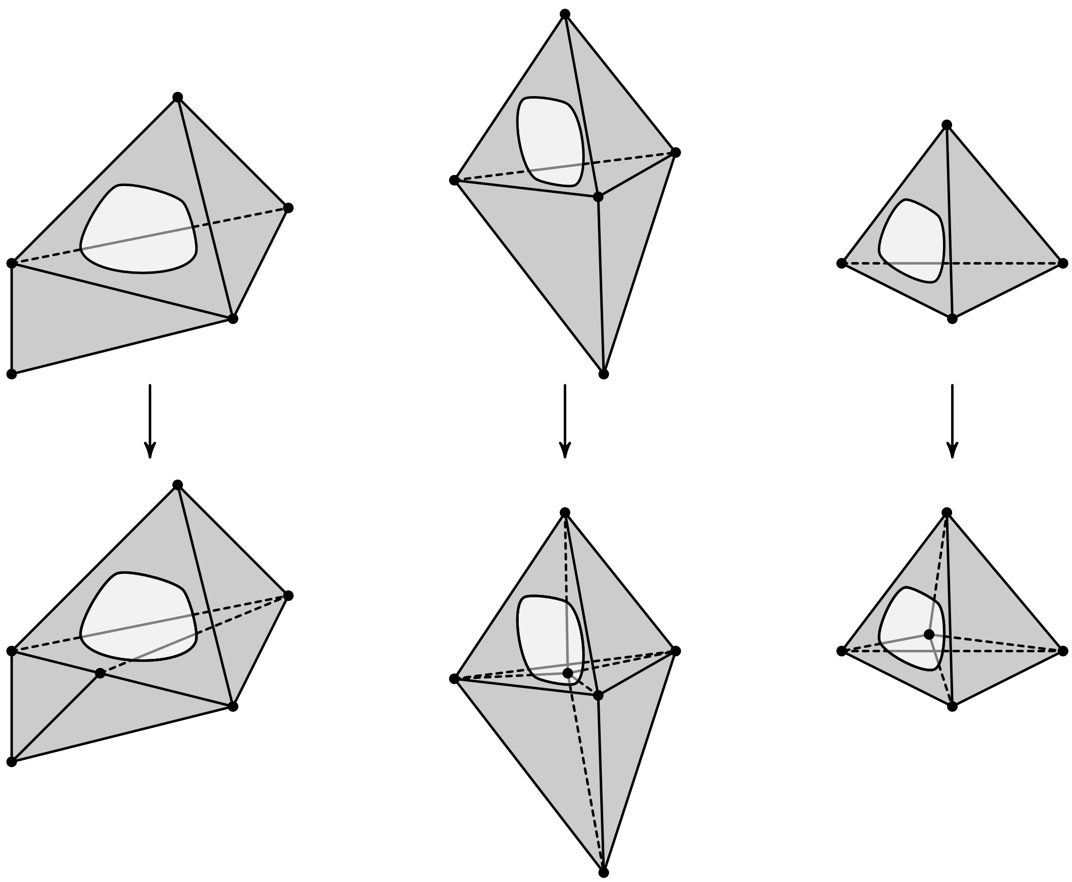

Figure 9.4.1. Subdivisions of 1-dimensional, 2-dimensional and 3-dimensional faces

by edge subdivision operations then the 'natural' homomorphism $H_2(T) \to H_2(U)$ is an isomorphism. (Cf. Assertion 6.4.1 (c).)

(b) The isomorphism of part (a) maps $w_1(T)$ to $w_1(U)$.

For this reason the notation $H_2(N)$ and $w_1(N)$ is defined.

9.4.7. (a) Any class in $H_2(N)$ can be represented by a closed connected 2-manifold (not necessarily orientable). More precisely, any 2-cycle in T is homologous to some triangulation of a closed connected 2-submanifold of some triangulation obtained from T by edge subdivision operations. (Cf. Problems 6.3.7 (b) and 14.9.3.)

(b) The class $w_1(N) \in H_2(N)$ can be represented by a closed connected orientable 2-manifold.

For a closed n-manifold N analogously one defines the class $w_1(N) \in H_{n-1}(N)$ as the obstruction to orientability, or as the (incomplete) obstruction to parallelizability. E.g. by Assertion 8.7.6 we have $w_1(\mathbb{R}P^{2k+1}) = 0$,

$w_1(\mathbb{R}P^{2k}) \neq 0$ and $w_1(\mathbb{C}P^n) = 0$ (observe that $H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}_2$ and $H_{2n-1}(\mathbb{C}P^n) = 0$).

9.4.8. Let M and N be closed manifolds.

(a) The manifold N is orientable if and only if $w_1(N) = 0$.

(b) The complement in N of (a neighborhood of) any cycle representing $w_1(N)$ is orientable.

(c) We have $w_1(N \times S^1) = w_1(N) \times S^1$.

(d) We have $w_1(M \times N) = w_1(M) \times N + M \times w_1(N)$.

Hint to 9.4.7. (a) (Banana and pineapple trick, cite[Figure I.26]HAMS.) Take any 2-cycle a in T. Every edge of T is adjacent to an even number of faces of a. 'Separating' these faces in pairs, we obtain a 2-cycle in some refinement T' of T, homologous to a, and represented by a 2-hypergraph K, whose every point has a neighborhood in K isomorphic to the cone over a disjoint union of circles. There are only finitely many points for which the number of circles is larger than one. For every of these points, 'separate' the cones that correspond to different circles. We obtain a 2-cycle in some refinement T'' of T', homologous to a and represented by a closed connected 2-submanifold.

9.5. Plan of the proof of the Stiefel Theorem

The following result is the most important step in the proof of the Stiefel Theorem 9.1.3, while for non-orientable 3-manifolds, this result is also interesting in its own right.

Lemma 9.5.1 (Obstruction)**.** *Let N be a closed 3-manifold. For any sufficiently small triangulation T of N there exist*

- *a linear space $H_1(T)$ over $\mathbb{Z}_2$,*
- *an element $w_2(T) \in H_1(T)$, and*
- *a non-degenerate bilinear map $\cap\colon H_1(T) \times H_2(T) \to \mathbb{Z}_2$*

such that the following properties hold.

(*Completeness*) *there is a pair tangent to N if and only if $w_2(T) = 0$;*

(*Heredity*) *there is a pair tangent to N on a closed connected 2-submanifold F in T if and only if $w_2(T) \cap [F] = 0$.*

The group $H_1(T)$ and the element $w_2(T)$ appear naturally in an attempt to construct a pair of fields (in § 9.6 and § 9.7, analogous to § 4.7—4.11 and § 6.1—6.4).

The 1-**dimensional homology group** $H_1(T)$ with coefficients in $\mathbb{Z}_2$ is the 1-dimensional homology group of the union of 2-dimensional faces of the triangulation T, for the definition see § 6.4 (even though this group appears here in the solution of a *different* problem!). Therefore, for many assertions for 2-hypergraphs (for example, 6.4.1 (c)), analogous assertions are also true for triangulations of 3-manifolds.

9.5.2. (a) For two homeomorphic triangulations T, T' of a 3-hypergraph we have $H_1(T) \cong H_1(T')$.

(b,c,d) Find $H_1(T)$ for some triangulation T of $S^1 \times S^2$, $(S^1)^3$, $\mathbb{R}P^3$.

Hint. This is analogous to Assertion 9.4.5.

For more computations of the group $H_1(T)$, see § 10.5, § 11.5.

The *second Stiefel—Whitney class* $w_2(T)$ is defined before Problem 9.7.4.

A multiplication $\cap$ (the intersection of homology classes) is defined analogously to § 6.7 (for details, see § 10.7). Its non-degeneracy means that for every $\alpha \in H_1(T) - \{0\}$ there exists $\beta \in H_2(T)$ such that $\alpha \cap \beta = 1 \in \mathbb{Z}_2$. The non-degeneracy of $\cap$ follows from the Poincaré Duality Theorem 10.8.1 (b).

Proof of the Stiefel Theorem 9.1.3. We can assume that the 3-manifold N is closed. Take a triangulation of N as in the Obstruction Lemma 9.5.1. The group $H_2(N)$ is finite. Hence, by Assertions 9.4.7 (a) there exists a refinement T of this triangulation such that every element of $H_2(T)$ can be represented by a triangulation of a closed connected 2-manifold. Take any closed connected 2-submanifold F in T. Since N is orientable, by the Submanifold Lemma 9.2.1 and heredity $w_2(T) \cap [F] = 0$. Then non-degeneracy of the multiplication $\cap$ implies $w_2(T) = 0$. So by the completeness there is a pair tangent to N. Since N is orientable, it follows that there is a triple tangent to N. □

9.6. Intuitive description of the obstruction class*

The description of the second Stiefel—Whitney class w_2 of a 3-manifold given in this subsection is not used later in the book. The description is 'global', without using a triangulation. This class is the $\mathbb{Z}_2$-homology class of the union of those closed curves on which some general position pair of tangent vector fields is linearly dependent.

We now give the details of this description. Denote by Σ the subset of $(\mathbb{R}^3)^2$ consisting of all linearly dependent pairs of vectors. The subset of $(\mathbb{R}^3)^2$ consisting of all pairs of vectors such that the first coordinate of one of the vectors is non-zero, is 6-dimensional (i. e. has codimension 0). The intersection of this subset with Σ can be described by two independent equations: the determinants formed by the first and the second column, and by the first and the third column must be zero. Therefore the intersection is 4-dimensional (i. e. has codimension 2). Analogously, considering the second and the third coordinate, we obtain that Σ is a union of three 4-dimensional sets. So Σ is 4-dimensional (i. e. has codimension 2).

A pair of vector fields on $\mathbb{R}^3$ is the same as a map $\mathbb{R}^3 \to (\mathbb{R}^3)^2$. The subset of $\mathbb{R}^3$ on which the pair is linearly dependent is the preimage of the codimension 2 subset Σ. Hence if the pair of vector fields is *in general position* then the preimage is a submanifold of codimension 2, i. e. is a disjoint union of closed curves.

So for a general position pair of tangent vector fields on a 3-manifold, the subset of the manifold on which the pair is linearly dependent is a disjoint union of closed curves. The *homology class* of this union (as defined later in this subsection or in § 10.6) is called the second Stiefel—Whitney class.

We hope that the reader has some intuitive understanding of the above-used notion of general position. Let us though reduce it to the notion of general position for vector subspaces. A triple of maps of manifolds to a manifold is *in general position* if all their intersections in pairs and in triples are locally diffeomorphic to the corresponding intersections of vector subspaces in general position. A pair of tangent vector fields is *in general position* if the corresponding sections of the tangent bundle together with the zero section are in general position.

9.7. Proof of the Obstruction Lemma

The **dual polyhedral decomposition** is defined analogously to the definition next to Fig. 4.8.1, see Fig. 9.7.1. We choose a triangulation T of a closed 3-manifold. In every tetrahedron x of the triangulation, we choose a point x^*. For every (2-dimensional) face f of the triangulation, we join by a *dual* edge f^* the chosen points in two tetrahedra having the face f in common. The intersection of this edge with the union U of all faces of the triangulation must consist of exactly one point, which lies inside the face f. For every edge a of the triangulation, a *dual* polygon a^* is a 2-dimensional curvilinear polygon whose edges are all dual edges that correspond to those faces of the triangulation that contain the edge a. The intersection of the dual polygon with the union of all edges of the triangulation must consist of exactly one point, which lies inside the edge a. The union of all dual polygons decomposes the 3-manifold into polyhedra (each polyhedron contains exactly one vertex of the triangulation.) The resulting polyhedral decomposition of the 3-manifold is called **dual** to T and is denoted by T^*.

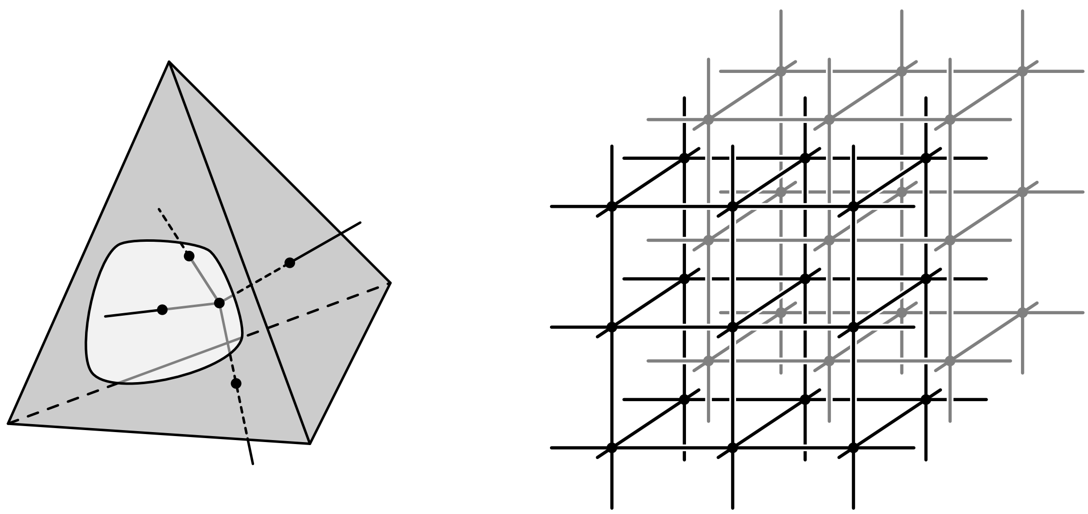

Figure 9.7.1. Dual polyhedral decompositions

The beginning of the proof of the Obstruction Lemma 9.5.1: *definition of the obstruction cycle.* Take a sufficiently fine triangulation of the given 3-manifold such that the angle between the tangent spaces at any two points in the same polyhedron of the dual decomposition is smaller than $\pi/2$. Let T be any refinement of this triangulation.

We first construct a pair of fields on the vertices of the dual decomposition T^*. Then we try to extend these fields to the union of all edges, then to the union of all faces, and finally to the union of all polyhedra.

The triangulation is very fine, hence the tangent spaces at different points of the edge can be identified with each other. Therefore a pair of fields on a part of the edge is the same as a map from this part of the edge to the space of all pairs of fields on $\mathbb{R}^3$, i. e. to SO_3. By Assertion 9.3.1, the pair of fields constructed on the vertices of the dual decomposition can be extended to the union of all edges of the dual decomposition.

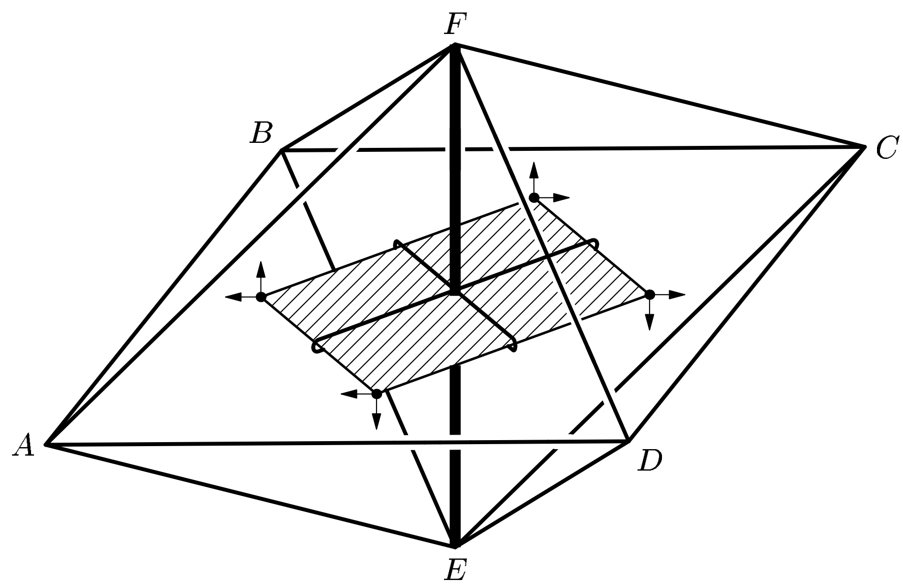

Figure 9.7.2. Extension of a pair of fields to an edge of the dual decomposition

Let us try to extend this pair of fields to a face a^* of the dual decomposition (see Fig. 9.7.2). The polyhedra are very small, hence the tangent spaces at different points of the same face can be identified with each other. Therefore a pair of fields on a part of the face is the same as a map from this part of the face to SO_3. If this map cannot be extended from the boundary ∂a^* to the whole face a^*, then the edge a of the original decomposition (the edge that 'pierces through' the face a^*) is colored red. Thus to any pair w of fields on the union of all edges of the dual decomposition there corresponds the set $\varepsilon(w)$ of red edges of the original decomposition. This set of edges is called the *obstruction cycle*.

9.7.1. (a) Find the obstruction cycle $\varepsilon(w)$ for $F \times S^1$, where F is a 2-manifold, and some w.

(b)* For any (sufficiently fine) triangulation of a 3-manifold, the union of all edges of its barycentric subdivision is $\varepsilon(w)$ for some pair of fields w. (Cf. Assertion 9.4.2.)

(c)* For any closed orientable 3-manifold, there exist a triangulation and a set of faces of its barycentric subdivision such that every edge of the barycentric subdivision is adjacent to an odd number of faces from the set. (A combinatorial solution to this problem together with part (b) will give a combinatorial proof of the Stiefel Theorem 9.1.3.)

9.7.2. (a) If $\varepsilon(w)=0$ then the pair of fields w can be extended to the union of all faces of the dual decomposition.

(b) If $\varepsilon(w)=0$ then the pair of fields w can be extended to the whole 3-manifold. (Use Assertion 9.3.3.)

(c) Outside $\varepsilon(w)$ there exists a pair of fields.

9.7.3. (a) Every vertex is adjacent to an even number of edges of $\varepsilon(w)$.

(b) Change the pair w of fields on an edge f^* of the dual decomposition by the non-trivial element of $\pi_1(SO_3)\cong\mathbb{Z}_2$. Then $\varepsilon(w)$ changes by adding the boundary of the dual to f^* face f of the original decomposition.

The **second Stiefel—Whitney class** is defined by

$$w_2(T):=[\varepsilon(w)]\in H_1(T).$$

This is well defined analogously to the construction of the first Stiefel—Whitney class (§ 6.4), of the Euler number (§ 4.8, 4.9) as well as of the invariants of vector fields (§ 4.11) and involutions (§ 7.3).

9.7.4. (a) (cf. Assertion 9.4.3 (c)) For any closed connected 2-manifold F we have $w_2(F\times S^1)=\rho_2\chi(F)[*\times S^1]$.

(b) State and prove version of Assertion 9.4.6 (b) for $w_2(T)$.

Part (a) holds by the solution of Problem 9.7.1 (a).

The completion of the proof of the Obstruction Lemma 9.5.1. The completeness follows from Assertions 9.7.2 (b) and 9.7.3 (b).

For the dual decomposition T^*, one analogously defines the group $H_1(T^*)$ and the class $w_2(T^*)$. The multiplication $\cap\colon H_1(T^*)\times H_2(T)\to\mathbb{Z}_2$ is defined analogously to § 6.7 (for details, see § 10.7).

(If we have already defined the dual decomposition, it is more economical to state the Obstruction Lemma in the language of $w_2(T^*)\in H_1(T^*)$ and the multiplication above. Then we can skip the next paragraph.)

For any class $\alpha\in H_1(T)$, there exists a class in $\alpha^*\in H_1(T^*)$ that is homologous to α in some triangulation that can be obtained from each of T and T^* by edge subdivision operations. The multiplication $\cap\colon H_1(T) \times H_2(T) \to \mathbb{Z}_2$ is well defined by the formula $\alpha \cap \beta := \alpha' \cap \beta$.

The non-degeneracy of the multiplication $\cap$ is the same as the Poincaré Duality Theorem 10.8.1 (b).

To prove the heredity, let $\omega \subset N$ be any obstruction 1-cycle in T^* (i. e. a representative of the class $w_2(T^*)$) that is the union of some edges of the decomposition T^*. Then $w_2(T^*) \cap [F] = \rho_2|\omega \cap F|$. If F admits a pair tangent to N, then $\omega \cap F = \varnothing$, so $w_2(T^*) \cap [F] = 0$. If $w_2(T^*) \cap [F] = 0$, then $\omega \cap F$ consists of an even number of points. Since F is connected, we can 'cancel' them by pairs.[24] □

9.7.5. State and prove a version of the Obstruction Lemma 9.5.1 for 3-manifolds with non-empty boundary.

Theorem 9.7.6. *For every closed* 3*-manifold* N*, we have* $w_1(N)^3 = 0$ *and* $w_2(N) = w_1(N)^2$.

Since $w_2 = w_1^2$, the equality $w_1^3 = 0$ is equivalent to the equality $w_2 w_1 = 0$. The equality $w_2 w_1 = 0$ follows by the heredity of the Obstruction Lemma 9.5.1, and Assertions 9.4.7.b, 9.3.4.a. Theorem 9.7.6 follows from Assertion 12.2.2 (a) and the fact that any 3-manifold admits an immersion in $\mathbb{R}^4$ (Theorem 12.1.4). It would be interesting to find a direct proof of Theorem 9.7.6 that does not use difficult Theorem 12.1.4. Such a proof could be based on the geometric or combinatorial interpretations (see Assertions 9.4.2, 9.7.1 (b) and 9.7.7). Hint: $w_2(N)|_F = w_1(N) \cap i_* w_1(F)$.

9.7.7. Let N be a closed 3-manifold. We call a closed 2-submanifold $F \subset N$ *characteristic* if the complement $N - F$ has an orientation that changes when we cross F.

We call a collection $S \subset N$ of closed smooth curves (i. e. a 1-submanifold) *characteristic* if the complement $N - S$ admits a pair of fields, and, for every 2-dimensional disc D that intersects S transversally in exactly one point, the pair of fields on ∂D cannot be extended to D (i. e. going 'around S' this pair of fields makes a homotopically non-trivial loop).

(a) There exists a characteristic 2-submanifold.

[24] Clearly, $w_2(T^*) \cap [F] = w_2(N)|_F$, see Assertion 9.3.6. Hence the heredity also follows from the completeness of the obstruction $w_2(N)|_F$.

(b) If F_1, F_2, F_3 are transversal characteristic 2-submanifolds then $|F_1 \cap F_2 \cap F_3|$ is even.

(c) There exist a characteristic collection of curves.

(d) If a closed 2-submanifold that has an orientable neighborhood in N and a characteristic curve S intersect transversally, then their intersection consists of an even number of points.

(e) If F_1, F_2 are transversal characteristic 2-submanifolds then $F_1 \cap F_2$ is a characteristic collection of curves.

(f) If a characteristic curve and a characteristic 2-submanifold intersect transversally, then their intersection consists of an even number of points.

Hint to 9.7.1. (a) Let v be a field on the union of all edges of some decomposition of the 2-manifold F such that the non-zero elements of the obstruction assignment are at the vertices $p_1, \dots, p_n$ of the dual decomposition and are equal to $\operatorname{sgn} \chi(F)$ (so that $n = |\chi(F)|$). Let v' be a unit vector field on S^1. Then, for the 'prismatic' decomposition of the product $F \times S^1$ and the pair (v, v'), the obstruction 1-cycle is the union of circles $p_i \times S^1$ for $i = 1, \dots, n$. To prove this, use Assertion 9.3.2 (e).

Hint to 9.7.2. (c) Consider a neighborhood of the union of the obstruction cycle with the set of all vertices of the dual polyhedral decomposition. The complement of this neighborhood is a neighborhood of the union of all 2-dimensional faces of the dual decomposition that are not red. A pair of fields on this complement can be extended to the complement of the obstruction cycle by Assertion 9.3.3.

Hint to 9.7.3. (a) For a given vertex of the dual polyhedral decomposition, consider the boundary sphere of the corresponding polyhedron in the original triangulation. The parity of the number of those 2-dimensional faces of this decomposition that are pierces through by red edges is equal to the sum of homotopy classes of maps from face boundaries to SO_3 and therefore is equal to zero. (In view of Assertion 9.3.2 (d), this argument can be modified to avoid the use of the operation of the sum.)

(c) Change a pair of fields on one edge of the dual polyhedral decomposition of the triangulation by a map $S^1 \to SO_3$ not homotopic to the map to a point. Then the numbers on all faces of the decomposition that are adjacent to this edge, change.

9.8. Characteristic classes for 4-manifolds

Formally speaking, this subsection is not used later in the book. The following problems mention homology groups $H_1(N;\mathbb{Z})$, $H_2(N)$, $H_3(N)$, classes $w_1(N)$, $w_2(N)$, $W_3(N)$ and the operation $\times$, which will appear naturally (and could be defined rigorously) in the process of solving these problems (analogously to the proof of the Obstruction Lemma 9.5.1). You do not need to know their definitions in advance. You can check your definitions using § 9.9, § 10.6.

In this subsection N is any closed connected 4-manifold, not necessarily orientable. Let $SO_4\subset\mathbb{R}^{16}$ be the space of positively oriented orthonormal frames in $\mathbb{R}^4$.

9.8.1. (a)We have $SO_4\cong SO_3\times S^3$. Moreover, for the 'standard' inclusion $SO_3\to SO_4$ we have $(SO_4,SO_3)\cong(SO_3\times S^3,SO_3\times *)$.

(b) There exists a map $p:SO_4\to S^3$ such that $p^{-1}(0,0,0,1)=SO_3$ and p has the local triviality property analogous to Assertion 8.10.7 (b) (and hence the lifting properties analogous to Assertions 8.10.7 (e, g)).

(c) Any map $S^2\to SO_4$ extends to a map $D^3\to SO_4$.

(d) There are exactly two homotopy classes of maps $S^1\to SO_4$.

(e) The composition of a map $S^1\to SO_3$ that is not homotopic to the map to a point and the 'standard' inclusion $SO_3\to SO_4$ is not homotopic to the map to a point.

9.8.2. (a) Define $H_2(N)$ and an obstruction $w_2(N)\in H_2(N)$ to the existence of a *triple* tangent to N.

(b) The obstruction $w_2(N)$ is incomplete.

(c) (cf. Assertion 9.7.2.c) The complement to (a neighborhood of) any non-empty 2-cycle representing $w_2(N)$ is 3-parallelizable.

In (b) a counterexample is given by $N=S^4$, which does not admit even one field. By Assertions 9.1.8.bb' we have $w_2(\mathbb{R}P^4)=0$ and $w_2(\mathbb{C}P^2)=[\mathbb{C}P^1]\neq 0$ (observe that $H_2(\mathbb{R}P^4)\cong H_2(\mathbb{C}P^2)\cong\mathbb{Z}_2$).

Theorem 9.8.3. *A closed connected* 4*-manifold* N *is*

(a) almost parallelizable if and only if N *is orientable and* $w_2(N)=0$.

(b) parallelizable if and only if N *is orientable,* $w_2(N)=0$ *and* $\chi(N)=\sigma(N)=0\in\mathbb{Z}$.

Part (a) (and Assertion 9.8.4.a below) is proved analogously to the Obstruction Lemma 9.5.1.

Comment. [Ma80] If N is orientable, then an obstruction to the extension of a quadruple tangent to N from N_0 to N is a pair of numbers (i. e. lies in $\pi_3(SO_4) \cong \mathbb{Z} \oplus \mathbb{Z}$, see § 14.5). These numbers are the Euler characteristic $\chi(N)$ and (up to a factor) the signature $\sigma(N)$ of the intersection form $\cap\colon H_2(N;\mathbb{Z}) \times H_2(N;\mathbb{Z}) \to \mathbb{Z}$ (which is defined analogously to the definition in § 6.7, see § 10.7).

9.8.4. (a) The following conditions are equivalent:
- $w_2(N) = 0$;
- the complement to some graph in N is 3-parallelizable;
- N is almost 3-parallelizable.
- $N_0 \times I$ is 4-parallelizable;
- $N_0 \times I^2$ is 5-parallelizable.

(b) A 3-manifold M (possibly with boundary) is 2-parallelizable if $M \times I$ is 3-parallelizable. (The converse is trivial.)

Use Assertion 9.8.1 (e) and its generalization to SO_n.

Part (b) shows that $\mathbb{R}P^2 \times D^2$ is not 3-parallelizable but $F \times D^2$ is 3-parallelizable for F the Klein bottle with a handle.

For a closed connected 2-manifold F denote $w_2(F) := \rho_2\chi(F)$.

9.8.5 (cf. Assertion 9.7.4.a)**.** (a) For any closed 3-manifold M, we have $w_2(M \times S^1) = w_2(M) \times S^1$.

(b) For any connected closed 2-manifold F and $p \in F$, we have $w_2(F \times S^1 \times S^1) = w_2(F)p \times S^1 \times S^1$.

(c) For any closed connected 2-manifolds F, F', and points $p \in F$, $p' \in F'$, we have

$$w_2(F \times F') = w_2(F)p \times F' + w_1(F) \times w_1(F') + F \times w_2(F')p'.$$

Part (b) follows by (a) and Assertion 9.7.4.a.

9.8.6. Let $V_{4,2} \subset \mathbb{R}^8$ be the Stiefel manifold of orthonormal ordered pairs of vectors in $\mathbb{R}^4$.

(a) $V_{4,2} \cong S^3 \times S^2$.

(b) Every map $S^1 \to V_{4,2}$ is homotopic to the map to a point.

(c) The composition of the inclusion $S^2 = V_{3,1} \to V_{4,2}$ and some homeomorphism from part (a) maps x to $((1,0,0,0), x)$.

9.8.7 (cf. Problems 9.8.2 and 9.8.4.a)**.** (a) Define $H_1(N;\mathbb{Z})$ and an obstruction $W_3(N) \in H_1(N;\mathbb{Z})$ to the existence of a *pair* tangent to N.

(b) The obstruction $W_3(N)$ is incomplete.

(c) The complement of a neighborhood of any non-empty graph representing $W_3(N)$ is 2-parallelizable.

(d) We have $W_3(N)=0$ if and only if N is almost 2-parallelizable.

(e) We have $2W_3(N)=0$.

(f) Is it correct that $\rho_2 W_3(N)=0$?

9.8.8. For closed connected 2-manifolds F, F', and points $p\in F$, $p'\in F'$, we have

$$\rho_2 W_3(F\times F') = w_2(F)p\times w_1(F') + w_1(F)\times w_2(F')p'.$$

9.8.9. (a,b) State and prove versions of Problems 9.8.2.ab for connected 4-manifolds with non-empty boundary.

(c) A connected 4-manifold X with non-empty boundary is parallelizable if and only if X is orientable and $w_2(X)=0\in H_2(X,\partial)$.

(e—h) State and prove versions of Problem 9.8.7 for connected 4-manifolds X with non-empty boundary, the group $H_1(X,\partial;\mathbb{Z})$ and the class $W_3(X)\in H_1(X,\partial;\mathbb{Z})$.

9.8.10. Assume that N is orientable.

(a) If $w_2(N)=0$ then $W_3(N)=0$.

(b) If $w_2(N)$ can be represented by an orientable 2-manifold then $W_3(N)=0$.

(c)* The class $w_2(N)$ can be represented by an orientable 2-manifold.

Part (a) follows by Assertions 9.8.3.a and 9.8.7 (d). Part (b) follows by Assertion 9.8.11.c.

9.8.11. Assume that N is orientable. Define the *Bockstein homomorphism* $\beta : H_2(N)\to H_1(N;\mathbb{Z})$ as follows. We can represent a class $a\in H_2(N)$ by a closed 2-manifold $F\subset N$ (by Assertion 14.9.3 (c)). Let βa be the homology class in N of any integer lift of any 1-cycle on F representing $w_1(F)$: $i_{N,*}w_1(F)=\rho_2\beta[F]$, where $i_N : F\to N$ is the inclusion. Cf. the formula $w_1(N)=\rho_2\beta[N]$ of Assertions 10.5.9.bcd.

(a) This β is well-defined.

(b) This is equivalent to the definition before Assertion 11.8.2.

(c) We have $W_3(N)=\beta w_2(N)$.

9.8.12 (cf. Assertion 6.7.3 (a,b))**.** (a) For any $a\in H_2(N)$, we have $a\cap a = w_2(N)\cap a$.

(b) We have $w_2(N)\cap w_2(N)=\rho_2\chi(N)$.

Hint to 9.8.1. (c) Take any map $f : S^2 \to SO_4$. Its composition $p \circ f : S^2 \to S^3$ is null-homotopic by Theorem 8.1.7.a. Take a homotopy from $p \circ f$ to the map to $(0, 0, 0, 1)$. Analogously to Assertion 8.10.7.cdf, by (b) the homotopy lifts to a homotopy from f to some map $f' : S^2 \to SO_4$. We have $f'(S^2) \subset p^{-1}(0, 0, 0, 1) = SO_3$. Hence f' extends to a map $D^3 \to SO_3$ by Assertion 9.3.2 (f). Then f extends to a map $D^3 \to SO_4$.

(d,e) Deductions of (d,e) from (b) and Assertion 9.3.2 (d) are analogous to (c).

(Compare these deductions to hint to 9.9.2.a, and to the deduction of Assertions 8.10.7. (e, g) from Assertion 8.10.7 (b); see a generalization in § 14.5).

Hint to 9.8.5. (a) Analogous to the solution of Problem 9.7.1 (a). Let v' be a unit tangent vector field on S^1. Take some triangulation of the 3-manifold M. Let u, v be a pair tangent to M on the union of all edges. Denote by ω the obstruction 1-cycle. Take the 'prismatic' cell subdivision of $M \times S^1$. Then for the triple (u, v, v') tangent to $M \times S^1$ the obstruction 2-cycle is $\omega \times S^1$

(c) Take a point $p \in F$ and a pair u, v tangent to F as in the sketch of the proof of Assertion 9.3.6.c (in §9.3). Take analogous point $p' \in F$ and a pair u', v' tangent to F'. Analogously to Assertion 9.1.8.c take the triple $(u, v + v', u')$ tangent to $F \times F'$. Now complete this sketch analogously to Assertion 9.1.8.c and to the last paragraph of the sketch of the proof of Assertion 9.3.6.c (in §9.3).

Hint to 9.8.8. Take a point $p \in F$ and a pair u, v on F as in the sketch of the proof of Assertion 9.3.6.c (in §9.3). Take analogous point $p' \in F$ and a pair u', v' on F'. Take the pair $(u + v', v + u')$ (linearly dependent at some points and) tangent to $F \times F'$.

The pair u, v is linearly independent outside $p \cup \omega$. The analogous statement holds for the pair u', v'. The pair $(u + v', v + u')$ is linearly independent on $\omega \times \omega'$. Hence the pair $(u + v', v + u')$ is linearly independent outside $p \times \omega' \cup \omega \times p' \cup p \times p'$.

So the obstruction 1-cycle representing W_3 can only contain oriented edges of $p \times \omega \cup \omega \times p'$. The edges of $p \times \omega'$ (respectively, of $\omega \times p'$) are contained in the obstruction 1-cycle with the coefficient whose parity is $\chi(F)$ (respectively, $\chi(F')$). This proves the required formula.

9.9. Characteristic classes for n-manifolds

The following theorem generalizes the Euler-Poincaré Theorem 4.6.2, the Orientability Theorem 6.1.1 (on $w_1(N)$, see also § 9.4), the Hopf Theorem 8.7.4 (on $W_n(N) := \chi(N) \in \mathbb{Z}$ for connected N), the Obstruction Lemma 9.5.1 (on $w_2(N)$) and some results in § 9.8.

In this section N is a (smooth compact) closed n-manifold. Recall that $\mathbb{Z}_{(n-k)}$ is $\mathbb{Z}$ for even $n-k$, and $\{0,1\}$ for odd $n-k$.

Theorem 9.9.1 (Obstruction)**.** *There are Stiefel—Whitney classes* $w_1(N) = W_1(N) \in H_{n-1}(N;\mathbb{Z}_2)$, $W_n(N) \in H_0(N;\mathbb{Z})$, *and*

$$W_{n-k+1}(N) \in H_{k-1}(N;\mathbb{Z}_{(n-k)}) \quad \textit{for} \quad 1 < k < n$$

such that the following properties hold.

(a) If N is k parallelizable, then $W_{n-k+1}(N) = 0$.

(b) We have $W_{n-k+1}(N) = 0$ if and only if the complement of some $(k-2)$-complex in N is k-parallelizable.

(c) If $n-k$ is even, then $2W_{n-k+1}(N) = 0$.

The converse to (a) is false, see Assertions 9.8.2 (b) and 9.8.7 (b).

The group $H_{k-1}(N;\mathbb{Z}_{(n-k)})$ and the class $W_{n-k+1}(N)$ appear naturally when we attempt to construct a tangent k-tuple, by extension from lower dimensional skeleta to higher dimensional skeleta, analogously to the Obstruction Lemma 9.5.1 (see below).

Sketch of the intuitive definition of the class $W_{n-k+1}(N)$ using general position. Consider a k-tuple (possibly degenerate) tangent to N. By general position, the subset of the manifold on which the k-tuple is linearly dependent is a union of $(k-1)$-submanifolds. If $n-k$ is even, then there is a 'natural' orientation on these submanifolds. Their union represents a $(k-1)$-*cycle* with coefficients in $\mathbb{Z}_{(n-k)}$. The class $W_{n-k+1}(N)$ is defined as the *homology class* of this cycle (see § 10.6 for definition of cycle and homology).

Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$.

9.9.2. (a) For any $j < n-k$, every map $S^j \to V_{n,k}$ is homotopic to the map to a point.

(b) Define a map $f\colon S^{n-k} = V_{n-k+1,1} \to V_{n,k}$ as 'appending $k-1$ vectors'. If $k>1$ and $n-k$ is odd, then every map $S^{n-k} \to V_{n,k}$ is homotopic to either the map to a point or to f. If $k=1$ or $n-k$ is even, then every map $S^{n-k} \to V_{n,k}$ is homotopic to the composition of the map f with some map $g\colon S^{n-k} \to S^{n-k}$, and for different $\deg g$ such compositions are not homotopic.

Sketch of the rigorous definition of the class $W_{n-k+1}(N)$: beginning. By Assertion 9.9.2 (a), a k-tuple tangent to N exists on the $(n-k)$-skeleton of some triangulation of N. By Assertion 9.9.2 (b), the obstruction to the extension of the k-tuple to the $(n-k+1)$-skeleton is an assignment of elements of $\mathbb{Z}_{(n-k)}$ to the $(k-1)$-cells of the dual cell decomposition. Analogously to Assertion 9.7.3.a, single out the *cycles* among all assignments of elements of $\mathbb{Z}_{(n-k)}$.

9.9.3. The obstruction assignment is a cycle.

Sketch of the rigorous definition of the class $W_{n-k+1}(N)$: completion. Then we define which cycles are *homologous*. For details see § 10.6. The group of homology classes of cycles is called the $(k-1)$-*dimensional homology group* $H_{k-1}(N;\mathbb{Z}_{(n-k)})$. (For computations of this group see § 10.5, § 11.5.) The **Stiefel—Whitney class** $W_{n-k+1}(N)$ is the homology class of the obstruction assignment. The definitions of this group and this class involve the triangulation, but in fact they only depend on N by Theorem 10.6.8 on PL-invariance and analogously to Assertion 9.4.6 (b).

Let $w_s(N) := \rho_2 W_s(N) \in H_{n-s}(N;\mathbb{Z}_2)$. These classes are easier to compute.[25] It is convenient to set $w_s(N) = 0$ for $s > n$, and define $w_0(N) = [N] \in H_n(N;\mathbb{Z}_2)$ to be the class represented by the union of all n-cells of some decomposition of N.

How do we express the Stiefel—Whitney classes of a product of manifolds in terms of the Stiefel—Whitney classes of these manifolds?

Theorem 9.9.4 (Whitney—Wu Formula; cf. Assertions 8.8.3 (b), 9.4.8.d, 9.8.5 and 9.8.8)**.** *If M and N are closed manifolds, then* $w_s(M \times N) = \sum\limits_{k=0}^{s} w_k(M) \times w_{s-k}(N)$.

[25] В [Pr14′, задаче 11.10] пропущено условие нечетности числа k. Замечание после задачи 11.10 в [Pr14′] не обосновано (и, видимо, неверно). Из равенства нулю приведения по модулю 2 элемента абелевой группы, имеющего порядок 2, не вытекает, что этот элемент нулевой. Пример: элемент $2 \in \mathbb{Z}_4$ ненулевой, хотя имеет порядок 2, и его приведение по модулю 2 нулевое.

Denote $1 := [N] = w_0(N)$; this notation is convenient since $[N] \cap x = x$ for any s and $x \in H_s(N)$. The *total Stiefel—Whitney class* of N is defined as

$$w(N) := 1 + w_1(N) + w_2(N) + \ldots \in H_n(N) \oplus H_{n-1}(N) \oplus H_{n-2}(N) \ldots$$

In this notation, the Whitney—Wu Formula can be rewritten as $w(M \times N) = w(M) \times w(N)$.

Some heuristics to the Whitney—Wu Formula. Let $m := \dim M$ and $n := \dim N$. An m-tuple $u_1, \ldots, u_m$ (possibly linearly dependent at some points) on M tangent to M, and consisting of pairwise orthogonal vectors is called *characteristic* if for every $k = 0, 1, \ldots, m-1$ there is a non-empty k-cycle ω_{m-k} (in some triangulation of M) representing $w_{m-k}(M)$ such that the linear dependence set of $u_1, \ldots, u_k$ is $\bigcup\limits_{j=0}^{k-1} \omega_{m-j}$. Assume that there is a characteristic m-tuple $u_1, \ldots, u_m$ on M tangent to M. (A k-tuple with analogous properties is presumably constructed by induction on k starting with $k = 1$, but we do not want to work out the details.) Assume that there is an analogous n-tuple $v_1, \ldots, v_n$ on N. On $M \times N$ take the following $(m+n+1-s)$-tuple tangent to $M \times N$:

$$u_m, \ldots, u_s,\ u_{s-1} + v_1, \ldots, u_1 + v_{s-1},\ v_s, \ldots, v_n.$$

Considering the degeneracy set of this tuple, we obtain the required formula.

9.9.5. (a) Compute $w((\mathbb{R}P^2)^k)$.

(b) (Riddle) Compute the Stiefel—Whitney classes of a product of several closed 2-manifolds.

Hint: a closed 2-manifold could be either orientable, or non-orientable of even Euler characteristic, or non-orientable of odd Euler characteristic.

9.9.6. (a) For any $s = 0, 1, \ldots, n$, we have $w_s(\mathbb{R}P^n) = 0$ if and only if $\binom{n+1}{s}$ is even.

(b) If $\binom{m}{s}$ is even for every $s = 1, 2, \ldots, m-1$ then m is a power of two.

Part (a) is reformulated as $w(\mathbb{R}P^n) = (1 + a)^{n+1}$, where $a = [\mathbb{R}P^{n-1}] \in H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}_2$ is the generator. Part (a) follows from Lemma 13.3.3.d, Assertion 13.2.10.c and the Whitney—Wu Formula 13.4.3 (b); see a different proof in [St40], see also Theorem 12.6.1. It would be interesting to have a direct proof of (a), at least for $s = 2$: *we have* $w_2(\mathbb{R}P^n) = 0$ *if and only if either* $n = 1$ *or* $n \equiv 0, 3 \mod 4$.

Theorem 9.9.7. *For any triangulation of a closed n-manifold N, the union of all k-dimensional simplices of its barycentric subdivision is a k-cycle, which represents the class $w_{n-k}(N)$.*

Hint to 9.9.2. (This text is obtained by editing a draft by A. Miroshnikov.) (a) Индукция по k.

База $k = 1$ справедлива ввиду $V_{n,1} \cong S^{n-1}$ и теоремы 8.1.7.a.

Переход индукции от $k - 1$ *к* k. Возьмем произвольное отображение $f : S^j \to V_{n,k}$. Определим отображение $p : V_{n,k} \to S^{n-1}$ формулой $p(\vec{e}_1, \dots, \vec{e}_k) = \vec{e}_k$. Возьмем гомотопию $S^j \times I \to S^{n-1}$ отображения $p \circ f$ к отображению в точку $\vec{1} := (\underbrace{0, \dots, 0}_{n-1 \text{ раз}}, 1)$.

Для отображения p выполнено свойство локальной тривиальности, а значит, и свойство поднятия гомотопии (доказательство аналогично доказательству утверждений 8.10.7.bcf). Поэтому взятая гомотопия поднимается до гомотопии $S^j \times I \to V_{n,k}$ между f и некоторым отображением $f' : S^j \to V_{n,k}$. Имеем $f'(S^j) \subset p^{-1}(\vec{1}) = i(V_{n-1,k-1})$, где *стандартное вложение* $i : V_{n-1,k-1} \to V_{n,k}$ определено формулой

$$(\vec{e}_1, \dots, \vec{e}_{k-1}) \mapsto \Big((\vec{e}_1, 0), \dots, (\vec{e}_{k-1}, 0), \vec{1}\Big).$$

Поэтому определено отображение $i^{-1} \circ f' : S^j \to V_{n-1,k-1}$. По предположению индукции оно гомотопно отображению в точку. Тогда $f' = i \circ i^{-1} \circ f'$ гомотопно отображению в точку. Значит, и f гомотопно отображению в точку.

(See a generalization in § 14.5.)

10.3. Higher-dimensional manifolds

The star of a vertex A of a hypergraph K is the subhypergraph formed by all faces containing this vertex:

$$F(\mathrm{st}_K A) := \{\alpha \in F(K) \ : \ A \in \alpha\}.$$

A hypergraph is called a **triangulation of an n-manifold** (or locally Euclidean) if the star of every its vertex is homeomorphic to D^n. The homeomorphism class of a triangulation of an n-manifold is called *piecewise-linear (PL) n-manifold* (or *n-manifold* to be short). A PL n-manifold is called *connected*, *orientable* and so on, if some (or equivalently any) hypergraph representing this n-manifold is connected, orientable and so on.

The **boundary** ∂T of a triangulation T of an n-manifold is the union of all $(n-1)$-faces that are contained in the only n-face. In this book manifolds are allowed to have non-empty boundary. Triangulation T of a manifold is the *triangulation of a closed manifold* if boundary of T is empty.

10.3.1. (a) Any $(n-1)$-face of a triangulation of an n-manifold is contained in one or two n-faces.

(b) *The link* of a vertex A of a hypergraph K is the hypergraph formed by all faces not containing A but contained in some face containing A:

$$F(\mathrm{lk}_K\, A) := \{\sigma \in F(K) \ : \ A \notin \sigma \subset \alpha \ni A \text{ for some } \alpha \in F(K)\}.$$

A hypergraph is a triangulation of an n-manifold if and only if the link of every vertex is homeomorphic to S^{n-1} or to D^{n-1}. (Cf. the Sphere Recognition Theorem 5.3.3.)

(c) Give an example of a 3-hypergraph which is not a triangulation of a 3-manifold, but the link of whose every vertex is connected, and for whose every edge $\{u, v\}$ the simplices containing this edge form a 'chain'

$$\{u, v, a_1, a_2\},\ \{u, v, a_2, a_3\},\ \ldots,\ \{u, v, a_{n-1}, a_n\},\ \{u, v, a_n, a_1\}.$$

The *cone* $\mathrm{Con}\, K$ over a graph $K = (V, E)$ is the 2-hypergraph whose set of vertices is $V \cup \{c\}$, $c \notin V$, and whose faces are $\{c, i, j\}$, for each $\{i, j\} \in E$. The cone over a hypergraph is defined analogously.

The **cellular decomposition** of an n-hypergraph K is a sequence $K_0 \subset K_1 \subset \ldots \subset K_n = K$ of subhypergraphs of K such that K_{k-1} is cellular $(k-1)$-subhypergraph in K_k for every $k = 1, \ldots, n$. The subhypergraph K_k is called the *k-(dimensional) skeleton* of the cellular decomposition.

For example, the set of k-skeleta of an n-hypergraph, $k = 0, 1, \ldots, n$, is a cellular decomposition of this hypergraph. Given a polyhedral decomposition of an n-submanifold in $\mathbb{R}^m$ (cf. §4.5, §8.6) one can construct a cellular decomposition of an n-hypergraph whose geometric realization is this manifold.

10.4.1. Construct cellular decompositions with unique 3-cell (i.e. with connected complement to 2-skeleton) for examples from Problem 10.4.4 below.

The dual polyhedral decomposition and the dual cellular decomposition are defined analogously to §9.7. For the first of these two definitions we need the following assertion.

10.4.2. (a) Let T be a triangulation of an n-manifold. Then the subcomplex of T formed by all faces containing certain k-face a is isomorphic to the join of a and certain $(n-k-1)$-complex PL homeomorphic to the sphere S^{n-k-1} or to the ball D^{n-k-1}.

(b) Дайте строгое определение клеточного разбиения, двойственного к триангуляции.

Указание. Дайте и используйте строгое определение барицентрического подразбиения.

The **Euler characteristic** of an n-hypergraph K is the alternating sum of the numbers V_k of k-faces:

$$\chi(K) := V_0 - V_1 + \ldots + (-1)^n V_n.$$

10.4.3. (a) Deleting (the interior of) an n-face decreases the Euler characteristic by $(-1)^n$.

(b) (Riddle) Guess and prove the formula for the Euler characteristic of a union.

(c) The Euler characteristics of PL homeomorphic hypergraphs are equal. I.e. the Euler characteristic is preserved under subdivision of an edge.

10.6. General definition of homology groups

We present a simplified definition of homology groups accessible to non-specialists in topology. Simple properties can be proved using this definition. For proving more advanced properties one may need more abstract reformulation, or more general definition. E.g. for invariance under deformation retraction (Assertion 10.6.3.b) and for topological invariance (Theorem 10.6.8) one needs *singular homology*, while for Poincaré duality (§§10.8,10.9) one needs a reformulation via a chain complex. See also expository papers [MNS, ADN+].

We give a definition of homology groups independent of motivating examples from the preceding chapters, where this notion appears.

In this subsection X, Y, A are arbitrary simplicial complexes. (There are analogous definitions and results for cellular decompositions.)

A **modulo 2 k-cycle** in X is a set x of k-faces such that every $(k-1)$-face is contained in an even number of faces from x. Consider the sum (modulo 2) operation on modulo 2 k-cycles in X.

If $\dim X = k$, then the *modulo 2 homology group* $H_k(X)$ is the group of modulo 2 k-cycles in X.

In a general complex X two modulo 2 k-cycles are *homologous* (modulo 2) if their sum (=difference) is the sum of boundaries of some $(k+1)$-faces. The **modulo 2 homology group** $H_k(X)$ is the group of homology classes of modulo 2 k-cycles in X.

The homology class of a cycle x is denoted by $[x]$.

10.6.1. $H_0(X) \cong \mathbb{Z}_2^{c(X)}$, where $c(X)$ is the number of connected components of X.

10.6.2. (a) $H_s(D^n) = 0$ for $s > 0$.

(b) $H_s(\mathrm{Con}\, X) = 0$ for $s > 0$, where Con denotes the cone.

10.6.3. (a) If $X \searrow A$, then $H_s(X) \cong H_s(A)$ for every s. Moreover, then the inclusion $A \to X$ induces an isomorphism $H_s(A) \to H_s(X)$

(b) A subset $A \subset X$ is called a **deformation retract** of the set $X \subset \mathbb{R}^m$ if there exist a homotopy $f_t \colon X \to X$ such that $f_0 = \mathrm{id}\, X$, $f_1(X) \subset A$ and $f_1(a) = a$ for every $a \in A$. (Cf. Assertions 6.5.5 (b) and 14.1.5.)

If A is a deformation retract of X, then the inclusion $A \to X$ induces an isomorphism $H_s(A) \to H_s(X)$ for every s.

(c) A collapsing $X \searrow A$ generates a deformation retraction $X \to A$.

Below use without proof Assertion 10.6.3.b.

10.6.4. (a) For every $n>0$ the group $H_s(S^n)$ is 0 for $s\neq 0,n$ and is isomorphic to $\mathbb{Z}_2$ for $s=0,n$.

(b) For every closed connected PL n-manifold N we have $H_n(N)\cong\mathbb{Z}_2$.

(c) For every connected PL n-manifold N with non-empty boundary $H_n(N)=0$.

10.6.5. (a) For every $n>0$ the group $H_s(S^n\vee S^n)$ is 0 for $s\neq 0,n$ and is $\mathbb{Z}_2^2$ for $s=n$. Describe the generators of $H_n(S^n\vee S^n)$.

(b) For every $s>0$ we have $H_s(X\vee Y)\cong H_s(X)\oplus H_s(Y)$. Describe the isomorphism.

(c) For every $s\geqslant 0$ we have $H_s(X\sqcup Y)\cong H_s(X)\oplus H_s(Y)$.

(d) Find $H_s(S^n\times S^n)$. Describe the generators.

Definition of $\mathbb{C}P^n$ as a smooth $2n$-submanifold of $\mathbb{R}^d$ for some d is given analogously to the case of $\mathbb{R}P^n$ (Example 8.6.3.a). Use without proof Remark 10.3.8.

10.6.6. (a) The group $H_s(\mathbb{R}P^n)\cong\mathbb{Z}_2$ is generated by the class $[\mathbb{R}P^s]$ whenever $0\leqslant s\leqslant n$.

(b) Find $H_s(\mathbb{C}P^n)$ for $0\leqslant s\leqslant 2n$. Describe the generators.

10.6.7. Homology groups of PL homeomorphic complexes are isomorphic.

Theorem 10.6.8 (topological invariance). *Homology groups of topologically homeomorphic complexes are isomorphic.*

10.6.9. (a) Let Y be the complex obtained from X by a finite number of identifications of pairs of points. Then the quotient map $h: X\to Y$ induces an isomorphism $H_s(X)\cong H_s(Y)$ for every $s\geqslant 2$.

(b) For every $s\geqslant 0$ there is an isomorphism $H_s(X)\cong H_{s+1}(\Sigma X)$, where Σ denotes the suspension.

10.6.10. We have $\chi(X)=\sum_s(-1)^s\dim H_s(X)$.

For a simplicial map $f\colon X\to Y$ and an s-cycle C in X define the *image* f_*C to be the set of all s-faces σ in Y for which there is an odd number of s-faces τ in C such that $f(\tau)=\sigma$.

10.6.11. For a simplicial map $f: X\to Y$

(a) the image of any s-cycle in X is an s-cycle in Y;

(b) the correspondence $C \mapsto f_*C$ gives a well-defined map $f_* : H_s(X) \to H_s(Y)$

(c) we have $(\mathrm{id}\, X)_* = \mathrm{id}\, H_s(X)$;

(d) we have $(f \circ g)_* = f_* \circ g_*$ for a simplicial map $g : Y \to Z$ between complexes.

The map of (b) is called the **induced homomorphism**

$$f_* = H_s(f)\colon H_s(X) \to H_s(Y).$$

Sometimes we shorten f_* to f.

10.6.12. Denote $V_l^n = S_1^n \vee S_2^n \vee \ldots \vee S_l^n$. For a simplicial (w.r.t. some triangulation) map $g\colon V_l^n \to V_m^n$ denote by $d_{pq}(g)$ the degree modulo 2 of the composition $S_p^n \xrightarrow{\subset} V_l^n \xrightarrow{g} V_m^n \xrightarrow{\text{retraction}} S_q^n$. Then the induced homomorphism $g_*\colon H_n(V_l^n) \to H_n(V_m^n)$ is a linear map whose matrix in standard bases is $d_{qp}(g)$.

Definition of integer cycle, boundary, and integer homology group.

For $k > 0$, an *orientation* of a k-simplex is an ordering of its vertices up to an even permutation. An *orientation* of a 0-simplex (i.e. of a vertex) is assignment of $+1$ or -1 to this 0-simplex. Alternatively, an *orientation* of a k-simplex is a basis in a linear span of this simplex, up to orientation-preserving (in the sense of linear algebra) linear transformation. An *oriented simplex* is a simplex with some orientation. For an oriented simplex α denoted by $-\alpha$ the same simplex with the opposite orientation.

For $k > 0$ let $\sigma = (\sigma_0, \ldots, \sigma_{k+1})$ be an oriented k-simplex on vertices $\sigma_0, \ldots, \sigma_{k+1}$. For any $j \in \{0, \ldots, k+1\}$ denote by $\widehat{\sigma}_j$ the oriented $(k-1)$-face obtained by deleting σ_j from $(\sigma_0, \ldots, \sigma_{k+1})$. The oriented k-simplex σ *comes in (comes out of)* its oriented $(k-1)$-face if the orientation of the $(k-1)$-face coincides with $(-1)^j\widehat{\sigma}_j$ (with $(-1)^{j-1}\widehat{\sigma}_j$). Thus coming in / out of $(k-1)$-face depends on the orientation of the $(k-1)$-face, but the properties described below do not depend of this orientation. The (oriented) **boundary** of σ is $\partial\sigma := \sum\limits_{j=0}^{k}(-1)^j\widehat{\sigma}_j$.

Let X be a simplicial k-complex whose k-faces are oriented. An assignment of integers to oriented k-faces of X is a (simplicial) **integer k-cycle** in X if for every oriented $(k-1)$-face the sum of integers assigned to incoming oriented k-faces equals the sum of integers assigned to outcoming oriented k-faces. This is equivalent to the boundary of this assignment being zero, where the boundary is the homomorphism from assignments to integers defined as above on the basis. E.g. the boundary of an oriented $(k+1)$-face is an integer k-cycle.

Consider the componentwise sum operation on integer k-cycles in X.

If $\dim X = k$, then the *integer homology group* $H_k(X;\mathbb{Z})$ is the group of integer k-cycles in X.

In a general complex X two integer k-cycles are *homologous* if their difference is a linear combination with integer coefficients of boundaries of some $(k+1)$-faces. The **integer homology group** $H_k(X;\mathbb{Z})$ is the group of homology classes of integer k-cycles in X.

For a cell complex X integer cycle, boundary, and integer homology group are defined analogously. (The alternative definition of the orientation is used.)

10.6.13. State and prove the analogues of Problems 10.6.1—10.6.12 for homology with $\mathbb{Z}$-coefficients.

One can define homology with coefficients in $\mathbb{Z}_p$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ analogously. Further we sometimes specify coefficients $\mathbb{Z}_2$ in notation. Yet if we omit coefficients, we assume them to be $\mathbb{Z}_2$.

Hint to 10.6.2. (a) This follows from (b).

(b) $\partial \operatorname{Con} x = x$.

Hint to 10.6.7. The proof is analogous to that of Assertion 6.4.1 (c).

Hint to 10.6.9. (b) Every $(s+1)$-cycle in ΣX is homologous to the suspension of some s-cycle in X.

10.7. Definition of the intersection product in homology

In this section N is a PL n-manifold (see §10.3). After Poincare one studies the *intersection number* of *transverse* submanifolds or chains in N. The intersection number gives a bilinear *intersection product*

$$\cap_N = I_N = \cdot_N = \lambda_N \colon H_k(N;R) \times H_{n-k}(N;R) \to R$$

defined on the homology of N (here $R = \mathbb{Z}_2$ or, if N is oriented, we can take $R = \mathbb{Z}$). Cf. §§6.6,6.7. For $n = 2k$ this is the *intersection* *form* of N. The intersection product is closely related to the notions of characteristic classes (§9), linking form (§10.9), and signature (§11.4). These are important invariants used in the classification of manifolds.

In this section T is a triangulation (or a cellular decomposition) of N, and T^* is the dual decomposition (§10.4).

Define the **modulo 2 intersection product**

$$\cap_{N,2} : H_k(N) \times H_{n-k}(N) \to \mathbb{Z}_2 \quad \text{by} \quad [x] \cap_{N,2} [y] := |x \cap y| \mod 2,$$

where x and y are modulo 2 k-cycle in T and $(n-k)$-cycle in T^*.

Lemma 10.7.1 (cf. Assertion 6.7.1)**.** *This product is*
(a) well-defined; (b) bilinear; (c) symmetric for $n = 2k$.

Proof of (a). The product $H_k(T) \times H_{n-k}(T^*) \to \mathbb{Z}_2$ is well-defined because

(i) the intersection of a k-cycle modulo 2 in T and the boundary of an $(n-k+1)$-cell of T^* consists of an even number of points;

(ii) the intersection of the boundary of a $(k+1)$-cell of T and an $(n-k)$-cycle modulo 2 in T^* consists of an even number of points.

In this paragraph we prove assertion (i); assertion (ii) is proved analogously. Let σ be a k-face of T, and τ^* an $(n-k+1)$-face of T^*. Denote by τ the $(k-1)$-face of T dual to τ^*. We have $\sigma \cap \partial\tau^* \neq \varnothing$ if and only if $\sigma \supset \tau$. So (i) is equivalent to the above definition of a modulo 2 k-cycle in T.

The proof is completed using the PL invariance of homology (Assertion 10.6.7), and the analogue of Assertion 6.7.1.d for an n-manifold. □

For a proof of (c) we need the equivalence of the above definition of the intersection form to a different definition.

10.7.2. (a) Find the intersection product $H_1 \times H_2 \to \mathbb{Z}_2$ (i.e. find its matrix in some basis) of $S^1 \times S^2$, $(S^1)^3$, $\mathbb{R}P^3$.

(b) Find the intersection form (i.e. find its matrix in some basis) of $S^k \times S^k$, $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$.

(c) We have $[\mathbb{R}P^k] \cap_{\mathbb{R}P^n} [\mathbb{R}P^{n-k}] = 1$.

For this, use without proof the following Lemma 10.7.3 and Assertion 8.6.6.d. The result of Problem 10.7.2.b shows that for each $k = 1, 2, 4$ there are a smooth $2k$-manifold N and an element $x \in H_k(N)$

such that $x^2 = 1$. For other k this is false, see a classical proof in [KS21, footnote 1].

Let V and W be k- and $(n-k)$-submanifolds of N. They are (more precisely, the pair V, W is) called *transversal* if for any $x \in V \cap W$ there exists a closed neighborhood Ox of x in N, and a PL homeomorphism $\varphi : Ox \to [-1,1]^n$ such that

$$\varphi(V \cap Ox) = [-1,1]^k \times 0^{n-k} \quad \text{and} \quad \varphi(W \cap Ox) = 0^k \times [-1,1]^{n-k}.$$

Lemma 10.7.3. *Let V and W be closed transversal k- and $(n-k)$-submanifolds of N. Then $[V] \cap_N [W]$ equals the parity of $|V \cap W|$.*

Sketch of a proof. Cf. [INI, Theorem 2.1] on intersection number of immersions. A simpler proof of Lemma 10.7.3 is given by

$$[V] \bigcap_N [W] = [V \cap OW] \bigcap_{OW} [W] = [V \cap OW] \bigcap_{OV \cap OW} [W \cap OV] = |V \cap W|_2.$$

Here

• OV, OW are tubular (regular) neighborhoods of V, W,

• the following intersection products are defined analogously to the above:

$$\cap_{OW} : H_k(OW, \partial OW) \times H_{n-k}(OW) \to \mathbb{Z}_2$$

$$\cap_{OV \cap OW} : H_k(OV \cap OW, \partial OW) \times H_{n-k}(OV \cap OW, \partial OV) \to \mathbb{Z}_2$$

$$\cap : H_k(I^k \times I^{n-k}, \partial I^k \times I^{n-k}) \times H_{n-k}(I^k \times I^{n-k}, I^k \times \partial I^{n-k}) \to \mathbb{Z}_2;$$

• the last equality holds by the transversality and because $[I^k \times 0] \cap [0 \times I^{n-k}] = 1$. □

For a $2k$-manifold N denote by $\operatorname{rk} N = \operatorname{rk} \cap_N$ the rank of the intersection form of N. The result of Problem 10.7.2.b shows that $\operatorname{rk}(S^k \times S^k) = 2$ and $\operatorname{rk} \mathbb{R}P^2 = \operatorname{rk} \mathbb{C}P^2 = \operatorname{rk} \mathbb{H}P^2 = 1$.

10.7.4 (monotonicity)**.** If $N_1 \subset N_2$ are PL $2k$-manifolds, then $\operatorname{rk} N_1 \leqslant \operatorname{rk} N_2$.

10.7.5. (a) $\operatorname{rk}(M_1 \sqcup M_2) = \operatorname{rk} M_1 + \operatorname{rk} M_2$.

(b) $\operatorname{rk}(M_1 \# M_2) = \operatorname{rk} M_1 + \operatorname{rk} M_2$.

(c) There are PL 4-manifolds intersecting by the 4-ball, having the same rank $r > 0$, and whose union has the same rank r. (Then $\operatorname{rk}(M_1 \cup M_2) < \operatorname{rk} M_1 + \operatorname{rk} M_2$.)

(d) If two PL $2k$-manifolds intersect by the $2k$-ball, then $\operatorname{rk}(M_1 \cup M_2) \leqslant \operatorname{rk} M_1 + \operatorname{rk} M_2$.

(e) The rank of manifolds is not additive: if $M_1 = M_2 = S^1 \times I$ and $M_1 \cup M_2 = S^1 \times S^1$, then $\operatorname{rk}(M_1 \cup M_2) = 1 > 0 = \operatorname{rk} M_1 + \operatorname{rk} M_2$.

Lemma 10.7.6 (superadditivity)**.** *Let M_1 and M_2 be compact orientable $2k$-manifolds, and $M_1 \cup M_2$ the union along some boundary components. Then* $\operatorname{rk}(M_1 \cup M_2) \geqslant \operatorname{rk} M_1 + \operatorname{rk} M_2$.

Proof. Let M_1' be the complement in M_1 to a collar of ∂M_1. Then by Assertions 10.7.4 and 10.7.5.a)

$$\operatorname{rk}(M_1 \cup M_2) \geqslant \operatorname{rk}(M_1' \sqcup M_2) = \operatorname{rk} M_1 + \operatorname{rk} M_2.$$

□

Definition of the integer intersection product for oriented N. Take oriented dual faces σ of T and σ^* of T^* intersecting at a point S.

If T is a triangulation of a smooth manifold N, then N is contained in $\mathbb{R}^d$ for some $\mathbb{R}^d$. In the tangent space of N at S take a base of the tangent subspace corresponding to the orientation of σ. Take an analogous base for σ^*. If the ordered pair of these bases forms the orientation of N, the orientations on σ and on σ^* are said to be *agreeing*.

Assume that T is a triangulation of a PL manifold N. Denote $k := \dim \sigma$. Let T' be the barycentric subdivision of T, one of whose vertices is S. Take an ordering $(S, A_1, \ldots, A_k)$ of vertices of a k-face of T' contained in σ, corresponding to the orientation of σ. Analogously, take an ordering $(S, B_1, \ldots, B_{n-k})$ of vertices of an $(n-k)$-faces of T' contained in σ^*, corresponding to the orientation of σ^*. The vertices of the k-face and of the $(n-k)$-face form together an n-face of T'. Then $(S, A_1, \ldots, A_k, B_1, \ldots, B_{n-k})$ is an ordering of vertices of the n-face. If this ordering forms the orientation of N, the orientations on σ and on σ^* are said to be *agreeing*.

Analogously one defines agreeing orientations on faces of T and of T^* when T is a cellular decomposition.

Take agreeing orientations on faces of T and of T^*. In this definition we make summations over all oriented k-faces σ of T. Take an integer k-cycle $x = \sum_\sigma x_\sigma \sigma$ in T. Analogously, take an integer $(n-k)$-cycle

$y = \sum_\sigma y_{\sigma^*}\sigma^*$ in T^*. Define the *integer intersection product*

$$\cap_{N;\mathbb{Z}} : H_k(N;\mathbb{Z}) \times H_{n-k}(N;\mathbb{Z}) \to \mathbb{Z} \quad \text{by} \quad [x] \cap_{N;\mathbb{Z}} [y] := \sum_\sigma x_\sigma y_{\sigma^*}.$$

Analogously to the modulo 2 case, the product of an integer k-cycle and a boundary of an $(n-k+1)$-face is zero. This and the PL invariance of homology (Theorem 10.6.8) imply that the integer intersection product is well-defined.

The integer intersection product is bilinear. Hence it vanishes on torsion elements. Thus it descends to a bilinear (integer) intersection pairing

$$H_k(N;\mathbb{Z})/\mathrm{Torsion} \times H_{n-k}(N;\mathbb{Z})/\mathrm{Torsion} \to \mathbb{Z}.$$

on the free modules.

10.7.7. (a) We have $x \cap_{N;\mathbb{Z}} y = (-1)^{k(n-k)} y \cap_{N;\mathbb{Z}} x$.

(For a proof we need the equivalence of the above definition of the integer intersection product to a different definition.) Hence for $n = 2k$ the form $\cap_{N;\mathbb{Z}}$ is symmetric when k is even, and is skew-symmetric when k is odd.

(b) For every odd k, $2k$-manifold N and $x \in H_k(N;\mathbb{Z})$ we have $x^2 = 0$.

(c) For every even k there are $2k$-manifold N and $x \in H_k(N;\mathbb{Z})$ such that $x^2 = 2$.

Lemma 10.7.8. *Let V and W be closed oriented transversal k- and $(n-k)$-submanifolds of N. Then $[V] \cap_{N;\mathbb{Z}} [W]$ equals the sum $V \cdot W$ of signs of the intersection points of V, W.*

By the rank of a bilinear form on a $\mathbb{Z}$-module we mean its rank over $\mathbb{Q}$. Then the integer analogues of Assertions 10.7.4-10.7.6 hold.

Analogously formula $[a] \cap [b] := a \cap b$ gives a well-defined bilinear *intersection product* $H_s(N) \times H_t(N) \to H_{s+t-n}(N)$.

10.7.9. (a) $[\mathbb{R}P^s] \cap [\mathbb{R}P^t] = [\mathbb{R}P^{s+t-n}] \in H_{s+t-n}(\mathbb{R}P^n)$.

(b) The set $(H_1 \oplus \ldots \oplus H_{3r})((\mathbb{R}P^3)^r)$ with operation of summation and multiplication is generated by these operations from elements $a_1, \ldots, a_r$, where a_i is represented by the Cartesian product of $\mathbb{R}P^2$ on i-th place and $\mathbb{R}P^3$ on other places. All relations of polynomials in $a_1, \ldots, a_r$ are generated by $a_i^4 = 0$.

10.7.10*. If $n=4$ and N is closed orientable, then $x^4=0$ for any $x\in H_3(N)$.

10.7.11*. Let N be a closed oriented connected 4-submanifold of a closed oriented connected 6-manifold M. Denote by $\overline{e}\in H_2(N;\mathbb{Z})$ the obstruction to construction on N of a non-zero vector field tangent to M and normal to N. Then $[N]^3=\overline{e}\cap\overline{e}\in\mathbb{Z}$.

Hint to 10.7.4. Take the 'plumbing union' (cf. the definition after Assertion 11.9.3) of two copies of punctured $\mathbb{C}P^2$.

Hint to 10.7.7. (c) Take $N=S^k\times S^k$ and $x=[S^k\times 0]+[0\times S^k]$.

Hint to 10.7.11. Assume $N_1,N_2\subset M$ are submanifolds close to N in general position to each other and to N. Then

$$[N]^3=[N_1]\cap[N_2]\cap[N]=\#(N_1\cap N_2\cap N)=$$
$$=\#[(N_1\cap N)\cap(N_2\cap N)]=\overline{e}\cap\overline{e}.$$

Here $\#$ denotes the algebraic sum of the intersection points in M, and $N_1\cap N$, $N_2\cap N$ are *oriented* intersections in N.

10.8. Poincaré duality for 3-manifolds

Theorem 10.8.1 (Poincaré duality modulo 2)**.** *For any triangulation T of a closed 3-manifold*

(a) *(easy part) $H_1(T)\cong H_2(T)$;*

(b) *the product $\cap\colon H_s(T)\times H_{3-s}(T)\to\mathbb{Z}_2$ is non-degenerate for each $s=1,2$.*

An alternative definition of homology groups via a chain complex. For $s=0,1,2,\dots$ denote by c_s the number of s-faces of K. Denote by $C_s=C_s(K)$ the group of assignments of zeros and units to s-faces (with componentwise summation). Clearly, $C_s\cong\mathbb{Z}_2^{c_s}$.

For an arbitrary edge a denote by $\partial_0 a$ the assignment of units to vertices of this edge and zeros to all other vertices. 'Extend' ∂_0 to the linear map $\partial_0\colon C_1\to C_0$. Similarly, for an arbitrary s-face a denote by $\partial_{s-1}a$ the assignment of units to $(s-1)$-faces of the boundary of a. 'Extend' ∂_{s-1} to the linear map $\partial_{s-1}\colon C_s\to C_{s-1}$.

Groups $\partial_{s-1}^{-1}(0)\subset C_s$ and $\partial_s C_{s+1}\subset C_s$ are called the groups of *s-cycles* and of *s-boundaries* respectively. We have

$$H_0(K):=C_0/\partial_0 C_1\quad\text{and}\quad H_s(K):=\partial_{s-1}^{-1}(0)/\partial_s C_{s+1}.$$

10.8.2. Any boundary is (indeed) a cycle: $\partial_s\partial_{s+1} = 0$.

Proof of Theorem 10.8.1.a. Recall that the number of s-faces is denoted by c_s. The dimension of the linear space $\partial_0^{-1}(0)$ of 1-cycles equals $c_1 - \operatorname{rk}\partial_0$. Analogously for the 1-boundaries we have $\dim \partial_1(\mathbb{Z}_2^{c_2}) = \operatorname{rk}\partial_1$. Thus $\dim H_1(T) = c_1 - \operatorname{rk}\partial_0 - \operatorname{rk}\partial_1$.

Using the dual decomposition T^* instead of T, define analogously to §10.6 numbers c_{i*} and maps $\partial_{i*} \colon \mathbb{Z}_2^{c_{i+1,*}} \to \mathbb{Z}_2^{c_{i*}}$ for $i = 1, 2$. We obtain analogously $\dim H_2(T^*) = c_{2*} - \operatorname{rk}\partial_{1*} - \operatorname{rk}\partial_{2*}$.

Clearly, $c_{2*} = c_1$. It is also clear that for faces α, β of the triangulation T the condition $\alpha \subset \beta$ is equivalent to the condition $\beta^* \subset \alpha^*$. Hence the matrices of ∂_{2*} and ∂_{1*} (in the standard bases) equal the *transposed* matrices of ∂_0 and ∂_1, respectively. So $\operatorname{rk}\partial_{2*} = \operatorname{rk}\partial_0$ and $\operatorname{rk}\partial_{1*} = \operatorname{rk}\partial_1$. Hence, $\dim H_1(T) = \dim H_2(T^*) = \dim H_2(T)$. □

Sketch of proof of Theorem 10.8.1.b. First suppose that $s = 2$. It suffices to prove the theorem for N connected. By Assertion 9.4.7 (a) any class $\alpha \in H_2(T)$ can be represented by some triangulation F of a connected closed 2-manifold. If $\alpha \neq 0$ then $N - F$ is connected (otherwise F is null-homologous as the boundary of any connected component of $N - F$). Choose a small arc transversally intersecting F at a unique point. Since $N - F$ is connected, we can join the ends of this arc by a polygonal line outside F. The union of this arc and this polygonal line is a 1-cycle which transversally interest F at a unique point. So the homology class of this 1-cycle is the required one.

The case $s = 1$ follows from the case $s = 2$ and part (a). □

For every finitely generated abelian group G denote by

- $T = TG \subset G$ the *torsion*, i.e., the subgroup of elements of finite order;
- $F = FG$ the *free part*, i.e., the group G/TG.

10.8.3. For any closed orientable 3-manifold N

(a) the order of any non-zero element of the group $H_2(N;\mathbb{Z})$ is infinite;

(b) $H_2(N;\mathbb{Z}) \cong FH_1(N;\mathbb{Z})$.

Hint. (a) Assume to the contrary that there exist a 3-chain y and a 2-cycle z such that $\partial y = kz$ for some integer $k > 1$. The multiplicity (in the chain y) of a 3-simplex not contained in y equals zero. So this multiplicity is divisible by k. If the multiplicity of some 3-simplex is divisible by k, then the multiplicity of any adjacent 3-simplex is also divisible by k. So $y = ky_1$. Then $\partial y_1 = z$. Hence, $[z] = 0 \in H_1(N;\mathbb{Z})$.

10.9. Poincaré duality for n-manifolds

Theorem 10.9.1 (Poincaré duality modulo 2, easy part)**.** *For any closed n-manifold N we have $H_s(N) \cong H_{n-s}(N)$.*

The proof is analogous to that of Theorem 10.8.1 (a).

Theorem 10.9.2 (Poincaré duality, easy part)**.** *For any closed n-manifold N*

- *the free parts of the groups $H_s(N;\mathbb{Z})$ and $H_{n-s}(N;\mathbb{Z})$ are isomorphic;*
- *the torsion subgroups of the groups $H_s(N;\mathbb{Z})$ and $H_{n-s-1}(N;\mathbb{Z})$ are isomorphic.*

The proof is analogous to those of Assertion 10.8.3 and Theorem 10.9.1, see details in [ST34, §69].[26]

Theorem 10.9.3 (Poincaré duality modulo 2)**.** *For any closed n-manifold N the product $\cap\colon H_s(N) \times H_{n-s}(N) \to \mathbb{Z}_2$ is non-degenerate.*

Proof. [27] We use orthogonal complements with respect to the modulo 2 intersection product $I_{T,2} : C_s(T) \times C_{n-s}(T^*) \to \mathbb{Z}_2$. It suffices to prove that

$$^{\perp}Z_{n-s}(T^*) = B_s(T) \quad \text{and} \quad Z_s(T)^{\perp} = B_{n-s}(T^*).$$

Let us prove the left-hand equality; the right-hand equality is proved analogously. Since $I_{T,2}$ is non-degenerate, we only need to check that $B_s(T)^{\perp} = Z_{n-s}(T^*)$. The inclusion $B_s(T)^{\perp} \supset Z_{n-s}(T^*)$ is obvious. The opposite inclusion follows because if $I_{N,2}(\partial c, d) = 0$ for an $(s+1)$-cell c of T and a chain $d \in C_{n-s}(T^*)$, then ∂d does not involve the cell c^* dual to c. □

For a closed orientable n-manifold N define the **linking product**

$$\mathrm{lk}\colon TH_s(N;\mathbb{Z}) \times TH_{n-1-s}(N;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

[26] Sometimes the easy part of Poincaré duality is proved using the intersection number. This only make the proof more complicated. However, the intersection number is useful, for example, to prove the following 'hard part' of Poincaré duality.

[27] Cf. Assertion 6.7.5 (e). The proof of Theorem 10.8.1 can be generalized to the cases $s = 1, n-1$ of Theorem 10.9.3. One should use Assertion 14.9.3 (b) instead of Assertion 9.4.7 (a). But this approach can not be generalized to other cases.

by the formula

$$\mathrm{lk}([a],[b]) := \left\{\frac{a\cap B}{k}\right\}, \quad \text{where} \quad \partial B = kb.$$

Sketch of a proof that the linking product is well-defined. Independence of the choice of the chain B. Assume $\partial B' = \partial B = kb$. Since $\partial(B'-B)=0$ and a has finite order, we have $a\cap B' - a\cap B = a\cap(B'-B) = 0$.

Independence of the choice of the chain b follows from the independence of the choice of the chain B, because $\partial(B+kc) = k(b+\partial c)$.

Independence of choice of cycle a. We have

$$(a+\partial A)\cap B - a\cap B = \partial A\cap B = \pm A\cap\partial B = \pm kA\cap b.$$

$\square$

10.9.4. (a) We have $\mathrm{lk}(a,a) = 1/2$ for the generator $a\in H_1(\mathbb{R}P^3)$.

(b) Find the linking product for $L(p,q)$.

(c) The linking product of a class of order A and a class of order B is a class of order $\gcd(A,B)$.

(d) For $n=2s+1$ we have $\mathrm{lk}(\alpha,\beta) = \pm\,\mathrm{lk}(\beta,\alpha)$.

Theorem 10.9.5 (Poincaré duality). *For any closed orientable n-maniflod N*

- *the integer product $\cap$ is unimodular (i.e. for any $\alpha\in H_s(N;\mathbb{Z})$ not divisible by any integer greater than 1 there exists $\beta\in H_{n-s}(N;\mathbb{Z})$ such that $\alpha\cap\beta = 1\in\mathbb{Z}$);*
- *the linking product* lk *is non-degenerate*[28].

Sketch of a proof. (The textbook [ST34, §69, §71, Proposition 2] and text by S. Avvakumov were used to write this sketch.) Choose a triangulation and the dual cellular decomposition. For every s choose the natural base of the group of s-chains of the triangulation, and the dual base of the group of $(n-s)$-chains of the dual decomposition.

[28]This 'hard part' of the Poincaré duality Theorems 10.9.3, 10.9.5 often either is not proved (for example, in [FF89] Theorem 6 of §17 in p. 148 is claimed to be trivial), or is proved using cohomology and universal coefficient formula, which makes the proof more complicated. Cohomology is indeed useful to work with differential forms, to study algebraic geometry or homotopy topology of manifolds with boundary or arbitrary complexes. In many textbooks cohomology of *manifolds* is introduced much earlier than the problems for which cohomology is necessary. As a result, cohomology is used to make proofs more complicated.

11.5. The Mayer—Vietoris sequence

Here we discuss an analogue of the inclusion/exclusion principle for homology.

The Mayer—Vietoris sequence

$$\to H_s(A\cap B)\xrightarrow{i_A\oplus i_B} H_s(A)\oplus H_s(B)\xrightarrow{I_A\oplus I_B} H_s(A\cup B)\xrightarrow{\gamma} H_{s-1}(A\cap B)\to$$

is a sequence of groups and homomorphisms defined as follows. The homomorphisms $i_A\oplus i_B$ and $I_A\oplus I_B$ are the sums of inclusion-induced homomorphisms. The homomorphism γ is any of the following compositions:

$$H_s(A\cup B)\xrightarrow{j} H_s(A\cup B, A)\xrightarrow{\text{ex}} H_s(B, A\cap B)\xrightarrow{\partial} H_{s-1}(A\cap B)\quad\text{and}$$

$$H_s(A\cup B)\xrightarrow{j} H_s(A\cup B, B)\xrightarrow{\text{ex}} H_s(A, A\cap B)\xrightarrow{\partial} H_{s-1}(A\cap B).$$

11.5.1. (a) If $H_s(A\cap B)=0=H_{s-1}(A\cap B)$, then $I_A\oplus I_B$ is an isomorphism.

(b) If $H_{s+1}(A\cup B)=0=H_s(A\cap B)$, then $i_A\oplus i_B$ is an isomorphism.

(c) The compositions defining γ are equal.

(d) If $H_s(A)=H_s(B)=0=H_{s-1}(A)=H_{s-1}(B)$, then γ is an isomorphism.

Let N be a codimension c submanifold of a manifold M and $y\in H_s(M)$ is represented by an s-cycle Y transverse to N. Define $y\cap N:=[Y\cap N]\in H_{s-c}(N)$. Analogously define the **restriction homomorphism** (or intersection) $H_s(M,\partial)\to H_{s-c}(N,\partial)$. (For specialists: this is the homological version of restriction: $y\cap N:=\mathrm{PD}((\mathrm{PD}y)|_N)$.) For inclusion $i\colon N\to M$ we have

$$y\cap[N]=i_*(y\cap N)\in H_{s-c}(M).$$

Sketch of a proof of the surjectivity in Assertion 11.5.1 (b) *for n-manifolds A and B intersecting by their common boundary.* Take arbitrary cycles a in A and b in B. There exists a chain C in $A\cup B$ such that $\partial C=a+b$. Let $c:=C\cap(A\cap B)$ Then $i_A[c]=a$ and $i_B[c]=b$.

Theorem 11.5.2. *The Mayer—Vietoris sequence is exact.*
(For $\mathbb{Z}$-coefficients one has $(-I_B)$ instead of I_B.)

11.5.3. For a matrix $M \in SL_2(\mathbb{Z})$ let f_M be the corresponding linear automorphism of the torus T. Calculate the homology groups of the space

(a) $N \cup_{f_M} D^2 \times S^1$, where N is a given 3-manifold with boundary $\partial N = T$;

(b) $T \times I/(x,0) \sim (f_M(x),1)$.

11.6. Alexander—Pontryagin duality

In this section we assume N to be a subhypergraph in some triangulation of the space $\mathbb{R}^m$ or of the sphere S^m.

11.6.1. (a) (*Higher-dimensional Jordan Theorem*) For any closed $(m-1)$-submanifold $N \subset \mathbb{R}^m$ the complement $\mathbb{R}^m - N$ is non-connected and consists exactly of two connected components.

(b) For any subhypergraph $N \subset \mathbb{R}^m$ we have $H_m(N) = 0$.

П. (a) можно доказать аналогично (кусочно-линейной) теореме Жордана для плоскости. Приведем более абстрактный способ изложить это доказательство, полезный для обобщений.

Denote by ON the regular neighborhood of N in $\mathbb{R}^m$ (see definition in §10.5). Let $C_N := S^m - \operatorname{Int} ON$. Let $p\colon ON \to N$ be the retraction.

Рассмотрим следующие гомоморфизмы:

$$H_m(C_N,\partial) \stackrel{\text{ex}}{\to} H_m(S^m, ON) \stackrel{\partial}{\to} H_{m-1}(ON) \stackrel{p_*}{\to} H_{m-1}(N).$$

Здесь ex и p_* — изоморфизмы (второй — ввиду гомотопической инвариантности гомологий). Рассмотрим точную последовательности пары (S^m, ON) (точнее, ее следующий фрагмент):

$$H_m(ON) \stackrel{i}{\to} H_m(S^m) \stackrel{j}{\to} H_m(S^m, ON) \stackrel{\partial}{\to} H_{m-1}(ON) \stackrel{i}{\to} H_{m-1}(S^m).$$

Получаем, что ∂ — эпиморфизм, не являющийся изоморфизмом. Используя это для коэффициентов $\mathbb{Z}_2$, ввиду $H_{m-1}(N;\mathbb{Z}_2) \ne 0$ получаем, что C_N несвязно.

Утверждение 11.6.1.b доказывается аналогично при помощи изоморфизмов ex, p_* и точной последовательности пары (S^m, ON):

$$H_{m+1}(S^m, ON) \stackrel{\partial}{\to} H_m(ON) \stackrel{i}{\to} H_m(S^m) \stackrel{j}{\to} H_m(S^m, ON).$$

Perhaps Alexander, trying so to distinguish knots (see Assertion 10.1.3), proved part (a) of the following assertion.

11.6.2. (a) For any closed non-self-intersecting polygonal line $N \subset \mathbb{R}^3$ we have[31] $H_1(\mathbb{R}^3 - N; \mathbb{Z}) \cong \mathbb{Z}$.

(b) For any closed connected orientable 2-submanifold (i.e. for the sphere with handles) $N \subset \mathbb{R}^4$ we have $H_1(\mathbb{R}^4 - N; \mathbb{Z}) \cong \mathbb{Z}$.

(c) If $A \subset S^m$ is a connected m-submanifold, then $H_{s+1}(A, \partial) \cong$ $\cong H_s(S^m - \mathrm{Int}\, A)$ for every $s = 0, 1, \dots, m-2$.

Hints. Parts (a,b) (and Theorem 11.6.3) доказываются при помощи точной последовательности пары (S^m, ON) (see Assertion 11.6.4) and applying Lefschetz duality 11.2.3 (a) to m-manifold C_N. Part (c) is proved using the excision isomorphism 11.2.1 (b) and the exact sequence of pair (S^m, C_A): $H_{s+1}(A, \partial) \overset{\text{ex}}{\cong} H_{s+1}(S^m, C_A) \overset{\partial}{\cong} H_s(C_A)$.

This lead Alexander to the discovery of the Alexander duality 11.6.3, which also generalizes the Euler formula for plane graphs, and Assertions 11.6.1, 11.6.2. In the rest of this section s is any integer from 0 to $m-1$.

Theorem 11.6.3 (Alexander duality)**.** *We have*

$$\widetilde{H}_s(N) \cong \widetilde{H}_{m-s-1}(S^m - N).$$

If N is an orientable manifold, then

- *the free parts of the groups $\widetilde{H}_s(N; \mathbb{Z})$ and $\widetilde{H}_{m-s-1}(S^m - N; \mathbb{Z})$ are isomorphic*;
- *the torsion subgroups of the groups $\widetilde{H}_s(N; \mathbb{Z})$ and $\widetilde{H}_{m-s-2}(S^m - N; \mathbb{Z})$ are isomorphic.*

Proposition 11.6.4. *We have $\widetilde{H}_s(N) \cong H_{s+1}(C_N, \partial)$.*

More precisely, the following compositions are equal and are isomorphisms:

$$H_{s+1}(C_N, \partial) \overset{\text{ex}}{\to} H_{s+1}(S^m, ON) \overset{\partial}{\to} H_s(ON) \overset{p_*}{\to} H_s(N),$$

$$H_{s+1}(C_N, \partial) \overset{\partial}{\to} H_s(\partial C_N) \xrightarrow{p_*} H_s(N).$$

[31] In this text we did not define the homology groups of non-compact spaces. A reader may give such definition by himself/herself or replace $S^m - N$ by C_N everywhere.

§ 12. Non-embeddability and non-immersibility

It startled the well informed by being a new and fantastic idea they had never encountered. It startled the ignorant by being an old and familiar idea they never thought to have seen revived.

G. K. Chesterton. The Man Who Knew Too Much

12.1. Introduction and Main Results

In 1935 Hopf announced the results of Stiefel on the collections of tangent vector fields, and his invention of characteristic classes (§ 9). This happened at the International topology conference in Moscow. It turned out that around 1934 Hassler Whitney also naturally arrived at the definition of characteristic classes in the course of his study of the embeddability problem (§11.1).

We work in the smooth category; that is, all manifolds, vector fields, and maps are assumed smooth, while the word 'smooth' is omitted.

Theorem 12.1.1 (Whitney)**.** (a) *Any n-dimensional manifold is embeddable into $\mathbb{R}^{2n}$ and immersible in $\mathbb{R}^{2n-1}$.*

(b) *If n is a power of two, then $\mathbb{R}P^n$ is not immersible in $\mathbb{R}^{2n-2}$ and not embeddable into $\mathbb{R}^{2n-1}$.*

Part (a) is not proved in this book, see the proofs in [Ad93, Pr14′]. Part (b) follows from Assertion 12.1.2 (a).

Proposition 12.1.2. (a) *If $\mathbb{R}P^n$ is embeddable into $\mathbb{R}^m$ or immersible in $\mathbb{R}^{m-1}$, then $\binom{m}{n}$ is even.*[36]

(b) *If $n = 2^{n_1} + \ldots + 2^{n_k}$ is the binary expansion of n, then $\mathbb{R}P^{2^{n_1}} \times \ldots \times \mathbb{R}P^{2^{n_k}}$ is not immersible in $\mathbb{R}^{2n-k-1}$ and not embeddable into $\mathbb{R}^{2n-k}$.*

(c) *If $\mathbb{R}P^n$ is immersible in $\mathbb{R}^{n+1}$, then either $n+1$ or $n+2$ is a power of two.*

[36] Therefore, $\binom{i}{n}$ is even for any $i = m, m+1, \ldots, 2n$, and moreover, $\binom{i}{s}$ is even for any $s = 1, 2, \ldots, n$ and any $i = m, m+1, \ldots, 2s$; the latter follows from the former and from the Pascal identity, hence not giving any new information.

This assertion follows from the Whitney Obstruction Lemma 12.2.3 (b) and Assertion 12.2.4 (c). For part (b) one also needs Assertion 12.2.4 (b), and for part (c) one needs Assertion 12.3.2.

Conjecture 12.1.3 (Massey). *Denote by $\alpha(n)$ the number of 1's in the binary expansion of n. Then any n-dimensional manifold is immersible*[37] *in $\mathbb{R}^{2n-\alpha(n)}$ and embeddable into $\mathbb{R}^{2n+1-\alpha(n)}$.*

Theorem 12.1.4. *Let N be a manifold of dimension n. If n is not a power of two, or if N in non-orientable, or if N is not closed, then N is embeddable into $\mathbb{R}^{2n-1}$ and (for $n \geqslant 3$) immersible in $\mathbb{R}^{2n-2}$.*

See the references after [Sk08, Theorem 2.4 (a)]. Of the proof of Theorem 12.1.4, we will only outline the easier part, the proof of the Massey Theorem 12.7.1. The harder part is a partial converse of the Whitney Obstruction Lemma 12.2.3 (b), see survey [Sk08, Theorem 2.12].

12.1.5. (a) The product of any k 2-manifolds is immersible in $\mathbb{R}^{3k}$ (and embeddable into $\mathbb{R}^{3k+1}$).

(b) $(\mathbb{R}P^2)^k$ is not immersible in $\mathbb{R}^{3k-1}$ (and not embeddable into $\mathbb{R}^{3k}$).

(c) $\mathbb{C}P^2$ is not immersible in $\mathbb{R}^5$ (and not embeddable into $\mathbb{R}^6$).

For part (a) one needs immersability of any 2-manifold in $\mathbb{R}^3$. For part (b,c) one needs the Whitney Obstruction Lemma 12.2.3 (b, c), the result of Problem 12.2.4 (a), and the facts that $w_1(\mathbb{R}P^2) \neq 0$, $w_2(\mathbb{R}P^2) \neq 0$, $w_1(\mathbb{C}P^2) = 0$ and $w_2(\mathbb{C}P^2) \neq 0$. See also Assertion 11.1.2 (e). The lowest dimension of the Euclidean space in which a given product of 2-manifolds is immersible (embeddable) is determined in [ARS01].

Hint to 12.1.5.b and to 12.2.4.a. Denote $a := [\mathbb{R}P^1] \in H_1(\mathbb{R}P^2)$ and $N_k := (\mathbb{R}P^2)^k$. Since $w(\mathbb{R}P^2) = 1 + a + a^2$, we obtain $\overline{w}(\mathbb{R}P^2) = 1 + a$.

We have $\overline{w}_2(N_2) = a \times a \neq 0$, where the equality holds since $\overline{w}(N_2) = (1 \times 1 + a \times 1)(1 \times 1 + 1 \times a) = \ldots + a \times a$ by the Whitney—Wu formula 12.2.4.b, and the 'non-equality' holds since $(a \times a) \cap (a \times a) = 1 \neq 0 \in \mathbb{Z}_2$.

Analogously $\overline{w}_k(N_k) = a^{\times k} \neq 0$.

[37] I have to warn the reader that some experts at a conference asserted that the proof [Co85] of this conjecture is not complete. As far as I know, no public criticism has appeared.

12.2. Collections of normal fields

The proofs of non-embeddability and non-immersibility are based on considerations of *collections of normal vector fields* (on a submanifold, or for an immersion). While studying the obstructions for the existence of such collections, Whitney introduced the normal (dual) Stiefel—Whitney classes of a manifold. Since the time of Whitney's work these classes play a great role in topology and differential geometry. A generalization is the theory of vector bundles (see § 13; even though formally § 13 does not depend on § 12, it helps to work a bit with collections of normal fields to motivate the notion of a vector bundle from § 13).

In this and the following sections, N is any closed connected n-manifold, $w_s := w_s(N)$, and $f\colon N \to \mathbb{R}^m$ is any immersion.

A **normal vector field** to f is a collection of vectors $v(x)$ at points $x \in f(N)$, vectors normal to the image $f(Ox)$ of some neighborhood Ox of x (shortly: to f), and depending continuously on $x \in N$.

A normal vector field need not exist. E.g. it does not exist for the Möbius band in $\mathbb{R}^3$.

Proposition 12.2.1. *For any immersion $f\colon N \to \mathbb{R}^{2n+1}$, there exists a normal vector field to f.*

Proposition 12.2.2. *(a) If N immerses in $\mathbb{R}^{n+1}$, then $w_2 = w_1^2$;*
(b) If N immerses in $\mathbb{R}^{n+2}$, then $w_3 = w_1^3$;
(c) If N immerses in $\mathbb{R}^{n+3}$, then $w_4 + w_2^2 + w_2 w_1^2 + w_1^4 = 0$.

Lemma 12.2.3 (Whitney Obstruction)**.** *(a) There exist unique classes $\overline{w}_s(N) \in H_{n-s}(N)$, $s = 0, 1, \ldots, n$, for whose sum $\overline{w}(N)$ one has $\overline{w}(N) \cap w(N) = 1$ (see the definition of $w(N)$ after Theorem 9.9.4).*
(b) If N immerses in $\mathbb{R}^m$, then $\overline{w}_s(N) = 0$ for any $s > m - n$.
(c) If N embeds into $\mathbb{R}^m$, then $\overline{w}_s(N) = 0$ for any $s \geqslant m - n$.

Comments on the proof. Part (a) can be easily proven by induction on s. The following equalities follow:

$$\overline{w}_1(N) = w_1, \quad \overline{w}_2(N) = w_2 + w_1^2, \quad \overline{w}_3(N) = w_2 + w_1^3,$$
$$\overline{w}_4(N) = w_4 + w_2^2 + w_2 w_1^2 + w_1^4.$$

Therefore, Assertions 12.2.2 are special cases of part (b).

Part (b) is non-trivial. The ideas of the proof of non-immersibility are shown in §12.3 and §12.4 in special cases, Assertions 12.2.2. The proof is sketched in §12.5.

The classes $\overline{w}_s(N)$ are called the **normal Stiefel—Whitney classes**.

12.2.4. (a) Compute $\overline{w}((\mathbb{R}P^2)^k)$.

(b) $\overline{w}(M \times N) = \overline{w}(M) \times \overline{w}(N)$.

(c) For any s, $0 \leqslant s \leqslant n$, we have $\overline{w}_s(\mathbb{R}P^n) = 0$ if and only if $\binom{n+s}{s}$ is even. (Use without proof Assertion 9.9.6.a.)

Proposition 12.2.5. (a) *When $m \geqslant 3n/2 + 1$ or $m \leqslant n + 3$, any embedding of S^n into $\mathbb{R}^m$ admits a normal $(m-n)$-tuple* [Ke59].

(b) *For any $n = 4l - 1 \geqslant 7$ and $m = 4l + 2, 4l + 3, \dots, 6l - 1$, there exists an embedding of S^n into $\mathbb{R}^m$ that does not admit a normal $(m-n)$-tuple* [Ha66, 6.8].

(c) *Any embedding of a closed orientable 3-manifold into $\mathbb{R}^6$ admits a normal triple.*

(d) *No embedding $\mathbb{C}P^2 \to \mathbb{R}^8$ admits a normal quadruple.*

Embeddings from (c,d) exist by the Whitney Theorem 12.1.1 (a).

Part (a) is proved for $m \leqslant n + 2$ in §8.6, while for $m \geqslant 3n/2 + 1$ part (a) follows from the Kervaire Theorem 15.2.4. Part (d) follows from $w_2(\mathbb{C}P^2) \neq 0$ and from the Whitney—Wu formula 12.6.3 (b). We do not prove part (a) for $m = n + 3 \geqslant 7$, part (b), and the general case of part (c). (The latter is proved in [Sk06m] but was known before.)

Outline of the proof of part (c) *for $N = S^3$.* By Normal Field Theorem 8.7.8 there is a unit normal vector field. Prove that the obstruction to existence of a unit vector field normal both to $f(S^3)$ and to the constructed normal field, vanishes.

A complete solution of the following problem is not known.

The Hirsch problem. For what m and what manifolds N any embedding $N \to \mathbb{R}^m$ admits a normal $(m-n)$-tuple?

In the following sections, if a vector field in $\mathbb{R}^m$ on $f(N)$ is not explicitly called tangent or normal, then the vector field is not assumed to be tangent to N (or rather df-image of such) or normal to f.

In what follows the obstructions are defined analogously to Obstruction Lemma 9.5.1 and to the solution of Problem 8.9.1, see §§ 6, 8.8, 9.7, 9.9.

12.3. Non-immersibility in codimension 1

In this section we prove Assertion 12.2.2 (a) and state its generalization (Proposition 12.3.2) which is proved analogously. First we illustrate the idea by proving Assertion 12.3.1.

The *normal Stiefel—Whitney* class

$$\overline{w}_1(f) \in H_{n-1}(N)$$

is defined as the (complete) obstruction to the existence of a orientations, continuosly parametrized by $x \in N$, on the normal to f spaces at points $f(x) \in f(N)$. Equivalently, this is the (incomplete) obstruction to the existence of $(m-n)$-tuple normal to f. For $m = n+1$, cf. Problems 4.10.4 and 8.9.1.

12.3.1. We have $\overline{w}_1(f) = w_1$.

Both classes vanish simultaneously, since vanishing of the tangent (normal) class is equivalent to the existence of agreeing orientations in tangent (normal) spaces. The equality of classes is a stronger statement.

Assertion 12.3.1 follows because

• any orientation on an n-face of N gives an orientation on normal spaces to this face, and

• agreeing orientations on two adjacent n-faces of N give agreeing orientations on normal spaces.

Below we present alternative arguments. They are more complicated, but they could be generalized to more complicated situations.

Sketch of a proof of Assertion 12.3.1. Take sufficiently small triangulation of N. Take n-tuples tangent to N at vertices. Take those $(m-n)$-tuples normal to f at the images of vertices whose orientations agree with the orientations of tangent n-tuples, and of $\mathbb{R}^m$. The tangent n-tuples extend to an edge if and only if the normal n-tuples extend to the edge. □

Sketch of a heuristic for Assertion 12.3.1. Denote by $x_1(f) \in H_{n-1}(N)$ the obstruction to the existence of an m-tuple in $\mathbb{R}^m$ on $f(N)$. We have

$$0 = x_1(f) = \overline{w}_1(f) + w_1.$$

Here the first equality holds since the required m-tuple does exist.

Let us prove the second equality. Take general position n-tuple $e_1, \dots, e_n$ tangent to N, and $(m-n)$-tuple $\nu_1, \dots, \nu_{m-n}$ normal to f. Take the m-tuple $e_1, \dots, e_n, \nu_1, \dots, \nu_{m-n}$ in $\mathbb{R}^m$ on $f(N)$. This is a general position m-tuple. This m-tuple is linearly dependent exactly at the points where either $e_1, \dots, e_n$ or $\nu_1, \dots, \nu_{m-n}$ is linearly dependent. So the homology class $x_1(f)$ of the linear dependence set of the m-tuple is $\overline{w}_1(f) + w_1$. □

Sketch of a proof of Assertion 12.2.2 (a). Let $f\colon N \to \mathbb{R}^{n+1}$ be an immersion. Denote by $x_2(f) \in H_{n-2}(N)$ the obstruction to the existence of an n-tuple in $\mathbb{R}^{n+1}$ on f (for $n=2$ the construction of $x_2(f) \in \mathbb{Z}_2$ is analogous to Assertions 9.3.5 and 9.3.6). We have

$$0 = x_2(f) = w_2 + w_1\overline{w}_1 = w_2 + w_1^2, \quad \text{where}$$

- the first equality holds since such an n-tuple exists;
- the last equality holds by Assertion 12.3.1.

Let us prove the second equality. A *characteristic* tuple is defined in the heuristic to the Whitney—Wu formula 9.9.4. Take a characteristic n-tuple $v_1, \dots, v_n$ tangent to N on N. Take a vector field ν normal to f, which is zero on some $(n-1)$-subcomplex $\overline{\omega}_1^*$ representing the class $\overline{w}_1$ ($\overline{\omega}_1^*$ is a subcomplex of the cellular decomposition dual to the triangulation used in the construction of the characteristic n-tuple). Denote $\widehat{v} := df(v)$. The n-tuple

$$U := \nu + \widehat{v}_n, \widehat{v}_{n-1}, \dots, \widehat{v}_1 \quad \text{in} \quad \mathbb{R}^{n+1} \quad \text{on} \quad f$$

is linearly dependent exactly at the points where

- either the $(n-1)$-tuple $v_1, \dots, v_{n-1}$ is linearly dependent,
- or $\nu = 0$ and the n-tuple $v_1, v_2, \dots, v_n$ is linearly dependent.

So the set of linear dependence of U is $\omega_2 \cup (\omega_1 \cap \overline{\omega}_1^*)$. Now the second equality follows analogously to the last paragraph of the proof of Assertion 9.3.6.c. □

Proposition 12.3.2. *If N immerses in $\mathbb{R}^{n+1}$, then $w_s = w_1^s$ for any $s = 1, 2, \dots, n$.*

12.4. Non-immersibility in codimension 2

The *normal Stiefel—Whitney class*

$$\overline{w}_2(f) \in H_{n-2}(N)$$

is defined as the obstruction to the existence of an $(m-n-1)$-tuple normal to f. If $m=n+2$, then this is the obstruction to the existence of a non-zero normal field.

12.4.1. We have $\overline{w}_2(f)=w_2+w_1^2$.

The proof is analogous to that of Assertion 12.2.2 (a).

Sketch of the proof of Assertion 12.2.2 (b). Let $f\colon N\to\mathbb{R}^{n+2}$ be an immersion. Denote by $x_3(f)\in H_{n-3}(N)$ the obstruction to the existence of an n-tuple in $\mathbb{R}^{n+2}$ on f. We have

$$0=x_3(f)=w_3+w_2\overline{w}_1(f)+w_1\overline{w}_2(f)=w_3+w_1^3,\quad\text{where}$$

- the first equality holds since the required triple exists;
- the last equality holds by Assertions 12.3.1 and 12.4.1.

Let us prove the second equality. Take a characteristic n-tuple $v_1,\dots,v_n$ tangent to N on N. Take also a pair ν_1,ν_2 normal to f such that

- $\nu_1=0$ on some $(n-2)$-subcomplex $\overline{\omega}_2$ representing the class $\overline{w}_2(f)$;
- $\nu_2\perp\nu_1$ and $\nu_2=0$ on the union of $\overline{\omega}_2$ and some $(n-1)$-subcomplex $\overline{\omega}_1$ representing the class $\overline{w}_1(f)$.

(Here $\overline{\omega}_2$ and $\overline{\omega}_1$ are subcomplexes of the cellular decomposition dual to the triangulation used in the construction of the characteristic n-tuple.)

The n-tuple

$$U:=\nu_1+\widehat{v}_n,\nu_2+\widehat{v}_{n-1},\widehat{v}_{n-2},\dots,\widehat{v}_1\quad\text{in}\quad\mathbb{R}^{n+2}\quad\text{on}\quad f$$

is linearly dependent exactly at the points where

- either $v_1,\dots,v_{n-2}$ are linearly dependent,
- or $\nu_2=0$ and $v_1,\dots,v_{n-1}$ are linearly dependent,
- or $\nu_1=0$ and $v_1,\dots,v_n$ are linearly dependent.

So the set of linear dependence of U is $\omega_3\cup(\omega_2\cap\overline{\omega}_1)\cup(\omega_1\cap\overline{\omega}_2)$. This gives the required formula for the homology class $x_3(f)$ of the linear dependence set of U. $\square$

12.5. Proof of the Whitney Obstruction Lemma

Sketch of the proof of the Whitney Obstruction Lemma 12.2.3 (b). Take an immersion $f\colon N\to\mathbb{R}^m$. For $k\leqslant m-n$, the *normal Stiefel–Whitney* *class*

$$\overline{w}_k(f) \in H_{n-k}(N)$$

is defined as the obstruction to the existence of an $(m-n+1-k)$-tuple normal to f. For $k > m-n$, this class is assumed to be zero.

Denote

$$\overline{w}(f) := 1 + \overline{w}_1(f) + \overline{w}_2(f) + \ldots \in H_n(N) \oplus H_{n-1}(N) \oplus H_{n-2}(N) \oplus \ldots$$

For $k \leqslant n$, denote by $x_k \in H_{n-k}(N)$ the obstruction to the existence of an $(m+1-k)$-tuple on f in $\mathbb{R}^m$. Denote

$$x := 1 + x_1 + \ldots + x_n \in H_n(N) \oplus H_{n-1}(N) \oplus \ldots \oplus H_0(N).$$

One proves that $1 = x = \overline{w}(f) \cap w(N)$ analogously to Assertions 12.2.2, 12.3.1 and 12.4.1, and the Whitney—Wu formula 9.9.4).

The equality $\overline{w}(f) \cap w(N) = 1$ expresses the chain of equalities

$$\overline{w}_1(f) = w_1, \quad \overline{w}_2(f) = w_2 + w_1^2, \quad \overline{w}_3(f) = w_3 + w_1^3, \ldots$$

For $m = n+3$ (Proposition 12.2.2 (c)), the next inequality is as follows:

$$0 = x_4(f) = w_4 + w_3\overline{w}_1(f) + w_2\overline{w}_2(f) + w_1\overline{w}_3(f) = \\ = w_4 + w_3 w_1 + w_2(w_2^2 + w_1) + w_1(w_3 + w_1^3).$$

Since $\overline{w}(f) \cap w(N) = 1$, by the Whitney Obstruction Lemma 12.2.3 (a) we obtain $\overline{w}(f) = \overline{w}(N)$ (in particular, the classes $\overline{w}_k(f)$ do not depend on f). Therefore, $\overline{w}_k(N) = 0$ for $k > m-n$. □

12.5.1. Let N be a closed n-manifold.[38]

(a) If there exists an immersion $N \to \mathbb{R}^{m+k}$ admitting a normal k-tuple, then $\overline{w}_{m-n+1}(N) = 0$.

(b) One has $\overline{w}_n(N) = 0$.

Sketch of the proof of the Whitney Obstruction Lemma 12.2.3 (c). This is a generalization of §6.8, see details in [Sk08, § 2], cf. the case

[38] The strengthening (a) of the Whitney Obstruction Lemma 12.2.3 (b) is equivalent to the lemma itself via the (difficult) Smale—Hirsch Theorem 15.3.6. Part (b) follows also from the (non-trivial) Whitney Theorem 12.1.1 (a) and Whitney Obstruction Lemma 12.2.3 (c).

$m = 2n$ considered in § 4. Let $f\colon N \to \mathbb{R}^m$ be a general position map (PL or smooth). Then the set $\Sigma(f)$ of its self-intersections supports a cycle (with coefficients mod 2). The homology class $[\Sigma(f)] \in H_{2n-m}(N)$ of this cycle does not depend on f. One can prove that $[\Sigma(f)] = \overline{w}_{m-n}(N)$. If f is an embedding, then $\Sigma(f) = \varnothing$, so $\overline{w}_{m-n}(N) = [\Sigma(f)] = 0$. ☐

12.6. Triviality of tangent classes*

Theorem 12.6.1. *If N is a closed n-manifold, and $w_1 = w_2 = \ldots = w_{[n/2]} = 0$, then $w_s = 0$ for all s.*

For $n \leqslant 4$, this follows from Surface Classification Theorem 5.6.1 (see the end of §5.7), Assertion 10.4.5 (b), the Stiefel Theorem 9.9.7, Theorem 9.8.3 (a), and Assertion 9.8.12 (b). We will outline the proof for arbitrary n using the Whitney—Wu formula 12.6.3.b.

12.6.2. If N is a closed n-submanifold in a closed orientable $(n+1)$-manifold M, then

$$w_1(M)|_N = 0 \quad \text{and} \quad w_2(M)|_N = w_2 + w_1^2 \in H_{n-2}(N).$$

For $n = 2$, this is Assertion 9.3.6.c (cf. Assertions 12.2.2 (a) and 12.4.1). The proof in the general case is analogous.

12.6.3. Let N be a closed n-submanifold in a closed $(n+c)$-manifold M.

(a) Analogously to §9.9, construct the obstruction

$$\overline{w}_{c-k+1,M} = \overline{w}_{c-k+1,M}(N) \in H_{n+k-c-1}(N)$$

to the existence of a family of k linearly independent vector fields on N, tangent to M and normal to N.

(b) **The Whitney—Wu formula** (special case):

$$w_s(M)|_N = w_s + w_{s-1}\overline{w}_{1,M} + \ldots + w_1\overline{w}_{s-1,M} + \overline{w}_{s,M}.$$

This equality is shortly written as $w(M)|_N = w(N)\overline{w}_M(N)$.

Comments on the proof of Theorem 12.6.1. *First we prove that $w_3 = 0$ for $n = 4$* (similarly one obtains $w_4 = 0$). We obtain $w_3 = 0$ by Assertion 14.9.3 (b) because for any closed 3-submanifold $F \subset N$ we have

$$w_3 \cap [F] = w_3(F) + w_2(F)\overline{w}_{1,N}(F) = 0 \in \mathbb{Z}_2, \quad \text{where}$$

• the first equality holds by the Whitney—Wu formula 12.6.3 (b) since $\overline{w}_{s,N}(F) = 0$ for $s \geqslant 2$;

• the second equality follows because

$$\overline{w}_{1,N}(F) = w_1(F) \quad \text{and} \quad w_3(F) = 0 = w_2(F)w_1(F)$$

(see Assertion 10.4.5 (b) and Theorem 9.7.6).

Consider the general case. Set $k = [n/2]$. Let us prove that $w_{k+1} = 0$ (similarly one obtains that $w_{k+2} = 0, \ldots, w_n = 0$). For any $(k+1)$-submanifold F one has

$$w_{k+1} \cap [F] = w_{k+1}(F) + w_k(F)\overline{w}_1(F) + \ldots + w_1(F)\overline{w}_k(F) + \overline{w}_{k+1,N}(F) = \\ = \overline{w}_{k+1}(F) = 0.$$

• The first equality follows from the Whitney—Wu formula 12.6.3 (b) for $s = k+1$, since the same formula for $s \leqslant k$ together with the hypothesis of the theorem implies that $\overline{w}_{s,N}(F) = \overline{w}_s(F)$ for $s \leqslant k$.

• The second equality follows because $\overline{w}_{k+1,N}(F) = 0$ and $w(F)\overline{w}(F) = 1$.

• The third equality is Assertion 12.5.1 (b).

In order to make the proof of the equality $w_{k+1} \cap x = 0$ work for a class x that is non-realizable by a submanifold, one needs to define the Stiefel—Whitney classes for the 'normal bundle' of this class. The attempts to do so lead to the definition of the *Steenrod squares*[39], cf. the following subsection.

12.7. Powers of two and the Stiefel—Whitney classes*

For the Stiefel—Whitney classes of an arbitrary closed manifold interesting relations hold.

Theorem 12.7.1 (Massey)**.** *For any closed smooth n-manifold N,*
(a) *if N is non-orientable, then $\overline{w}_{n-1}(N) = 0$* [Ma62];
(b) *if $q < \alpha(n)$, then $\overline{w}_{n-q}(N) = 0$* [Ma60].

Part (b) for $q = 0$ is Assertion 12.5.1 (b). For the proof, the following assertions are needed. This proof in interesting for its use of the

[39]Perhaps, this is how they were invented. The corresponding work of Steenrod is devoted to a different problem, and contains a formal definition of the Steenrod squares without motivation.

13.2. Locally trivial fibrations

An S^1*-action* on a complex K is a PL map $t : S^1 \times K \to K$ such that $t(zw, x) = t(z, t(w, x))$ for any $z, w \in S^1$ and $x \in K$. Denote $zx := t(z, x)$. Then $(zw)x = z(wx)$.

An S^1-action is *free* if $zx \neq wx$ for any $x \in K$ and distinct $z, w \in S^1$.

For a free S^1-action t on K, and any $x \in K$ identify with each other the points zx for all $z \in S^1$. The space obtained via this identification is (the body of) a complex (use this fact without proof). This complex K/t is called the *quotient complex* of K by the S^1-action.

13.2.1. Define free S^1-actions t on the following spaces so that

(a) $K \times S^1/t \cong K$ for a complex K;

(b) $S^3/t \cong S^2$;

(c) $S_\nu N/t \cong N$, where N is an orientable n-submanifold of an orientable $(n+2)$-manifold, and $S_\nu N$ is defined below;

(d) $SN/t \cong N$, where N is an orientable 2-manifold, and SN is defined below.

Hint to (b). Let $t(z, w) := zw$.

Let N be a smooth submanifold of $\mathbb{R}^d$ (or of a smooth manifold M). Denote by T_xN the tangent space to N at a point $x \in N$. Define *the tangent space* of N and the *the spherical tangent space* of N by

$$TN := \{(x, v) \in N \times T_xN\} \quad \text{and} \quad SN := \{(x, v) \in N \times T_xN : \ |v| = 1\}.$$

Define the **tangent bundle** $\tau_N : TN \to N$ of N by $\tau_N(x, v) := x$.

Define *the tubular neighborhood* of N

$$D_\nu N := \{(x, v) \in N \times T_xM \ : \ v \perp T_xN, \ |v| \leqslant 1\}.$$

Define *the boundary of the tubular neighborhood* of N

$$S_\nu N := \{(x, v) \in N \times T_xM \ : \ v \perp T_xN, \ |v| = 1\}.$$

A *tubular neighborhood* of N in M is also the image of a smooth embedding $D_\nu N \to M$ sending $D_\nu N \cap (N \times 0)$ to N.

Theorem 13.2.2 (Tubular Neighborhood). *Any closed smooth submanifold has a tubular neighborhood.*

Let $f\colon N\to\mathbb{R}^m$ be a smooth immersion. Define

$$E_\nu(f):=\{(x,v)\in N\times\mathbb{R}^m\ :\ v\perp T_{f(x)}f(N)\}.$$

Define the **normal bundle** $\nu_f\colon E_\nu(f)\to N$ of f by $\nu_f(x,v):=x$.

Recall that $\mathbb{R}P^n$ is the space of all lines $\mathbb{R}^{n+1}$ passing through the origin. Define

$$E(\zeta_n):=\{(l,v)\in\mathbb{R}P^n\times\mathbb{R}^{n+1}\ :\ v\in l\}.$$

Define the **tautological bundle** $\zeta_n\colon E(\zeta_n)\to\mathbb{R}P^n$ by $\zeta_n(l,v):=l$.

Maps $p_j:E_j\to B$, $j=1,2$, are said to be **isomorphic (fiberwise equivalent)** if there is a homeomorphism $\varphi\colon E_1\to E_2$ such that $p_1=p_2\circ\varphi$. Notation: $p_1\cong p_2$.

13.2.3. (a) The space $E(\zeta_1)$ is homeomorphic to the Möbius band, and ζ_1 is isomorphic to the projection onto its middle circle.

(b) The map ζ_n is isomorphic to the normal bundle of $\mathbb{R}P^n$ in $\mathbb{R}P^{n+1}$.

(c) The tangent bundle τ_N is isomorphic to the normal bundle of the diagonal in $N\times N$.

A map $p\colon E\to B$ is called a **(locally trivial) fibration** with the fiber F if for every point $b\in B$ there is a neighborhood Ob such that $p|_{p^{-1}Ob}:p^{-1}Ob\to Ob$ is isomorphic to the projection $Ob\times F\to Ob$. The spaces B,E are called the *base* and *the total space* of the fibration. Cf. Local Triviality Lemma 8.10.7 (b).

For example,

- the **trivial fibration** is the projection $B\times F\to B$.
- any covering is a fibration.

13.2.4. (a) Define a fibration $Kl\to S^1$ from the Klein bottle with the fiber S^1.

(b) The tangent, normal and tautological bundles are fibrations. (They are called bundles because they have richer structure than just fibrations, see §13.4.)

(c) Let $f\colon K\to K$ be a PL homeomorphism of a complex K. Let

$$S^1\mathbin{\tilde\times_f}K:=\frac{K\times I}{(x,0)\sim(f(x),1)_{x\in K}}$$

be the complex obtained from $K \times I$ by identifying the points $(x, 0)$ and $(f(x), 1)$. Define $p : S^1 \widetilde{\times}_f K \to S^1$ by $p[x, t] = [x]$. Prove that this is a fibration over S^1 with the fiber K.

For any free smooth S^1-action the projection to the quotient space is a fibration with the fiber S^1.

13.2.5. (a) There exists a 3-manifold not homeomorphic to $(S^1)^3$, but which is simultaneously the total space of a fibration over $S^1 \times S^1$ with the fiber S^1, and of a fibration over S^1 with the fiber $S^1 \times S^1$.

(b)* Let N and X be spheres with handles. If a 3-manifold is simultaneously the total space of a fibration over N with the fiber S^1, and the total space of a fibration over S^1 with the fiber X, but is neither homeomorphic to $N \times S^1$ nor to $X \times S^1$, then $N \cong X \cong S^1 \times S^1$.

A map $s\colon B \to E$ is called a **section** of a map $p\colon E \to B$ if $p \circ s = \mathrm{id}_B$. A tangent (normal) vector field is the same as a section of the tangent (normal) bundle.

13.2.6. Any fibration with the fiber S^1 over B has a section, if B is

(a) a graph; (b) D^2;

(c) a connected 2-manifold with non-empty boundary;

(d) D^3; (e) $S^1 \times D^2$; (f) S^3.

13.2.7. (a,b) For every n neither $\tau_{S^{2n}}|_{S S^{2n}}$ have a section, nor ζ_n have a section whose image is disjoint with $\mathbb{R}P^n \times 0$.

13.2.8. Any of the following fibrations is isomorphic to a trivial fibration.

(a-d) The bundles τ_{S^1}, τ_{S^3}, τ_{S^7}, and $\tau_{(S^1)^n}$.

(e) A fibration over D^n.

(f) A fibration over S^3 with the fiber S^1.

Neither of the bundles $\tau_{S^{2k}}$ and ζ_n is isomorphic to a trivial fibration by Assertions 13.2.7.ab and 13.2.9.ab.

13.2.9. If a fibration is isomorphic to a trivial fibration, then

(a) so is any its restriction.

(b) it has a section.

13.2.10. (a) A double covering has a section if and only if the covering is isomorphic to the trivial covering.

(b) (Riddle) For a double covering $p: E \to B$ of a closed n-manifold B construct a class $w_1(p) \in H_{n-1}(B)$ such that the covering has a section if and only if $w_1(p) = 0$.

(c) We have $w_1(\zeta'_n) = [\mathbb{R}P^{n-1}]$. (Here ζ'_n is the double covering which is the restriction to $\{(l, v) \in E(\zeta_n) \ : \ |v| = 1\}$ of the tautological bundle ζ_n. Use Assertion 13.2.3.b.)

Hint to 13.2.4. (a) Represent Kl by gluing the sides $\overrightarrow{AB}$ and $\overrightarrow{CD}$, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ of a square $ABCD$. When gluing the sides $\overrightarrow{BC}$ and $\overrightarrow{AD}$, one obtains an annulus (i.e. lateral surface of a cylinder) $S^1 \times I$. The two boundary circles are obtained by identifying the endpoints of the segment AB, and identifying the endpoints of the segment CD, respectively. In total, Kl is obtained by gluing the boundary circles $S^1 \times 0$ и $S^1 \times 1$ of the annulus $S^1 \times I$.

Hint to 13.2.5. (a) For $e \in \mathbb{Z} - \{0\}$ take the self-homeomorphism $\overline{e}$ of the torus obtained from the automorphism of the plane given by $(x, y) \mapsto (x + ey, y)$. Take the 3-manifold $S^1 \widetilde{\times}_{\overline{e}} (S^1 \times S^1)$.

(b) Find the homology of the 3-manifold via the fibration over N, and via the fibration over S^1. See Problem 11.5.3.

13.3. The sum and the product of fibrations

The *product* of maps $p_1 \colon E_1 \to B_1$ and $p_2 \colon E_2 \to B_2$ is the map

$$p_1 \times p_2 \colon E_1 \times E_2 \to B_1 \times B_2 \text{ defined by } (p_1 \times p_2)(x, y) := (p_1(x), p_2(y)).$$

13.3.1. (a) The product of fibrations with fibers F_1, F_2 is a fibration with fiber $F_1 \times F_2$.

(b) $\tau_{N_1 \times N_2} \cong \tau_{N_1} \times \tau_{N_2}$.

For maps $p_j \colon E_j \to B$, $j = 1, 2$ define

$$E(p_1 \oplus p_2) := \{(x, y) \in E_1 \times E_2 \ : \ p_1(x) = p_2(y)\}.$$

The (Whitney) **sum** of maps p_1 and p_2 is the map

$$p_1 \oplus p_2 \colon E(p_1 \oplus p_2) \to B \quad \text{defined by} \quad (p_1 \oplus p_2)(x, y) := p_1(x) = p_2(y).$$

(From §13.4 it is clear why this is called the sum, not the product.)

13.3.2. (a) The sum of fibrations with fibers F_1, F_2 is a fibration with fiber $F_1 \times F_2$.

(b) If N is a submanifold of V which is a submanifold of $\mathbb{R}^d$, then the normal bundle of N in $\mathbb{R}^d$ is the sum of the normal bundle of N in V, and the restriction to N of the normal bundle of V in $\mathbb{R}^d$.

From now on we denote by ε the 1-dimensional trivial fibration (whose base is evident from the context).

Lemma 13.3.3. *(a)* $\zeta_1 \oplus \zeta_1 \cong 2\varepsilon$; *(b)* $\tau_{S^n} \oplus \varepsilon \cong (n+1)\varepsilon$;

(c) $\tau_N \oplus \nu_f \cong d\varepsilon$ *for any immersion* $f: N \to \mathbb{R}^d$ *of a manifold* N;

(c') $\tau_N \oplus \nu_f \cong \tau_V|_N$ *for any immersion* $f: N \to V$ *of a manifold* N *to a manifold* V *(for which* ν_f *is defined analogously to the case* $V = \mathbb{R}^d$*; in particular,* $\tau_{\mathbb{R}P^{n+1}}|_{\mathbb{R}P^n} \cong \tau_{\mathbb{R}P^n} \oplus \zeta_n$*, which helps to invent the following formula);*

(d) $\tau_{\mathbb{R}P^n} \oplus \varepsilon \cong (n+1)\zeta_n$.

Sketch of the proof. (c) The required isomorphism of bundles is given by the formula $(x, v_1) \oplus (x, v_2) \mapsto (x, v_1 + v_2)$.

(d) Recall that $\mathbb{R}P^n$ is the space of the lines in $\mathbb{R}^{n+1}$ passing through the origin. A tangent vector at a point $l \in \mathbb{R}P^n$, $l \subset \mathbb{R}^{n+1}$, can be naturally identified with a linear map $l \to l^\perp \subset \mathbb{R}^{n+1}$. (A tangent vector can as well be identified with a point in $l^\perp$, but this identification will not be natural.) Then a pair of a tangent vector at the point $l \in \mathbb{R}P^n$ and a number can be naturally identified with a pair of linear maps $l \to l$ and $l \to l^\perp$. The latter pair can be naturally identified with a linear map $l \to \mathbb{R}^{n+1}$, that is, with an ordered $(n+1)$-tuple of linear functionals $l \to \mathbb{R}$. Given an scalar product in $\mathbb{R}^{n+1}$, a linear functional $l \to \mathbb{R}$ can be naturally identified with an element of l.

13.4. Vector bundles

A **vector bundle** of dimension n is a map $p\colon E \to B$ together with the structure of an n-dimensional vector space over $\mathbb{R}$ on the set $p^{-1}b$ for every point $b \in B$, satisfying the following *local triviality* assumption:

for every $b \in B$ there are a neighborhood $Ob \subset B$ and a homeomorphism $h_b\colon Ob \times \mathbb{R}^n \to p^{-1}Ob$ such that $p \circ h_b = \mathrm{pr}_1$, and for any $a \in Ob$ the restriction $h_b|_{a \times \mathbb{R}^n}$ is an isomorphism of vector spaces $\mathbb{R}^n$ and $p^{-1}a$.

One can define the structure of a vector bundle for τ_N, for ν_f, for ζ_n, as well as for the sum and the product of vector bundles.

The *zero* section maps each point $b \in B$ to the origin of the vector space $p^{-1}b$. A section is called *nowhere vanishing*, if no point $b \in B$ is mapped to the origin of the vector space $p^{-1}b$.

Vector bundles $p_j \colon E_1 \to B$, $j = 1, 2$, over the same base are called *isomorphic* if there is a homeomorphism $\varphi \colon E_1 \to E_2$ such that $p_2 \circ \varphi = p_1$, and the restriction $\varphi|_{p_1^{-1}b} \colon p_1^{-1}b \to p_2^{-1}b$ is an isomorphism of vector spaces for every $b \in B$.

13.4.1. The bundle τ_N is isomorphic to the trivial bundle if and only if N is parallelizable.

For a vector bundle $p : E \to B$ over a complex B, the Stiefel—Whitney characteristic class $w_i(p) \in H^i(B)$ is defined (analogously to §9.9, §12.5) as the obstruction to the existence of a collection of linearly independent sections. It is clear that $w_i(\tau_N) = w_i(N)$ and $w_i(\nu_f) = \overline{w}_i(N)$ for a manifold N and an immersion $f \colon N \to \mathbb{R}^m$.

13.4.2. (a) $w_i(\xi \oplus n\varepsilon) = w_i(\xi)$.

(b) $w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)$.

(c) If $\dim \xi = m$ and $\dim \eta = n$, then $w_{m+n}(\xi \times \eta) = w_m(\xi) \times w_n(\eta)$.

(d) If $\dim \xi = m$ and $\dim \eta = n$, then $w_{m+n}(\xi \oplus \eta) = w_m(\xi) w_n(\eta)$.

The *total Stiefel—Whitney class* of a vector bundle p is defined as $w(p) := 1 + w_1(p) + w_2(p) + \ldots$. E.g. Assertion 13.2.10.c means that

$$w(\zeta_n) = 1 + [\mathbb{R}P^{n-1}].$$

13.4.3. (a) $w(\xi \times \eta) = w(\xi) \times w(\eta)$.

(b) *The Whitney—Wu formula.* $w(\xi \oplus \eta) = w(\xi) w(\eta)$.

13.4.4. Let $B = \bigcup\limits_i U_i$ be an open cover of a hypergraph B and $\varphi_{ij} \colon U_i \cap U_j \to O_n$ be maps such that

$$\varphi_{ii} = \mathrm{id}, \quad \varphi_{ij} = \varphi_{ji}^{-1} \quad \text{and} \quad \varphi_{ik} = \varphi_{ij}\varphi_{jk}.$$

Set

$$E := \bigcup_{\{(x,s)=(x,\varphi_{ij}s)\}_{x \in U_i \cap U_j}} U_i \times \mathbb{R}^n \quad \text{and} \quad p[x,s] := x.$$

Define a structure of a vector bundle for p.

(Any vector bundle over a hypergraph B can be obtained using this construction. So this gives an equivalent definition of a vector bundle.)

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On the path of this book to a reader

Here we give details to 'publishing rights' in p. 2 of this file. As of May, 2022, no public reply from the Editorial Board or from Springer are available. Updates (e.g. a public reply, if available) will be presented here.

A. Skopenkov's letter to the Editorial board of Springer book series 'Moscow Lecture Notes' (Cc M. Peters). Dec 6, 2021.

Dear colleagues,

Hope you are fine and healthy.

Thank you for accepting for publication in 'Moscow Lecture Notes' series of Springer the book Algebraic Topology From a Geometric Standpoint, https://www.mccme.ru/circles/oim/obstructeng.pdf

I'm afraid Springer is disregarding this acceptance decision of the Editorial Board. The Publishing Agreement proposed by Springer in April does not make the Publisher committed to publishing the book. Martin Peters and I found a compromise in May. But our compromise is not realized, and the problem is still unresolved - in spite of my monthly reminders. Natalia Tsilevich did excellent urgent translation work in July, but neither is paid by Springer, nor has a legal document ensuring later payment.

Does Editorial Board have any means to ensure that its acceptance decision is fulfilled by Springer? This information is vital for authors submitting to 'Moscow Lecture Notes' series.

Best wishes, Arkadiy.

PS The translation went fast and was already completed as early as in July (only the introduction and sections 3,4 remained). The translation was stopped for reasons described above.

A. Skopenkov's letter to A. Gorodentsev and V. Bogachev, Editors of Springer book series 'Moscow Lecture Notes' (Cc M. Peters). Dec 15, 2021.

Dear Alexey and Vladimir Igorevich,

Upon request of Vladimir Igorevich I describe how Springer is disregarding the acceptance decision of the Editorial Board of 'Moscow Lecture Notes' series. On compromises, see my letter of 6 Dec.

Could the Editorial Board make minimal efforts supporting its acceptance decision? A possible way is to publicly support the authors' amends to the Agreement proposed by Springer (I am willing to send you the list of amends). The information on whether the acceptance decision of the Editorial Board is final, is vital for authors submitting to the 'Moscow Lecture Notes' series. So the result of your efforts (if you choose to do some) should be widespread throughout the scientific community.

(1) The Agreement proposed by Springer contains the following clause allowing the Publisher to terminate the Agreement without any losses. This makes the publisher not committed to publishing the book, and so makes the acceptance decision of the Editorial Board void.

****

11.2. If the Publisher, acting reasonably, decides that the Work is not suitable for publication in the intended market place and/or community or that there is no substantial market for the Work, or the economic circumstances of publication have substantially changed (in each case other than due to the Work not being of a suitable quality to justify publication) then the Publisher may at any time terminate this Agreement by giving one month's notice to the Author in writing.

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(2) The Agreement proposed by Springer does not contain a deadline for publication of the book (in terms of months after receipt of the translation). This makes the publisher not committed to publishing the book, and so makes the acceptance decision of the Editorial Board void.

(3) The Agreement proposed by Springer contains the following clause which makes the acceptance decision of the Editorial Board void.

****

13.1. This Agreement, and the documents referred to within it, constitute the entire agreement between the Parties with respect to the subject matter hereof and supersede any previous agreements, warranties, representations, undertakings or understandings. Each Party acknowledges that it is not relying on, and shall have no remedies in respect of, any undertakings, representations, warranties, promises or assurances that are not set forth in this Agreement.

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(4) The Agreement proposed by Springer does not specify the amount of, and the deadline for, Publisher's payment for translation. For this, the Agreement refers to the Translation Agreement, but gives no guarantee that the terms of that Translation Agreement will be acceptable to the author and other translator. Since the author should not sign such an Agreement, this makes the acceptance decision of the Editorial Board void.

Best Regards, Arkadiy.

A. Skopenkov's letter to V. Bogachev, Editor of Springer book series 'Moscow Lecture Notes' (Cc A. Gorodentsev and M. Peters). Dec 23, 2021.

Dear Vladimir Igorevich,

Thank you for your reply.

Why do you write that my suggestions have been taken into account in a modified contract? This is wrong as I explained in my letter of Dec 15: my suggestions on items (1)-(4) are not taken into account. I forwarded you the last list of my suggestions sent to M. Peters on Nov 17 (analogous suggestions to previous versions of the Publishing Agreement were sent earlier). I received no reply either accepting these suggestions, or stating that Springer would not change the contract, or proposing compromises.

Recall that

(*) Springer is disregarding the acceptance decision of the Editorial Board because the Publishing Agreement proposed by Springer does not make the Publisher committed to publishing the book.

This is justified in my letter of Dec 15 by items (1)-(4). You do not consider those items, so you could not refute the statement (*). You write that the Publishing Agreement proposed by Springer is standard, but again this does not refute the statement (*). If something bad is a standard practice, this does not make it good.

My real experience with Springer is poor. I spent an enormous amount of time correcting errors that appeared during typesetting of my paper in Arnold J. Math. In May M. Peters agreed to take my suggestions into account. As of December, neither this is done, nor he informed me that this would not be done. So publication of the book is unduly postponed for an uncontrolled amount of time. All positive parts of our collaboration with M. Peters are explicitly made void by clause 13.1 of the Agreement:

****

13.1. This Agreement, and the documents referred to within it, constitute the entire agreement between the Parties with respect to the subject matter hereof and supersede any previous agreements, warranties, representations, undertakings or understandings. Each Party acknowledges that it is not relying on, and shall have no remedies in respect of, any undertakings, representations, warranties, promises or assurances that are not set forth in this Agreement.

****

For the moment, I will not comment on the other part of your letter for the following reason. The above (and the rest of your letter) makes me suppose that you confused a responsible business discussion with an irresponsible tea-time talk. If I am wrong, then I am sorry, and I have the following suggestion.

We strongly need this discussion to be responsible. We do not have enough time to discuss premature ideas, whose invalidity becomes clear when their publication (or a mental experiment of publication) is suggested. So I inform you that our correspondence with the Editorial Board on this subject is public.

I will publish all my letters at https://www.mccme.ru/circles/oim/obstructeng.pdf . If you would not send me a public reply to my Dec 15 letter, then the best way is to treat the private reply as non-existent, and inform the community that there is no public reply. If you send me a public reply to my Dec 15 letter (please feel free to edit your private reply), then I will publish it. My reply, your further reply, etc will also be published; presumably the discussion will soon converge by revealing important questions (like Q1, Q2, Q3 below) and the Editors answering them. If I receive a letter not stated to be public, then I will delete it unread (to avoid confusion). If a part of such a public discussion would become obsolete, we could delete that part (only) by our mutual consent.

Such a public discussion would be very useful for potential authors of this book series. In particular, they would be grateful if the Editors could publicly answer the following questions:

(Q1) Is Agreement with the properties (1)-(4) from my Dec 15 letter absolutely standard for this book series?

(Q2) Is Springer not obliged to accept all recommendations of the Editorial Board for this book series?

(Q3) Do Editors advise the authors to sign the Agreement without reading it?

If there is no public answer, a potential author could only assume that the answer is 'yes'.

Such a public discussion would require much effort. So let us find a way to avoid it. E.g., discussion by skype / zoom / phone makes it easier to understand each other and to find compromises.

Best wishes, Arkadiy.

A. Skopenkov's letter to M. Peters, A. Gorodentsev, V. Bogachev, and Yu. S. Ilyashenko. Jan 30, 2022.

Dear Martin, Alexey, Vladimir Igorevich, and Yuliy Sergeevich,

Hope you are fine and healthy.

I am grateful to the Editorial Board of 'Moscow Lecture Notes' of Springer for accepting in January, 2021 for publication the book 'Algebraic Topology From Geometric Standpoint'. (Please see the electronic version of a part at https://www.mccme.ru/circles/oim/obstructeng.pdf.)

The translation was essentially rejected by Springer by sending an unacceptable publishing agreement, promising to make amends suggested by the author in May, 2021, and neither making amends nor informing the author that the amends are not accepted, by January, 2022.

So, however reluctantly, I inform you that this book is no longer submitted to Springer.

We do not have enough time to discuss premature ideas, whose invalidity becomes clear when their publication (or a mental experiment of publication) is suggested. So I inform you that our correspondence on this subject is public. My letters are published at https://www.mccme.ru/circles/oim/obstructeng.pdf. If I receive a letter not stated to be public, then I will delete it unread (to avoid confusion).

I am also open to private discussions by skype / zoom / phone.

Best wishes, Arkadiy.