Groups, ends and trees: exercises II

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Exercise 1. Give examples of groups which are not the fundamental group of a closed surface.

Exercise 2. Draw a picture of the Cayley graph of $\mathbf{Z}_3 * \mathbf{Z}_4$.

Exercise 3. Let $G = \langle S \rangle$ be a group generated by the set S. For any $x \in G$, define its *word length* $\|x\|_S$ to be the length of the shortest word in the alphabet $S \cup S^{-1}$ that represents the element x:

$$\|x\|_S = \min\{n \mid \exists\, s_1, \dots, s_n \in S \cup S^{-1} \text{ s.t. } x = s_1 \cdots s_n\}.$$

Show that the function $d_S : G \times G \longrightarrow G$ defined by $d_S(x, y) = \|x^{-1}y\|_S$ is a distance on G and coincides with the graph distance on the Cayley graph of G with respect to generating set S.

Exercise 4. Let $G = \langle S \rangle$ be a finitely generated group and let $\Gamma(G, S)$ be its Cayley graph, equipped with the graph distance d_S. Show that the multiplication in G defines a distance-preserving transitive action of G on $\Gamma(G, S)$: for any two vertices $x, y \in \Gamma(G, S)$, and for any $g \in G$, $d_S(g.x, g.y) = d_S(x, y)$.

Show that the quotient space for this action is homeomorphic to a bouquet of $\#S$ circles.

Exercise 5. Let S and S' be two different finite generating sets of a group G. Show that the Cayley graphs $\Gamma(G, S)$ and $\Gamma(G, S)$ are quasi-isometric.

Exercise 6. Show that the Cayley graph of a finite group is quasi-isometric to a point. More generally, every compact metric space is quasi-isometric to a point.

Exercise 7. Let $H \leq G$ be a finite index subgroup of G. Show that H is finitely generated if and only if G is.

Suppose that $G = \langle S \rangle$ and $H = \langle S' \rangle$ are finitely generated. Show that the Cayley graphs $\Gamma(G, S)$ and $\Gamma(H, S')$ are quasi-isometric.

Exercise 8. Show that quasi-isometry is an equivalence relation.

Exercise 9. Let (M, d_g) be a compact Riemannian manifold.

a) Show that it is possible to define a metric $\widetilde{d_g}$ on the universal cover $\widetilde{M}$ of M in such a way that the monodromy action of the fundamental group $\pi_1(M)$ on $(\widetilde{M}, \widetilde{d_g})$ preserves the distances: $\pi_1(M)$ is a subgroup of the group of isometries of $(\widetilde{M}, \widetilde{d_g})$.

b) Let $U \subset \widetilde{M}$ be a fundamental domain for the monodromy action: $U = \sigma(M)$, where $\sigma : M \longrightarrow \widetilde{M}$ is any continuous section of the covering projection $\widetilde{\pi} : \widetilde{M} \longrightarrow M$, that is σ is injective and $\pi \circ \sigma = id|_M$.

c) Show that the set

$$S = \{s \in \pi_1(M) \mid s \neq id \text{ and } s\bar{U} \cap \bar{U} \neq \emptyset\}$$

is finite.

d) Show that

$$\inf\{\widetilde{d_g}(\bar{U}, x\bar{U}) \mid x \in \pi_1(M) - (S \cup \{id\})\} =: 2d$$

is a minimum and strictly positive. Moreover, if $d_g(\bar{U}, x\bar{U}) < 2d$ then $x \in S \cup \{id\}$.

e) Let us fix $p \in \widetilde{M}$. For any $x \in \pi_1(M)$, denote by $[p, x.p]$ a geodesic path from p to $x.p$. Write

$$k := \lfloor \frac{\widetilde{d_g}(p, x.p)}{d} \rfloor$$

and let us take points

$$y_0 = p, y_1, \ldots, y_k, y_{k+1} = x.p$$

on the geodesic curve $[p, x.p]$, such that $\widetilde{d_g}(y_i, y_{i+1}) \leq d$ for any $i = 0, \ldots, k$. For any i, consider $h_i \in \pi_1(M)$ such that $y_i \in h_i\bar{U}$. Then $h_i^{-1}h_{i+1} \in S$.

This implies that S generates the fundamental group of M: $\pi_1(M) = \langle S \rangle$.

f) The set of points $\pi_1(M).p := \{x.p \mid x \in \pi_1(M)\}$ is discrete in $\widetilde{M}$ and there exists $D > 0$ such that the neighbourhood of radius $2D$,

$$B_{2D}(\pi_1(M).p) = \bigcup_{x \in \pi_1(M)} B_{2D}(x.p),$$

is the whole manifold $\widetilde{M}$.

g) For any $x \in \pi_1(M)$, let $m := \|x\|_S = d_S(id, x)$. Show that

$$\frac{1}{2D}\widetilde{d_g}(p, x.p) \leq m \leq k + 1 \leq \frac{1}{d}\widetilde{d_g}(p, x.p) + 1,$$

and hence $(\pi_1(M), d_S)$ and $(\widetilde{M}, \widetilde{d_g})$ are quasi-isometric. This result is a theorem proved indepen-denlty by Švarcz and Milnor.