French-Russian Laboratory “J.-V. Poncelet”
Moscow Center for Continuous Mathematical Education
Independent University of Moscow
Institute for Information Transmission Problems of RAS
Institute of Control Sciences of RAS

International Workshop

TROPICAL AND IDEMPOTENT MATHEMATICS

Moscow, Russia, August 26–31, 2012

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Moscow – 2012

ISBN 978-5-91910-143-7

G. L. Litvinov, V. P. Maslov, A. G. Kushner, S. N. Sergeev (Eds.) Tropical and idempotent mathematics. – Moscow: 2012 – 276 pages.

TEX editor: S. N. Tychkov.

This volume contains the proceedings of the International Workshop on Idempotent and Tropical Mathematics (Moscow, Russia, August 26–31, 2012).
This publication is supported by the RFBR grant 12-01-06083-g.

2000 Mathematics Subject Classification: 00B10, 81Q20, 06F07, 35Q99, 49L90, 46S99, 81S99, 52B20,52A41, 14P99.

PREFACE

Idempotent/tropical mathematics is a relatively new branch of mathematical sciences, which rapidly developed and gained popularity over the last two decades. It is closely related to many areas of mathematics and numerous applications. Tropical mathematics is a very imporatant part of idempotent mathematics, see an introductory lecture below. The literature on the subject is vast and includes numerous books and an all but innumerable body of journal papers.

The present book contains materials presented for the International Workshop TROPICAL AND IDEMPOTENT MATHEMATICS, Moscow, Russia, August 26–31, 2012. Our workshop has become traditional. Materials related with previous workshops are presented in volumes 377 and 495 of Contemporary Mathematics (American Mathematical Society, Providence, Rhode Island, 2005 and 2009).

It is our pleasure to thank all the institutions supporting the workshop: the Independent University of Moscow, French-Russian Laboratory "J.V. Poncelet", Moscow Center for Continuous Mathematical Education, A.A. Kharkevich Institute for Information Transmission Problems of RAS, V.A. Trapeznikov Institute of Control Sciences of RAS, Russian Fund for Basic Research, CNRS (France), and Dynasty Fund (Moscow), for their important support.

We are grateful to a number of colleagues, especially to E.S. Kryukova, A. P. Kuleshov, A.N. Sobolevski, and M.A. Tsfasman, for their great help. We thank all the authors of the volume and members of our "idempotent/max-plus, tropical community" for their contributions, help, useful contacts and discussions.

The editors
Moscow, August 2012.

Dequantization of mathematical structures and tropical/idempotent mathematics. An introductory lecture

G. L. Litvinov

Abstract A very brief introduction to tropical and idempotent mathematics is presented.

1 Introduction

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in sections 2 and 3). For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as "classical" or "semiclassical" objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$), see [9–13].

Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to the traditional

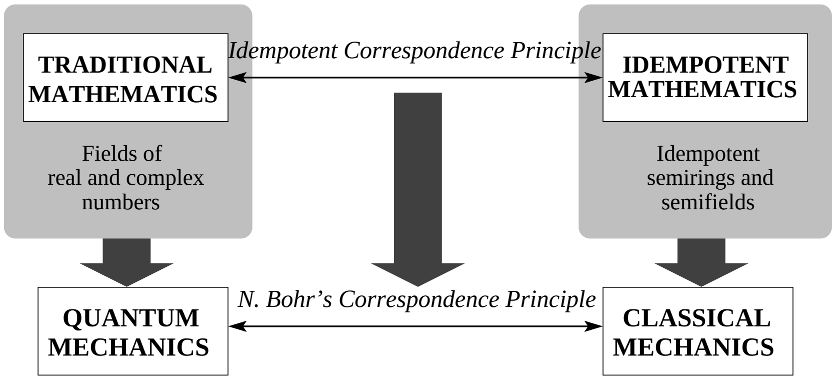

Fig. 1 Relations between idempotent and traditional mathematics.

algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [7,34,35,38–40]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of N.Bohr's correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar results over idempotent semirings. A systematic application of this correspondence principle (which is a basic paradigm in idempotent/tropical mathematcs) leads to a variety of theoretical and applied results [9–14, 20], see Fig.1.

The history of the subject is discussed, e.g., in [9]. There is a large list of references.

2 The Maslov dequantization

Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real and complex numbers. The so-called max-plus algebra $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be treated as a result of the *Maslov dequantization* of the semifield $\mathbf{R}_+$ of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x,$$

where $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$, see Fig.2. This logarithmic transform was used by many authors. Let the addition and multiplication

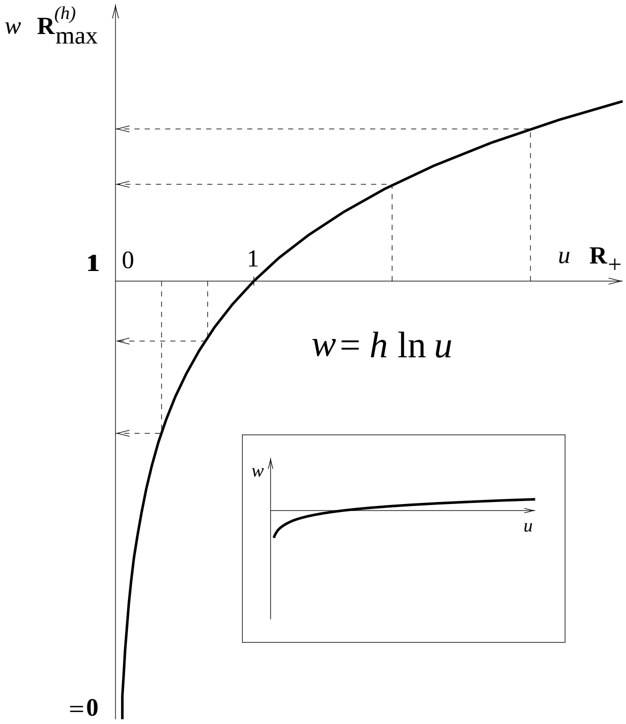

Fig. 2 Deformation of $\mathbf{R}_+$ to $\mathbf{R}^{(h)}$. Inset: the same for a small value of h.

operations be mapped from $\mathbf{R}_+$ to $\mathbf{R} \cup \{-\infty\}$ by Φ_h, i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,$$
$$\mathbf{0} = -\infty = \Phi_h(0), \quad \mathbf{1} = 0 = \Phi_h(1).$$

It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. Thus we get the semifield $\mathbf{R}_{\max}$ (i.e. the max-plus algebra) with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$ as a result of this deformation of the algebraic structure in $\mathbf{R}_+$.

The semifield $\mathbf{R}_{\max}$ is a typical example of an *idempotent semiring*; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element x of this semiring.

The semifield $\mathbf{R}_{\max}$ is also called a *tropical algebra.*The semifield $\mathbf{R}^{(h)} = \Phi_h(\mathbf{R}_+)$ with operations $\oplus_h$ and $\odot$ (i.e.+) is called a *subtropical algebra.*

The semifield $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with operations $\oplus = \min$ and $\odot = +$ $(\mathbf{0} = +\infty, \mathbf{1} = 0)$ is isomorphic to $\mathbf{R}_{\max}$.

The analogy with quantization is obvious; the parameter h plays the role of the Planck constant. The map $x \mapsto |x|$ and the Maslov dequantization for $\mathbf{R}_+$ give us a natural transition from the field $\mathbf{C}$ (or $\mathbf{R}$) to the max-plus algebra $\mathbf{R}_{\max}$. *We will also call this transition the Maslov dequantization.* In fact the Maslov dequantization corresponds to the usual Schrödinger dequantization but for imaginary values of the Planck constant (see below). The transition from numerical fields to the max-plus algebra $\mathbf{R}_{\max}$ (or similar semifields) in mathematical constructions and results generates the so called *tropical mathematics*. The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is the transition from basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called *idempotent mathematics*, i.e. mathematics over idempotent semifields and semirings. Recently new versions of the Maslov dequantization appeared, see, e.g. [41].

Remark. The term 'tropical' appeared in [37] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand V.P. Maslov used this term in 80s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, 'tropical' means 'over $\mathbf{R}_{\max}$ (or $\mathbf{R}_{\min}$)' and tropical algebras are $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. The terms 'max-plus', 'max-algebra' and 'min-plus' are often used in the same sense.

3 Semirings and semifields

Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

and

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all $x, y, z \in S$.

A *unity* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of a semiring S is an element (if it exists) $\mathbf{0} \in S$ such that

$\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with a neutral element $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible with respect to the multiplication. The theory of semirings and semifields is treated, e.g., in [5].

4 Idempotent analysis

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject is presented in the book of V. N. Kolokoltsov and V. P. Maslov [8] (a version of this book in Russian was published in 1994).

Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, and unit $\mathbf{1}$. The set S is supplied with the *standard partial order* $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. If the zero element exists, then all elements of S are nonnegative: $\mathbf{0} \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic "topological" concepts and results of idempotent analysis at the purely algebraic level; this line of reasoning was examined systematically in [9]– [24] and [3].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $B(X, S)$ of all functions $X \to S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\, dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum\limits_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus\limits_i \varphi(x_i) \odot \sigma_i = \max\limits_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An *idempotent* (or *Maslov*) *measure* on X is defined by the formula $m_\psi(Y) = \sup_{x\in Y} \psi(x)$, where $\psi \in B(X,S)$ is a fixed function. The integral with respect to this measure is defined by the formula

$$I_\psi(\varphi) = \int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x\in X} (\varphi(x) \odot \psi(x)). \qquad (2)$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order is opposite to the conventional order $\leq$, so in this case equation (2) assumes the form

$$\int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \inf_{x\in X} (\varphi(x) \odot \psi(x)),$$

where inf is understood in the sense of the conventional order $\leq$.

5 The superposition principle and linear problems

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings; this is V. P. Maslov's idempotent superposition principle, see [29–31]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F [1, 2]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [1, 2].

The linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$, which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),$$

where $x = (x_1, \ldots, x_N)$ are generalized coordinates, $p = (p_1, \ldots, p_N)$ are generalized momenta, m_i are generalized masses, and $V(x)$ is the potential. In this case the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

$$S = \int_{t_0}^{t} L(x(t), \dot{x}(t), t)\, dt,$$

where the integration is performed along the factual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of t and t_0 and arbitrary trajectories $x(t)$, the action functional $S = S(x(t))$ can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as elements of $\mathbf{R}_{\min}$. In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of e^{-S}, i.e. idempotent integral $\int_{\{paths\}}^{\oplus} e^{-S(x(t))} D\{x(t)\}$ with respect to the max-plus algebra $\mathbf{R}_{\max}$. Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleĭnik solution formula for the Hamilton–Jacobi equation.

Since $\partial S/\partial x_i = p_i$, $\partial S/\partial t = -H(p, x)$, the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \tag{3}$$

Quantization leads to the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \widehat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi, \tag{4}$$

where $\psi = \psi(x, t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbf{R}^N)$, and $\widehat{H}$ is the energy operator obtained by substitution of the momentum operators $\widehat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$ and the coordinate operators $\widehat{x}_i \colon \psi \mapsto x_i\psi$ for the variables p_i and x_i in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x,t) = a(x,t)e^{iS(x,t)/\hbar}$, and to keep only the leading order as $\hbar \to 0$ (the 'semiclassical' limit).

Instead of doing this, we switch to imaginary values of the Planck constant $\hbar$ by the substitution $h = i\hbar$, assuming $h > 0$. Thus the Schrödinger equation (4) turns to an analog of the heat equation:

$$h\frac{\partial u}{\partial t} = H\left(-h\frac{\partial}{\partial x_i}, \hat{x}_i\right) u, \tag{5}$$

where the real-valued function u corresponds to the wave function ψ. A similar idea (the switch to imaginary time) is used in the Euclidean quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of equation (5). Thus if u_1 and u_2 are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \tag{6}$$

Let $S = h\ln u$ or $u = e^{S/h}$ as in Section 2 above. It can easily be checked that equation (5) thus turns to

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N} \frac{1}{2m_i}\left(\frac{\partial S}{\partial x_i}\right)^2 + h\sum_{i=1}^{n}\frac{1}{2m_i}\frac{\partial^2 S}{\partial x_i^2}. \tag{7}$$

Thus we have a transition from (4) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{ReS/h}$, where ReS is the real part of S. Now let us consider S as a real variable. The equation (7) is nonlinear in the conventional sense. However, if S_1 and S_2 are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$

obtained from (6) by means of our substitution $S = h\ln u$. Here the generalized multiplication $\odot$ coincides with the ordinary addition and the generalized addition $\oplus_h$ is the image of the conventional addition under the above change of variables. As $h \to 0$, we obtain the operations of the idempotent semiring $\mathbf{R}_{\max}$, i.e., $\oplus = \max$ and $\odot = +$, and equation (7) turns to the Hamilton–Jacobi equation (3), since the third term in the right-hand side of equation (7) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over $\mathbf{R}_{\max}$. This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations see, e.g., [8, 14, 36]. Notice that if h is changed to $-h$, then we have that the resulting Hamilton–Jacobi equation is linear over $\mathbf{R}_{\min}$.

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. The idempotent linear functional analysis (see below) is a natural tool for investigation of those nonlinear infinite-dimensional problems that possess this property.

6 Convolution and the Fourier–Legendre transform

Let G be a group. Then the space $\mathcal{B}(G, \mathbf{R}_{\max})$ of all bounded functions $G \to \mathbf{R}_{\max}$ (see above) is an idempotent semiring with respect to the following analog $\circledast$ of the usual convolution:

$$(\varphi(x) \circledast \psi)(g) == \int_G^{\oplus} \varphi(x) \odot \psi(x^{-1} \cdot g)\, dx = \sup_{x \in G} (\varphi(x) + \psi(x^{-1} \cdot g)).$$

Of course, it is possible to consider other "function spaces" (and other basic semirings instead of $\mathbf{R}_{\max}$).

Let $G = \mathbf{R}^n$, where $\mathbf{R}^n$ is considered as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x)\, dx, \tag{8}$$

where $e^{i\xi \cdot x}$ is a character of the group G, i.e., a solution of the following functional equation:

$$f(x + y) = f(x) f(y).$$

The idempotent analog of this equation is

$$f(x + y) = f(x) \odot f(y) = f(x) + f(y),$$

so "continuous idempotent characters" are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (8) assumes the form

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^{\oplus} \xi \cdot x \odot \varphi(x)\, dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)). \tag{9}$$

The transform in (9) is nothing but the *Legendre transform* (up to some notation) [31]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics. The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [27].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution $\circledast$ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform.

The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory, see, e.g., [19, 26]. In particular, "idempotent" representations of groups and semigroups can be examined as representations of the corresponding convolution semirings (i.e. idempotent (semi)group semirings) in semimodules.

7 Idempotent functional analysis

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity take $S = \mathbf{R}_{\max}$ and let X be an arbitrary set. The idempotent integration can be defined by the formula (1), see above. The functional $I(\varphi)$ is linear over S and its values correspond to limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions φ and ψ is defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).$$

So it is natural to construct idempotent analogs of integral operators in the form

$$\varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x,y) \odot \varphi(y)\, dy = \sup_{y \in Y} \{K(x,y) + \varphi(y)\}, \tag{10}$$

where $\varphi(y)$ is an element of a space of functions defined on a set Y, and $K(x, y)$ is an S-valued function on $X \times Y$. Of course, expressions of this type are standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings which are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces it was proved that every 'good' linear operator (in the idempotent sense) can be presented in the form (10); this is an idempotent version of the kernel theorem of L. Schwartz; results of this type were proved by V. N. Kolokoltsov, P. S. Dudnikov and S. N. Samborskiĭ, I. Singer, M. A. Shubin and others. So every 'good' linear functional can be presented in the form $\varphi \mapsto \langle \varphi, \psi \rangle$, where $\langle , \rangle$ is an idempotent scalar product.

In the framework of idempotent functional analysis results of this type can be proved in a very general situation. In [16–19, 22, 24] an algebraic version of the idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms. The treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known theorems of Hahn-Banach, Riesz, and Riesz-Fisher) to idempotent analogs of A. Grothendieck's concepts and results on topological tensor products, nuclear spaces and operators. Abstract idempotent versions of the kernel theorem is formulated. Note that the passage from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many non-isomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) were obtained by G. Cohen, S. Gaubert, and J.-P. Quadrat. Idempotent functional analysis has received much attention in the last years, see, e.g., [3], [8]– [24] and works cited in [9].

8 The dequantization transform and the Newton polytopes

Let X be a topological space. For functions $f(x)$ defined on X we shall say that a certain property is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X. Suppose X is $\mathbf{C}^n$ or $\mathbf{R}^n$; denote by $\mathbf{R}^n_+$ the set $x = \{ (x_1, \ldots, x_n) \in X \mid x_i \geq 0$ for $i = 1, 2, \ldots, n$. For $x = (x_1, \ldots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \ldots, \exp(x_n))$; so if $x \in \mathbf{R}^n$, then $\exp(x) \in \mathbf{R}^n_+$.

Denote by $\mathcal{F}(\mathbf{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbf{C}^n$ such that $U \supset \mathbf{R}^n_+$. It is clear that $\mathcal{F}(\mathbf{C}^n)$ is a ring (and an algebra over $\mathbf{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbf{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|, \tag{11}$$

where h is a (small) real positive parameter and $x \in \mathbf{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to +0} \hat{f}_h(x), \tag{12}$$

if the right-hand side of (12) exists almost everywhere.

We shall say that the function $\hat{f}(x)$ is a *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbf{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to $\mathbf{R}^n_+$ only; so in fact the dequantization transform is constructed for functions defined on $\mathbf{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$.

Of course, similar definitions can be given for functions defined on $\mathbf{R}^n$ and $\mathbf{R}^n_+$. If $s = 1/h$, then we have the following version of (11) and (12):

$$\hat{f}(x) = \lim_{s \to \infty} (1/s) \log |f(e^{sx})|. \tag{12'}$$

Denote by $\partial \hat{f}$ the subdifferential of the function $\hat{f}$ at the origin.

If f is a polynomial or if $\hat{f}$ is a sublinear function we have

$$\partial \hat{f} = \{\, v \in \mathbf{R}^n \mid (v, x) \leq \hat{f}(x) \ \forall x \in \mathbf{R}^n \,\}. \tag{1}$$

It is well known that all the convex compact subsets in $\mathbf{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $\alpha, \beta \in \mathcal{S}$ the sum $\alpha \oplus \beta$ is the convex hull of the union $\alpha \cup \beta$; the product $\alpha \odot \beta$ is defined in the following way: $\alpha \odot \beta = \{\, x \mid x = a + b$, where $a \in \alpha, b \in \beta$, see Fig.3. In fact $\mathcal{S}$ is an idempotent linear space over $\mathbf{R}_{\max}$.

Of course, the Newton polytopes of polynomials in n variables form a subsemiring $\mathcal{N}$ in $\mathcal{S}$. If f, g are polynomials, then $\partial(\widehat{fg}) = \partial \hat{f} \odot \partial \widehat{g}$; moreover, if f and g are "in general position", then $\partial(\widehat{f+g}) = \partial \hat{f} \oplus \partial \widehat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this "traditional" semiring to the idempotent semiring $\mathcal{N}$.

Theorem 1 *If f is a polynomial, then the subdifferential $\partial \hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of f. For the semiring of polynomials with nonnegative coefficients, the transform $f \mapsto \partial \hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the Minkowski operations (see above).*

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets [23].

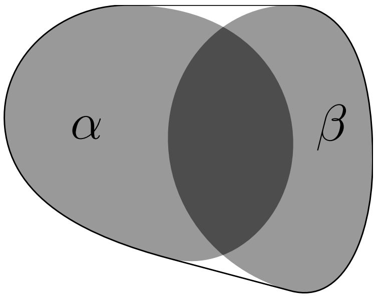

Fig. 3 Algebra of convex subsets.

9 Dequantization of set functions and measures on metric spaces [25]

Example 1. Let M be a metric space, S its arbitrary subset with a compact closure. It is well-known that a Euclidean d-dimensional ball B_ρ of radius ρ has volume

$$\mathrm{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1+d/2)}\rho^d,$$

where d is a natural parameter. By means of this formula it is possible to define a volume of B_ρ for any *real* d. Cover S by a finite number of balls of radii ρ_m. Set

$$v_d(S) := \lim_{\rho\to 0} \inf_{\rho_m<\rho} \sum_m \mathrm{vol}_d(B_{\rho_m}).$$

Then there exists a number D such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number D is called the *Hausdorff-Besicovich dimension* (or *HB-dimension*) of S, see, e.g., [28]. Note that a set of non-integral HB-dimension is called a fractal in the sense of B. Mandelbrot.

Theorem 1 *Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius ρ covering S. Then*

$$D(S) = \underline{\lim}_{\rho\to+0} \log_\rho(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of S. Set $\rho = e^{-s}$, then

$$D(S) = \underline{\lim}_{s\to+\infty} (1/s)\cdot\log\mathcal{N}_{exp(-s)}(S).$$

So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $\mathcal{N}_\rho(S)$.

Example 2. Let μ be a set function on M (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball B_ρ. Let $B_{x,\rho}$ be a ball of radius ρ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let $\rho = e^{-s}$ and

$$D_{x,\mu} := \underline{\lim}_{s \to +\infty} -(1/s) \cdot \log(|\mu_x(e^{-s})|).$$

This number could be treated as a dimension of M at the point x with respect to the set function μ. So this dimension is a result of a dequantization of the function $\mu_x(\rho)$, where x is fixed. There are many dequantization procedures of this type in different mathematical areas. In particular, V.P. Maslov's negative dimension (see [32]) can be treated similarly.

10 Dequantization of geometry

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [38]. Starting from the idempotent correspondence principle O. Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problem for constructing real algebraic varieties with prescribed properties and parameters) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, see [4, 39]. Then complex algebraic geometry was dequantized by G. Mikhalkin and the result turned out to be the same; this new 'idempotent' (or asymptotic) geometry is now often called the *tropical algebraic geometry*, see, e.g., [7, 14, 15, 21, 34, 35].

There is a natural relation between the Maslov dequantization and amoebas.

Suppose $(\mathbf{C}^*)^n$ is a complex torus, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ is the group of nonzero complex numbers under multiplication. For $z = (z_1, \ldots, z_n) \in (\mathbf{C}^*)^n$ and a positive real number h denote by $\mathrm{Log}_h(z) = h\log(|z|)$ the element

$$(h\log|z_1|, h\log|z_2|, \ldots, h\log|z_n|) \in \mathbf{R}^n.$$

Suppose $V \subset (\mathbf{C}^*)^n$ is a complex algebraic variety; denote by $\mathcal{A}_h(V)$ the set $\mathrm{Log}_h(V)$. If $h = 1$, then the set $\mathcal{A}(V) = \mathcal{A}_1(V)$ is called the *amoeba* of V; the amoeba $\mathcal{A}(V)$ is a closed subset of $\mathbf{R}^n$ with a non-empty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity suppose V is a hypersurface in $(\mathbf{C}^*)^n$ defined by a polynomial f; then there is a deformation $h \mapsto f_h$ of this polynomial generated by the Maslov dequantization and $f_h = f$ for $h = 1$. Let $V_h \subset (\mathbf{C}^*)^n$ be the

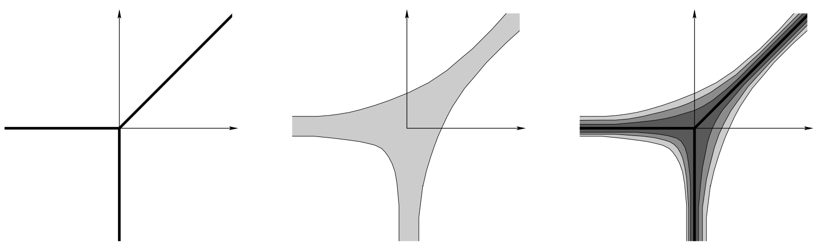

Fig. 4 Tropical line and deformations of an amoeba.

zero set of f_h and set $\mathcal{A}_h(V_h) = \mathrm{Log}_h(V_h)$. Then there exists a tropical variety $Tro(V)$ such that the subsets $\mathcal{A}_h(V_h) \subset \mathbf{R}^n$ tend to $Tro(V)$ in the Hausdorff metric as $h \to 0$. The tropical variety $Tro(V)$ is a result of a deformation of the amoeba $\mathcal{A}(V)$ and the Maslov dequantization of the variety V. The set $Tro(V)$ is called the *skeleton* of $\mathcal{A}(V)$.

Example 3. For the line $V = \{\, (x, y) \in (\mathbf{C}^*)^2 \mid x+y+1 = 0 \,\}$ the piecewise-linear graph $Tro(V)$ is a tropical line, see Fig.4(a). The amoeba $\mathcal{A}(V)$ is represented in Fig.4(b), while Fig.4(c) demonstrates the corresponding deformation of the amoeba.

11 Applications

There are very many important applications of tropical/idempotent mathematics including optimization and control, algebraic geometry, dynamic programming, differential equations, mathematical biology, mathematical physics and chemistry, transport and energoenergetic netwoks, interval analysis, mathematical economics, game theory, computer technology etc., see, e.g. [1,2,6,8,9,11,12,14, 15,20,21,29,30,33–36,38,40,41]. Applications of the idempotent correspondence principles to software and hardware design are examined, e.g. in [11,12,20]. Some applications are discucced in the present Proceedings.

References

1. B. A. Carré, *An algebra for network routing problems*, J. Inst. Appl. **7** (1971), 273-294.
2. B. A. Carré, *Graphs and networks*, The Clarendon Press/Oxford University Press, Oxford, 1979.
3. G. Cohen, S. Gaubert, and J.-P. Quadrat, *Duality and separation theorems in idempotent semimodules*, Linear Algebra and its Applications **379** (2004), 395–422. Also arXiv:math.FA/0212294.
4. I. M. Gelfand, M. M. Kapranov, and A. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, 1994.
5. J. S. Golan, *Semirings and their applications*, Kluwer Acad. Publ., Dordrecht, 1999.

6. J. Gunawardena (Ed.), *Idempotency*, Publ. of the Newton Institute, Vol. **11**, Cambridge University Press, Cambridge, 1998.
7. I. Itenberg, G. Mikhalkin, E. Shustin, *Tropical Algebraic Geometry*, Oberwolfach Seminars, Vol. **35**, Birkhäuser, Basel e.a., 2007.
8. V. Kolokoltsov and V. Maslov, *Idempotent analysis and applications*, Kluwer Acad. Publ., 1997.
9. G. L. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction*, Journal of Mathematical Sciences **140**, #3(2007), 426–444. Also arXiv:math.GM/0507014.
10. G. L. Litvinov, *Tropical mathematics, idempotent analysis, classical mechanics and geometry.* - in: Spectral Theory and Geometric Analysis M.Braverman et al., Eds., AMS Contemporary Mathematics, vol. 535, 2011, p. 159-186. See also E-print arXiv: 1005.1247 (http://arXiv.org)
11. G. L. Litvinov and V. P. Maslov, *Correspondence principle for idempotent calculus and some computer applications*, (IHES/M/95/33), Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, 1995. Also arXiv:math.GM/0101021.
12. G. L. Litvinov and V. P. Maslov, *Idempotent mathematics: correspondence principle and applications*, Russian Mathematical Surveys **51** (1996), no. 6, 1210–1211.
13. G. L. Litvinov and V. P. Maslov, *The correspondence principle for idempotent calculus and some computer applications.* — In [6], p. 420–443.
14. G. L. Litvinov and V. P. Maslov (Eds.), *Idempotent mathematics and mathematical physics*, Contemporary Mathematics, Vol. 377, AMS, Providence, RI, 2005.
15. G. L. Litvinov, V. P. Maslov and S. N. Sergeev (Eds.), *International workshop IDEMPOTENT AND TROPICAL MATHEMATICS AND PROBLEMS OF MATHEMATICAL PHYSICS*, Moscow, Independent Univ. of Moscow, vol. I and II, 2007. Also arXiv:0710.0377 and arXiv:0709.4119.
16. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Linear functionals on idempotent spaces: an algebraic approach*, Doklady Mathematics **58** (1998), no. 3, 389–391. Also arXiv:math.FA/0012268.
17. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Tensor products of idempotent semimodules. An algebraic approach*, Mathematical Notes **65** (1999), no. 4, 497–489. Also arXiv:math.FA/0101153.
18. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Idempotent functional analysis. An algebraic approach*, Mathematical Notes **69** (2001), no. 5, 696–729. Also arXiv:math.FA/0009128.
19. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, *Idempotent (asymptotic) analysis and the representation theory.* – In: V. A. Malyshev and A. M. Vershik (Eds.), Asymptotic Combinatorics with Applications to Mathematical Physics. Kluwer Academic Publ., Dordrecht et al, 2002, p. 267–278. Also arXiv:math.RT/0206025.
20. G. L. Litvinov, V. P. Maslov, A. Ya. Rodionov, A.N. Sobolevski, *Universal algorithms, mathematics of semirings and parallel computations.* - In: A. N. Gorban and D. Roose, Eds., Coping with Complexity: Model Reduction and Data Analysis, Lecture Notes in Computational Science and Engineering, Vol. 75, 2011, p. 63-89. See also E-print arXiv: 1005.1252 (http://arXiv.org).
21. G. L. Litvinov, S. N. Sergeev (Eds.), *Tropical and Idempotent Mathematics*, Contemporary Mathematics, Vol. 495, AMS, Providence, RI, 2009.
22. G. L. Litvinov and G. B. Shpiz, *Nuclear semimodules and kernel theorems in idempotent analysis: an algebraic approach*, Doklady Mathematics **66** (2002), no. 2, 197–199. Also arXiv :math.FA/0202026.
23. G. L. Litvinov and G. B. Shpiz, *The dequantization transform and generalized Newton polytopes.* — In [14], p. 181–186.
24. G. L. Litvinov and G. B. Shpiz, *Kernel theorems and nuclearity in idempotent mathematics. An algebraic approach*, Journal of Mathematical Sciences **141**, #4(2007), 1417–1428. Also arXiv:mathFA/0609033.
25. G. L. Litvinov and G. B. Shpiz, *Dequantization procedures related to the Maslov dequantization.* — In [15], vol. I, p. 99–104.
26. G. L. Litvinov and G. B. Shpiz, *Versions of the Engel theorem for semigroups.* The present Proceedings.
27. G. G. Magaril-Il'yaev and V. M. Tikhomirov, *Convex analysis: theory and applications*, Translations of Mathematical Monographs, vol. **222**, American Math. Soc., Providence, RI, 2003.

28. Yu. I. Manin, *The notion of dimension in geometry and algebra*, E-print arXiv:math.AG/05 02016, 2005.
29. V. P. Maslov, *New superposition principle for optimization problems.* — In: Seminaire sur les Equations aux Dérivées Partielles 1985/86, Centre Math. De l'Ecole Polytechnique, Palaiseau, 1986, exposé 24.
30. V. P. Maslov, *On a new superposition principle for optimization problems*, Uspekhi Mat. Nauk, [Russian Math. Surveys], **42**, no. 3 (1987), 39–48.
31. V. P. Maslov, *Méthodes opératorielles*, Mir, Moscow, 1987.
32. V. P. Maslov, *A general notion of topological spaces of negative dimension and quantization of their densities*, Math. Notes, **81**, no. 1 (2007), 157–160.
33. V. P. Maslov and S. N. Samborskii (Eds.), *Idempotent Analysis*, Adv. Soviet Math., **13**, Amer. Math. Soc., Providence, R.I., 1992.
34. G. Mikhalkin, *Enumerative tropical algebraic geometry in* $\mathbf{R}^2$, Journal of the ACM **18** (2005), 313–377. Also arXiv:math.AG/0312530.
35. G. Mikhalkin, *Tropical geometry and its applications*, Proceedings of the ICM, Madrid, Spain, vol. II, 2006, pp. 827–852. Also arXiv:math.AG/0601041v2.
36. I. V. Roublev, *On minimax and idempotent generalized weak solutions to the Hamilton–Jacobi Equation.* – In [14], p. 319–338.
37. I. Simon, *Recognizable sets with multiplicities in the tropical semiring.* Lecture Notes in Computer Science **324** (1988), 107–120.
38. O. Viro, *Dequantization of real algebraic geometry on a logarithmic paper.* — In: 3rd European Congress of Mathematics, Barcelona, 2000. Also arXiv:math/0005163.
39. O. Viro, *What is an amoeba?*, Notices of the Amer. Math, Soc. **49** (2002), 916–917.
40. O. Viro, *From the sixteenth Hilbert problem to tropical geometry*, Japan. J. Math. **3** (2008), 1–30.
41. O. Viro, *Hyperfields in tropical geometry I. Hyperfields and dequntization*, E-print arXiv:math.AG/1006.3034v2.

This work is supported by the RFBR grant 12–01–00886-a and the joint RFBR-CNRS grant 11–01–93106-a.

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Bose Condensate in the D-Dimensional Case

V. P. Maslov

Abstract In the paper, the problem of Bose condensation into the zero energy of particles is investigated using methods of number theory. We examine the D-dimensional case, in particular, for $D = 2$.

The author studied the relationship between the economy during a crisis and the Bose condensate, which corresponds to the bankruptcy [9]. Continuing the correspondence principle proposed by Irving Fisher, an economist and a disciple of Gibbs (this principle is the "fundamental law of economics"), where the amount of money M corresponds to the number of particles N, the author suggested to compare the chemical potential to the negative value of the nominal interest rate, which corresponds to Friedman's rule.

The issue of money accompanied the fall of the nominal interest to 0.5% following this dependence in which the small parameter $\frac{\mu}{T_d} N_d$ became equal to $\frac{1}{2D}$, where D stands for the "number of degrees of freedom", which can be fractional (in our case, this number is the dimension) [8].

In 1925, Einstein, when examining a work of Bose, discovered a new phenomenon, which he called the Bose condensate. A modern presentation of this discovery can be found in [1]. An essential point in this presentation is to define the entropy of the Bose gas. The definition is related to the dimension by means of the so-called "number of states" (cells), which is denoted by G_j in the book [1]. After this, the problem of minimizing the entropy is considered by using the Lagrange multipliers under two constraints, namely, for the number of particles and for energy. The number of states G_j is determined by the formula which mathematicians call the "Weyl relation;" it is described in detail in [2] in the "semiclassical case" in the section "Several degrees of freedom." The $2D$-dimensional phase space is partitioned into a lattice, and the number G_j is defined by the formula

$$G_i = \frac{\Delta p_j \Delta q_j}{(2\pi h)^D}. \tag{1}$$

The indeterminate Lagrange multipliers are expressed in terms of temperature and chemical potential of the gas.

Further, in [1], following Einstein, a passage to the limit is carried out as $N \to \infty$, which enables one to pass from sums to integrals. Then, in the section "Degenerate Bose gas," a point is distinguished which corresponds to the energy equal to zero. This very point is the point of Bose condensate on which excessive particles whose number exceeds some value $N_d \gg 1$ are accumulated at temperatures below the so-called degeneracy temperature T_d. The theoretical discovery of this point anticipated a number of experiments that confirmed this fact not only for liquid helium but also for a series of metals and even for hydrogen.

From a mathematical point of view, distinguishing a point in the integral is an incorrect operation if this point does not form a δ function. In particular, for the two-dimensional case, this incorrectness leads to a "theorem" formulated in various textbooks and claiming that there is no Bose condensate in the two-dimensional case.

In this paper, we get rid of this mathematical incorrectness and show that, both in the two-dimensional and in the one-dimensional case, the Bose condensate exists if the point introduced above is well defined.

The main idea of the author in the proof of the occurrence of the Bose condensate in the D-dimensional case is in a concordance between the chemical potential $\mu \to 0$ and the number of particles $N \to \infty$ when passing to the limit.

The phenomenon associated with the point of condensation holds only if the limit as $\mu \to 0$ depends on $N \to \infty$.

If we accept Einstein's remarkable discovery for the three-dimensional case and justify it in a mathematically correct way, then the Bose condensate in the two-dimensional case is equally correct mathematically. We dwell on the two-dimensional case below in particular detail.

Thus, we consider the case in which $N \gg 1$, but n is not equal to infinity. In the section "Ideal gas in the case of parastatistics" of the textbook by Kvasnikov [3], there is a problem (whose number in the book is (33)) which corresponds to the final parastatistics

$$n_j = \frac{1}{\exp\{\frac{\varepsilon_j-\mu}{T}\}-1} - \frac{k+1}{\exp\{(k+1)\frac{\varepsilon_j-\mu}{T}\}-1}, \qquad n_j = \frac{N_j}{G_j}. \tag{2}$$

In our case, we have $k = N_d$, and the point of condensate is $\varepsilon_0 = 0$.

By (1), it is clear that G_j is associated with the D-dimensional Lebesgue measure and, in the limit with respect to the coordinates Δq_j, gives the volume V in the space of dimension 3 and the area Q in the space of dimension 2. The passage with respect to the momenta Δp_j is also valid as $N \to \infty$ and $\mu > \delta > 0$, where δ is arbitrarily small.

Expanding (2) at the point $\varepsilon_0 = 0$ in the small parameter

$$x = (\mu N_d)/T_d,$$

where N_d stands for the number of particles corresponding to the degeneration and T_d for the degeneracy temperature, and writing

$$\xi = -\mu/T_d,$$

we obtain ($G_0 = 1$, see (12) below)

$$n_0 = \left\{\frac{1}{\exp\{\frac{-\mu}{T}\}-1} - \frac{N_d+1}{\exp\{(N_d+1)\frac{-\mu}{T}\}-1}\right\} = \frac{e^{\xi N_d}-1-(N_d+1)(e^{\xi}-1)}{(e^{\xi}-1)(e^{2N_d}-1)}$$
$$= \frac{N_d}{2}\frac{1+\frac{x}{6}+\frac{x^2}{4!}+\frac{x^3}{5!}+\dots}{1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{4!}+\dots} = \frac{N_d}{2}\left(1-\frac{x}{3}-\frac{11}{24}x^2-0.191x^3-\dots\right). \tag{3}$$

For example, if $x \to 0$, then $n_0 = N_d/2$, and hence the number n_0 in the condensate at $T = T_d$ does not exceed $N_d/2$. If $x = 1.57$, then $n_0 \approx N_d/10$. Certainly, this affects the degeneracy temperature, because this temperature can be expressed only in terms of the number of particles above the condensate, $\tilde{N}_d$, rather than in terms of the total number of particles N_d (which is equal to the sum of $\tilde{N}_d$ and of the number of particles in the condensate).

According to the concept of Einstein, at $T = T_d$ the condensate contains $o(N_d)$ particles. However, even this accumulation gives a δ function, albeit with a small coefficient (in the two-dimensional case, this coefficient is

$$\tilde{N}_d/\ln N_d,$$

and therefore it is $o(N_d)$).

To reconcile the notion of Bose statistics which is given in [1] with symmetric solutions of the N-particle Schrödinger equation, i.e., of the direct sum of N noninteracting Hamiltonians corresponding to the Schrödinger equation, and the symmetric solutions of their spectrum, it is more appropriate to assign to the cells the multiplicities of the spectrum of the Schrödinger equation in the way described in [4].

Consider the nonrelativistic case in which the Hamiltonian H is equal to

$$p^2/(2m),$$

where p stands for the momentum.

The comparison of G_i with the multiplicities of the spectrum of the Schrödinger equation gives a correspondence between the eigenfunctions of the N-partial Schrödinger equation that are symmetric with respect to the permutations of particles and the combinatorial calculations of the Bose statistics that are presented in [1].

A single-particle ψ-function satisfies the free Schrodinger equation with the Dirichlet conditions on the vessel walls. According to the classical Courant formula,

$$\lambda_j \sim \frac{2h^2}{m}\left(\frac{\pi^{D/2}\Gamma(D/2+1)}{V}\right)^{2/D} j^{2/D} \qquad \text{as} \quad j \to \infty, \tag{4}$$

where D stands for the dimension of the space, because the spectral density has the asymptotic behavior

$$\rho(\lambda) = \frac{V m^{D/2}\lambda^{D/2}}{\Gamma(D/2+1)(2\pi)^{D/2}h^D}(1+o(1)) \qquad \text{as} \quad \lambda \to \infty. \tag{5}$$

The asymptotics (4) is a natural generalization of this formula.

Using this very correspondence, we establish a relationship between the Bose–Einstein combinatorics [1], the definition of the N-particle Schrödinger equation, and the multiplicity of the spectrum of the single-particle Schrödinger equation.

The spectrum of the single-particle Schrödinger equation, provided that the interaction potential is not taken into account, coincides, up to a factor, with the spectrum of the Laplace operator. Consider its spectrum for the closed interval, for the square, and for the D-dimensional cube with zero boundary conditions. This spectrum obviously consists of the sum of one-dimensional spectra.

On the line we mark the points $i = 0, 1, 2, \dots$ and on the coordinate axes x, y of the plane we mark the points with $x = i = 0, 1, 2, \dots$ and $y = j = 0, 1, 2, \dots$. To this set of points (i, j) we assign the points on the line that are positive integers, $l = 1, 2, \dots$.

To every point we assign a pair of points, i and j, by the rule $i + j = l$. The number of these points is $n_l = l + 1$. This is the two-dimensional case.

Consider the 3-dimensional case. On the axis z we set $k = 0, 1, 2, \dots$, i.e., let

$$i + j + k = l$$

In this case, the number of points n_l is equal to

$$n_l = \frac{(l+1)(l+2)}{2}.$$

It can readily be seen, for the D-dimensional case, that the sequence of multiplicities for the number of variants

$$i = \sum_{k=1}^{D} m_k,$$

where m_k are arbitrary positive integers, is of the form

$$q_i(D) = \frac{(i+D-2)!}{(i-1)!(D-1)!}, \qquad \text{for } D = 2, \quad q_i(2) = i, \tag{6}$$

$$\sum_{i=1}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=1}^{\infty} q_i(D) N_i = E. \tag{7}$$

The following problem in number theory corresponds to the three-dimensional case $D = 3$ (cf. [1]):

$$\sum_{i=1}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=1}^{\infty} \frac{(i+2)!}{i!6} N_i = E, \qquad \frac{E}{\varepsilon} = M. \tag{8}$$

Write $M = E_d/\varepsilon_1$, where ε_1 stands for the coefficient in formula (4) for $j = 1$. Let us find E_d,

$$E_d = \int_0^{\infty} \frac{\frac{|p|^2}{2m}\, d\varepsilon}{e^{\frac{|p|^2}{2m}/T_d} - 1}, \tag{9}$$

where

$$d\varepsilon = \frac{|p|^2}{2m} \frac{dp_1 \dots dp_D \, dV_D}{(2\pi h)^D}. \tag{10}$$

Whence we obtain the coefficient α in the formula,

$$E_d = \alpha T_d^{2+\gamma} \zeta(1 + D/2) \Gamma(1 + D/2). \tag{11}$$

To begin the summation in (7) at the zero index (beginning with the zero energy), it is necessary to rewrite the sums (7) in the form

$$\sum_{i=0}^{\infty} N_i = N, \qquad \varepsilon \sum_{i=0}^{\infty} (q_i(D) - 1) N_i = E - \varepsilon N. \tag{12}$$

The relationship between the degeneracy temperature and the number $\tilde{N}_d$ of particles above the condensate for $\mu > \delta > 0$ (where δ is arbitrarily small) can be found for $D > 2$ in the standard way.

Thus, we have established a relationship between G_i in formula (1) (which is combinatorially statistical) and the multiplicity of the spectrum for the single-particle Schrödinger equation, i.e., between the statistical [1] and quantum-mechanical definitions of Bose particles.

For $D = 2$, the general problem reduces to a number theory problem.

Consider the two-dimensional case in more detail. There is an Erdős' theorem for a system of two Diophantine equations,

$$\sum_{i=1}^{\infty} N_i = N, \qquad \sum_{i=1}^{\infty} iN_i = M. \tag{13}$$

The maximum number of solutions of this system is achieved if the following relation is satisfied:

$$N_d = c^{-1} M_d^{1/2} \log M_d + aM_d^{1/2} + o(M_d^{1/2}), \qquad c = \pi\sqrt{2/3}, \tag{14}$$

and if the coefficient a is defined by the formula

$$c/2 = e^{-ca/2}.$$

The decomposition of M_d into one summand gives only one version. The decomposition M_d into M_d summands also provides only one version (namely, the sum of ones). Therefore, somewhere in the interval must be at least one maximum of the variants. Erdős had evaluated it (14) (see [7]).

If the number N increases and M is preserved in the problem (13), then the number of solutions decreases. If the sums (13) are counted from zero rather than from one, i.e., if we set

$$\sum_{i=0}^{\infty} iN_i = (M - N), \qquad \sum_{i=0}^{\infty} N_i = N, \tag{15}$$

then the number of solutions does not decrease and remains constant.

I'll try to explain this effect. The Erdős–Lehner problem [5] is to decompose M_d into $N \leq N_d$ summands. Let us expand the number 5 into two summands. We obtain $3 + 2 = 4 + 1$. The total number is 2 versions (this problem is known as "partitio numerorum"). If we include 0 to the possible summands, we obtain three versions: $5+0 = 3+2 = 4+1$. Thus, the inclusion of zero makes it possible to say that we expand a number into $k \leq n$ (positive integer) summands. Indeed, the expansion of the number 5 into three summands includes all the previous versions, namely, $5+0+0$, $3+2+0$, and $4+1+0$, and adds new versions, which do not include zero.

In this case, the maximum number of versions for the decomposition of the number 5 into N summands (there are two versions) is achieved at $N = 2$ and $N = 3$ (the two values for the maximum number of versions for N above the condensate).

In this case, the maximum does not change drastically [5]; however, the number of versions is not changed, namely, the zeros, i.e., the Bose condensate, make it possible that the maximum remains constant, and the entropy never decreases; after reaching the maximum, it becomes constant. This remarkable property of the entropy enables us to construct an unrestricted probability theory in the general case [8].

Let us turn to a physical definition.

Note first that, without changing the accuracy of the quantity whose logarithm is evaluated, we can replace $\log M_d$ by

$$(1/2)\log(\tilde{N}_d/Q).$$

Then

$$\sqrt{M_d} = \frac{2\tilde{N}_d/Q}{c^{-1}\log(\tilde{N}_d/Q) + a} + o\left(\frac{\tilde{N}_d}{Q}\right). \tag{16}$$

In our case, $\tilde{N}_d/Q$ corresponds to the number of particles above the condensate.

According to formula (11), in the two-dimensional case we must set $\gamma = 0$ and find the coefficient α. Then formula (16) gives us a relationship between $\tilde{N}_d$ and T_d due to the fact that the number of particles in the condensate is $o(\tilde{N}_d)$.

For details concerning the Bose condensate in the one-dimensional case, see [11], [12], and [13].

In fact, we have proved that there is a gap between $\mu > \delta > 0$ and $\mu = 0$. In the one-dimensional case, this gap in the spectrum is much wider than that in the two- and three-dimensional cases [11]. The consequences form a topic of another paper.

Remark 1 The author studied the relationship between the economy during a crisis and the Bose condensate, which corresponds to the bankruptcy [9]. Continuing the correspondence principle proposed by Irving Fisher, an economist and a disciple of Gibbs (this principle is the "fundamental law of economics"), where the amount of money M corresponds to the number of particles N, the author suggested to compare the chemical potential to the negative value of the nominal interest rate, which corresponds to Friedman's rule.

The issue of money accompanied the fall of the nominal interest to 0.5% following this dependence in which the small parameter $\frac{\mu}{T_d}N_d$ became equal to $\frac{1}{2D}$, where D stands for the "number of degrees of freedom", which can be fractional (in our case, this number is the dimension) [8].

In the paper [10] by E. M. Apfel'baum and V. S. Vorob'ev, taking into account the de Boer parameter, the values Z_c for helium were calculated experimentally, which, according to the author's rule

$$Z_c = \frac{\zeta(\frac{D}{2}+1)}{\zeta(\frac{D}{2})},$$

enables one to determine the number D of degrees of freedom and to experimentally verify whether or not there is an empirical relation of this kind between μ and T_d in thermodynamics. It was assumed that the Bose gas is not perfect (i.e., the Schrödinger equation with a potential is considered) and the value of $x = (\mu N_d)/T_d$ reflects the interaction between the particles, just as the Zeno line reflects the interaction between particles in classical thermodynamics [11].

Let us present heuristic considerations concerning the passage through the point T_d which were presented by the author in [14]. The author proved and used the Bose statistics in the case of a fractional number of degrees of freedom for classical thermodynamics, where to a value of T_d there corresponds the critical temperature. The author has shown that these values coincide. For an example describing the creation of a dimer, it is shown that, for $T = T_c$, one degree of freedom becomes " frozen", and we obtain two degrees of freedom rather than three. For a dimer with $T > T_c$, if the oscillational degrees of freedom are taken into account, then the number of degrees of freedom becomes equal to 6. Two degrees of freedom are obtained under the assumption that the oscillational degrees of freedom of the dimer are also "frozen" at $T = T_c$. If we suppose this heuristic supposition for the quantum case, then, for $T < T_d$, both dimers with two degrees of freedom and dimers with six degrees of freedom are created. This corresponds to the two-liquid Thiess–Landau model. In this case, the dimers with two degrees of freedom give the λ-point and the dimers with six degrees of freedom give superfluidity. Indeed, in the two-dimensional case we have

$$c_p \cong \frac{2T}{T_d}\int_0^\infty \frac{\xi\,d\xi}{e^\xi - 1} + \frac{T}{T_d}\int_0^\infty \frac{e^\xi \xi\,d\xi}{(e^{\xi-\mu/T}-1)^2} + O\biggl(\frac{T-T_d}{T_d}\biggr),$$

and we obtain a logarithmic divergence at the point $\xi = 0$ for $\mu \to 0$.

Thus, if we consider an N-particle Schrödinger equation whose eigenfunctions are symmetric under the permutations of the particles, then the parastatistic correction leads to the fact that $N/2$ particles are in the condensate for $T = T_d$ and $N = N_d$. For $N > N_d$, all the extra particles pass into the condensate state, which determines the dependence of the temperature T on N, and hence the dependence of N on the temperature for $T < T_d$ as well.

The case in which N is not so large as it is in statistical physics, i.e., the so-called mesoscopic state (see [14]), can also be of interest for us. In this case, let us use Fock's idea for the Hartree equations, which lead to the Hartree–Fock equations.

Namely, we consider the single-particle equation of the mean field (a self-consistent field) and apply (to the resulting "dressed" potential) the procedure of transition to the N-partial Schrödinger equation with a dressed potential, just as we proceeded above for the operator $\frac{h^2}{2m}\Delta$. Here we can consider two ways of investigation. The first way is the way used by Fock and which leads in the semiclassical limit to the Thomas–Fermi equations for the dressed potential. Another way is to consider the Hartree temperature equations (see [15]) and to obtain the Thomas–Fermi temperature equations in the classical limit.

Since the quantity T_d is small, it is easier to use the first way and to find the "dressed" potential.

Let $V(q - q')$ be a pairwise interaction potential such that $\int |V(r)|\, dr < \infty$. The dressed potential $W(q)$ is given by the formula

$$W(q) = U(q) + \int V(q - q')|\psi(q')|^2 \, dq',$$

where $U(q)$ stands for the external potential and $\psi(q')$ for an eigenfunction of the Schrödinger equation which depends on the "dressed" potential and is thus an equation with a "unitary" nonlinearity. The expansion of the equation in powers of h can be found by the method of complex germ up to $O(h^k)$, where k is an an arbitrarily large number[1] (see [23], where system (63) defines a complex germ; see also [24]– [30]). The superfluidity in nanotubes was confirmed experimentally.

The author thanks Professors G. I. Arkhipov, V. S. Vorob'ev, and V. N. Chubarikov for permanent discussions.

[1] For $U(q) \equiv 0$, one obtains Bogolyubov's famous equation [16]. The creation of dimers leads to the ultrasecond quantization, i.e., to the operators of creation and annihilation of pairs. This makes it possible to satisfy the boundary conditions in a capillary ([17]– [22]).

References

1. L. D. Landau and E. M. Lifshits, *Statistical Physics* (Nauka, Moscow, 1964) [in Russian].
2. L. D. Landau and E. M. Lifshits, *Quantum Mechanics* (Nauka, Moscow, 1976) [in Russian].
3. I. A. Kvasnikov, *Thermodynamics and Statistical Physics: Theory of Equilibrium Systems* (URSS, Moscow, 2002), Vol. 2 [in Russian].
4. V. P. Maslov, "Mathematical Aspects of Weakly Nonideal Bose and Fermi Gases on a Crystal Base", Funktsional. Anal. i Prilozhen. **37** (2), 16–27 (2003) [Functional Anal. Appl. **37** (2), (2003)].
5. P. Erdős, J. Lehner, "The Distribution of the Number of Summands in the Partitions of a Positive Integer," Duke Math. J. **8** (2), 335–345 (June 1941).
6. V. P. Maslov, "New Probability Theory Compatible with the New Conception of Modern Thermodynamics: Economics and Crisis of Debts," Russian Journal of Math. Physics **19** (1), 63–100 (2012).
7. P. Erdős, "On some asymptotic formulas in the theory of partitions," Bull. Amer. Math. Soc. **52**, 185–188, (1946).
8. V.P.Maslov, "Unbounded Probability Theory Compatible with the Probability Theory of Numbers," Math. Notes, **91** (5) 603–609, (2012).
9. V. P. Maslov, "Theorems on the Debt Crisis and the Occurrence of Inflation," Math. Notes, **85** (1) 146–150, (2009).
10. E. M. Apfelbaum, V. S. Vorob'ev, "Correspondence between the Critical and the Zeno-Line Parameters for Classical and Quantum Liquids," J. Phys. Chem. B, **113** (11), 3521–3526 (2009).
11. V. P. Maslov, "Mathematical conception of "phenomenological" equilibrium thermodynamics", Russ. J. Math. Phys. **18** (4), 363–370 (2011).
12. V. P. Maslov, "Demonstrativeness in Mathematics and Physics,", Russ. J. Math. Phys. **19** (2), 163–175 (2012).
13. V.P.Maslov, "Binodal for the New Ideal Gas and the Ideal Liquid," Math. Notes, **91** (6) 893–894, (2012).
14. V. P. Maslov, "Hypothetic λ-Point for Noble Gases," Russ. J. Math. Phys. **17** (4), 400–413 (2010).
15. V. P. Maslov, *Complex Markov Chains and the Feynman Path Integral for Nonlinear Equations* (Nauka, Moscow, 1976) [in Russian].
16. N. N. Bogolyubov, *On the Theory of Superfluidity*, in *Selected Works* (Naukova Dumka, Kiev, 1970), Vol. 2 [in Russian].
17. V. P. Maslov, "On the dependence of the criterion for superfluidity from the radius of the capillary," Teoret. Mat. Fiz. **143** (3), 307–327 (2005) [Theoret. and Math. Phys. **143** (3), 741–759 (2005)].
18. V. P. Maslov, "Resonance between one-particle (Bogoliubov) and two-particle series in a superfluid liquid in a capillary," Russ. J. Math. Phys. **12** (3), 369–379 (2005).
19. V. P. Maslov, "On the superfluidity of the classical fluid in a nanotube for even and odd numbers of neutrons in a molecule," Teoret. Mat. Fiz. **153** (3), 388–408 (2007) [Theoret. and Math. Phys. **153** (3), 1677–1696 (2007)].
20. V. P. Maslov, "On the superfluidity of classical liquid in nanotubes. I. Case of even number of neutrons," Russian J. Math. Phys. **14** (3), 304–318 (2007).
21. V. P. Maslov, "On the superfluidity of classical liquid in nanotubes. II. Case of odd number of neutrons," Russian J. Math. Phys. **14** (4), 401–412 (2007).
22. V. P. Maslov, "On the Superfluidity of Classical Liquid in Nanotubes. III," Russian J. Math. Phys. **15** (1), 61–65 (2008).
23. V. P. Maslov, "Quasi-particles associated with Lagrangian manifolds corresponding to semiclassical self-consistent fields. III," Russ. J. Math. Phys. **3** (2), 271–276 (1995).
24. V. P. Maslov and O. Yu. Shvedov, *The Method of Complex Germ.* (URSS, Moscow, 2000)[in Russian].
25. V. P. Maslov, *The Complex WKB Method in Nonlinear Equations* (Nauka, Moscow, 1977) [in Russian].
26. V. P. Maslov, *The Complex WKB Method for Nonlinear Equations I* (Birkhäuser Verlag, Basel–Boston–Berlin, 1994).
27. V. P. Maslov, "On an Integral Equation of the Form $u(x) = F(x) + \int G(x,\xi)u_+^{(n-2)/2}(\xi)d\xi/\int u_+^{(n-2)/2}(\xi)d\xi$ for $n = 2$ and $n = 3$", Mat. Zametki **55** (3), 96–108 (1994) [Math. Notes **55** (3–4), 302–311 (1994)].

28. V. P. Maslov, “Spectral Series, Superfluidity, and High-Temperature Superconductivity”, **58** (6), 933–936 (1995) [Math. Notes **58** (5–6), 1349–1352 (1995)].
29. V. P. Maslov and O. Yu. Shvedov, “The number of Bose-condensed particles in a weakly nonideal Bose gas,” Mat. Zametki **61** (5), 790–792 (1997) [Math. Notes, **61** (5), 661–664 (1997)].
30. V. P. Maslov, “On an averaging method for the quantum many-body problem,” Funktsional. Anal. i Prilozhen. **33** (4), 50–64 (1999) [Funct. Anal. Appl. **33** (4), 280–291 (2000)].

The research was supported by the RFBR grants 12–01–00886-a and 11–01–12058_ofi_m and by the Joint grant of RFBR and CNRS 11-01-93106_a.

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Fixed points of discrete convex monotone dynamical systems

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Abstract Convex, order preserving maps of $\mathbb{R}^n$ include at the same time tropical or max-plus linear maps, and affine order-preserving maps of $\mathbb{R}^n$. In these particular cases, the set of fixed points or additive eigenvectors and the convergence of the orbits are characterized respectively by the max-plus spectral theorem and the Perron-Frobenius theorem. We shall give a survey of the characterizations that have been obtained for general convex, order preserving maps of $\mathbb{R}^n$. We shall also discuss the possible extension to the setting of stationary solutions of Hamilton-Jacobi-Bellman equations.

This talk covers joint works with Stéphane Gaubert, Benoît David, and Bas Lemmens.

1 Convex, order-preserving maps as dynamic programming operators of stochastic control problems

A map f from $\mathbb{R}^n$ to itself is *order-preserving* if it preserves the partial order of $\mathbb{R}^n$, that is $f(v) \leq f(v')$ for all $v, v' \in \mathbb{R}^n$ such that $v \leq v'$, where $v \leq v'$ means $v_i \leq v'_i$ for all $i = 1, \ldots, n$. It is *convex* if all its coordinates $f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions. It is *additively homogeneous* if it commutes with the addition of a constant vector $f(\lambda + v) = \lambda + f(v)$ for all $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, where we use the notation $\lambda + v$ for the vector with entries $(\lambda + v)_i = \lambda + v_i$. It is *additively* *sub-homogeneous* if $f(\lambda + v) \leq \lambda + f(v)$ for all $v \in \mathbb{R}^n$ and $\lambda \geq 0$. An order-preserving additively sub-homogeneous map of $\mathbb{R}^n$ is necessarily nonexpansive for the sup-norm.

Moreover, it can be put in the form

$$f(v)_i := \max_{a \in \mathcal{A}_i} \left(\sum_{j=1}^{n} P_{ij}^{(a)} v_j + r_i^{(a)} \right) \qquad \forall i \in [n], \tag{1}$$

where $[n] := \{1, \dots, n\}$, and for all $i, j \in [n]$, $a \in \mathcal{A}_i$, $\mathcal{A}_i$ is a subset of $\mathbb{R}^n$, $r_i^{(a)} \in \mathbb{R}$ and $P_{ij}^{(a)} \geq 0$ with $\sum_{j=1}^{n} P_{ij}^{(a)} \leq 1$. Then, f can be interpreted as the dynamic programming operator of a stochastic control problem with state space $[n]$, and set of possible actions $\mathcal{A}_i$ in state $i \in [n]$, where $r_i^{(a)}$ is the additive reward at each time when the process is in state i and applies the action $a \in \mathcal{A}_i$, and $P_{ij}^{(a)}$ is the transition probability of the process from state i to state j, when applying the action $a \in \mathcal{A}_i$, multiplied by some discount factor (which is ≤ 1) depending on i and a. Hence, when f is contracting, its fixed point is the value function of an infinite horizon stochastic control problem, and the fixed point iteration associated to f coincides with the value iteration of this problem. When f is additively homogeneous, the nonlinear additive eigenvalue of f, meaning a constant $\rho \in \mathbb{R}$ such that there exists $v \in \mathbb{R}^n$ satisfying $f(v) = \rho + v$, is the value of the an ergodic stochastic control problem. These properties motivate the study of fixed points, additive eigenvalue and eigenvectors, and convergence of the orbits.

When f is not necessarily sub-homogeneous but still sends $\mathbb{R}^n$ to itself, it can still be put in the form (1) for some $P_{ij}^{(a)} \geq 0$. Then, f can be interpreted as the dynamic programming operator of a stochastic control problem, where now discount factors may be greater than 1, or equivalently discount rates may be greater than 0. Such a problem arises for instance in sustainable development, and in portfolio management.

2 A convex spectral theorem

Important particular cases of (1) include the max-plus linear maps which correspond to deterministic control problems, and affine maps which correspond to stochastic problems with no action or control. In these particular cases, the set of fixed points or additive eigenvectors and the convergence of the orbits are characterized respectively by the max-plus spectral theorem and the (reducible) Perron-Frobenius theorem. Inspired by these characterizations, the following result was shown in [2]. (The case when f is piecewise affine, or when the sets

A_i are finite, was already treated in works by Schweitzer, Federgruen (77,78), Romanovski (73), and Lanery (67) using different techniques of proof.)

Theorem 1 (Convex spectral theorem [2]) *Let f be order-preserving, additively homogeneous and convex, and assume $\mathcal{E}(f) \neq \emptyset$. Let C be the set of nodes of $\mathcal{G}^c(f)$ and $r : \mathbb{R}^n \to \mathbb{R}^C, x \mapsto x_C = (x_i)_{i \in C}$. Then*

- *r is an order-preserving and additively homogenous isomorphism and thus a sup-norm isometry from $\mathcal{E}(f)$ to its image $\mathcal{E}^c(f)$.*
- *$\mathcal{E}^c(f)$ is an inf-subsemilattice of $(\mathbb{R}^C, \leq)$ and is convex. Its dimension is at most the number of strongly connected components of $\mathcal{G}^c(f)$, with equality when f is piecewise affine.*
- *Let $c = c(\mathcal{G}^c(f))$, then for all $v \in \mathbb{R}^n$, $f^{kc}(v) - kc\lambda$ has a limit when $k \to \infty$.*

Let us explain the notations used in this theorem. Since f is nonexpansive, an eigenvalue λ is unique if it exists, then we denote by $\mathcal{E}(f) = \{v \in \mathbb{R}^n \mid \lambda + v = f(v)\}$ the set of corresponding additive eigenvectors. Since f is convex, consider the *subdifferential* of f at v defined by:

$$\partial f(v) = \{M \in \mathbb{R}^{n \times n} \mid \ f(w) - f(v) \geq M(w - v) \ \forall w \in \mathbb{R}^n\}.$$

Then $\partial f(v)$ is rectangular: $\partial f(v) = \partial f_1(v) \times \cdots \times \partial f_n(v)$, where matrices $M \in \mathbb{R}^{n \times n}$ are identified to the n-uple of their rows $(M_1, \dots, M_n)$. Moreover, $\partial f(v)$ is non-empty, convex, compact, and included in the set $\mathscr{S}_{nn}$ of $n \times n$ Markov matrices. For any rectangular subset $\mathcal{P}$ of $\mathscr{S}_{nn}$, we denote $\mathcal{G}^c(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} \mathcal{G}^c(P)$, where $\mathcal{G}^c(P)$ is the restriction of the directed graph $\mathcal{G}(P)$ of the Markov matrix P to the set of recurrent (or equivalently final) nodes of P. When f has an eigenvector, the graph $\mathcal{G}^c(\partial f(v))$ is independent of the choice of $v \in \mathcal{E}(f)$ [2], which allows us to denote it by $\mathcal{G}^c(f)$. Then, $c(\mathcal{G}^c(f))$ is the cyclicity of this graph, where the cyclicity of a strongly connected graph is the gcd of the lengths of its cycles, and the cyclicity of a general graph is the lcm of its strongly connected components.

After an easy transformation [2], one can also obtain from Theorem 1 a characterization of the set of fixed points and orbits of an order-preserving, additively sub-homogeneous and convex map f from $\mathbb{R}^n$ to itself. In that case, recurrent nodes of a sub-stochastic matrix P are the nodes belonging to a final class which has row sums equal to 1, then the critical graph of f may be empty, in which case the fixed point is unique.

In [3], we generalized the sub-homogeneous version of Theorem 1 to the case of maps f that are not necessarily nonexpansive, by replacing the set of fixed points of f by those satisfying a certain stability property, as follows. Namely, let $\mathcal{D}$ be an open and convex subset of $\mathbb{R}^n$ and f be an order-preserving and convex selfmap of D. Then, now $\partial f(v)$ is still included in the set $\mathcal{P}_{nn}$ of $n \times n$ matrices with nonnegative entries, but it may contain non substochastic matrices. We say that a fixed point v of f is *tangentially stable (t-stable)* if all the orbits of f'_v are bounded (or equivalently bounded from above), where f'_v is the directional derivative of f at v:

$$f'_v(w) := \lim_{t \to 0_+} \frac{f(v+tw) - f(v)}{t} = \sup_{P \in \partial f(v)} Pw \ \forall w \in \mathbb{R}^n \ .$$

This implies (but is not equivalent to) the property that all $P \in \partial f(v)$ are stable matrices. Moreover, if v is Lyapounov stable then v is t-stable, but the reverse implication fails. We denote by $\mathcal{E}_t(f)$ the set of tangentially stable fixed points of f. For any rectangular subset $\mathcal{P}$ of $\mathcal{P}_{nn}$, composed of stable matrices, we denote $\mathcal{G}^c(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} \mathcal{G}^c(P)$, where $\mathcal{G}^c(P)$ is the restriction of $\mathcal{G}(P)$ to the union of classes C of P such that $\rho(P_{CC}) = 1$. When $\mathcal{E}_t(f) \neq \emptyset$, then $\mathcal{G}^c(\partial f(v))$ is independent of the t-stable fixed point v of f, which allows us to denote it $\mathcal{G}^c(f)$.

Theorem 2 ([3]) *Let $\mathcal{D}$ be an open, convex and downward subset of $\mathbb{R}^n$, and $f : \mathcal{D} \to \mathcal{D}$ be order-preserving and convex. Assume $\mathcal{E}_t(f) \neq \emptyset$. Let C be the set of nodes of $\mathcal{G}^c(f)$ and $r : \mathbb{R}^n \to \mathbb{R}^C, x \mapsto x_C = (x_i)_{i \in C}$. Then*

- *r is an order-preserving isomorphism from $\mathcal{E}_t(f)$ to its image $\mathcal{E}_t^c(f)$.*
- *$\mathcal{E}_t^c(f)$ is an inf-subsemilattice of $(\mathbb{R}^C, \leq)$ and is convex. Its dimension is at most the number of strongly connected components of $\mathcal{G}^c(f)$.*
- *Let $c = c(\mathcal{G}^c(f))$, then the period of each t-stable periodic point of f divides c.*

We also proved a global convergence result, when $\mathcal{D} = \mathbb{R}^n$. For a convex selfmap f of $\mathbb{R}^n$, we denote by $\hat{f}$ its *recession map*:

$$\hat{f}(x) = \lim_{\lambda \to \infty} \frac{1}{\lambda} f(\lambda x) = \sup_{y \in \mathbb{R}^n} f(y+x) - f(y), \quad x \in \mathbb{R}^n.$$

Theorem 3 ([3]) *If $\hat{f}$ has all its orbits bounded from above, then f is nonexpansive with respect to some polyhedral norm.*

Moreover, if f has a fixed point, then every orbit of f converges to a Lyapunov stable periodic orbit of f whose period divides $c = c(\mathcal{G}^c(f))$.

3 Stationary solutions of Hamilton-Jacobi-Bellman equations

Fixed points equations of maps f of the form (1) can also be obtained as discretizations of Hamilton-Jacobi-Bellman partial differential equations associated to optimal control problems of diffusions, when using a monotone scheme. This suggests that similar results may hold for Hamilton-Jacobi-Bellman equations. The study of first order equations, corresponding to deterministic control problems and for which the semigroup is max-plus linear by the Maslov superposition principle, is the object of the so called Weak KAM theory, see in particular the results of Fathi and Siconolfi (05) which give characterizations of stationary solutions of Hamilton-Jacobi equations similar to the ones of max-plus spectral theorem. In [1], we considered a particular stochastic control problem in the n-dimensional torus, and proved a similar characterization as in Theorem 1, which generalize similar results obtained in the deterministic case by Rouy and Tourin (92), and Kolokoltsov and Maslov (97).One may thus ask if the same can occur for the possible negative discount case.

References

1. M. Akian, B. David, and S. Gaubert. Un théorème de représentation des solutions de viscosité d'une équation d'Hamilton-Jacobi-Bellman ergodique dégénérée sur le tore. *C. R. Acad. Sci. Paris, Ser. I*, 346:1149–1154, 2008.
2. M. Akian and S. Gaubert. Spectral theorem for convex monotone homogeneous maps, and ergodic control. *Nonlinear Analysis. Theory, Methods & Applications*, 52(2):637–679, 2003. See also `ttp://www.arXiv.org/abs/mat`/0110108.
3. M. Akian, S. Gaubert, and B. Lemmens. Stability and convergence in discrete convex monotone dynamical systems. *Journal of Fixed Point Theory and Applications*, 9:295–325, 2011. 10.1007/s11784-011-0052-1.

The work was partially supported by the joint RFBR-CNRS grant 11–01–93106-a.

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Gagliardo type Inequality: an idempotent point of view

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Abstract We discuss a result obtained in collaboration with A. Avantaggiati.

In [2] we established a Gagliardo inequality in Gaussian space.
The original result, established by Gagliardo in [3], holds for Lebesgue measure on Lipschitzian bounded sets. In [2] we adapted the original proof contained in [3] to Gaussian measure.
Let us recall briefly the result. Denote by $\mathbb{R}^N = \mathbb{R}_1 \times \dots \mathbb{R}_N$ with $R_j = \mathbb{R}$ as $j = 1, \dots N$.
For any fixed k with $1 \leq k \leq N$, we denote by σ a k-tuple of indices from the set $1, \dots, N$, i.e., $\sigma = (i_1, \dots, i_k)$ with $i_1 < i_2 < \dots < i_k$, and by σ' the $(N-k)$-tuple of the indices different from $i_1, i_2, \dots, i_k$ (in other words, complementary to $i_1, i_2, \dots, i_k$). Also x will be the N-vector $x = (x_1, x_2, \dots, x_N)$, x_σ will be the subvector $(x_{i_1}, x_{i_2}, \dots, x_{i_k})$, and $\sigma' = (i'_1, \dots, i'_{N-k})$.
Then $\mathbb{R}^\sigma = R_{i_1} \times \dots \mathbb{R}_{i_k}$ and $\mathbb{R}^N = \mathbb{R}^\sigma \times \mathbb{R}^{\sigma'}$. A function f defined in $\mathbb{R}^N$ is also written as $f(x) = f(x_1, x_2, \dots, x_N) = f(x_\sigma, x_{\sigma'})$, and also, if g is defined in $\mathbb{R}^\sigma$, $g_\sigma = g(x_\sigma)$. Similarly, the Gaussian measure is $d\gamma = (2\pi)^{-\frac{N}{2}} \exp(-\frac{1}{2}|x|^2)dx = \prod_{j=1}^N (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}|x_j|^2)dx$, and $d\gamma_\sigma = \prod_{j\in\sigma} (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}|x_j|^2)dx$, with $d\gamma = d\gamma_\sigma d\gamma_{\sigma'}$ and we have $\int_{\mathbb{R}^\sigma} d\gamma_\sigma(x_\sigma) = 1$. By $L^p(\mathbb{R}^N, d\gamma)$ we denote the L^p space with respect to $d\gamma$.
We denote by $\mathcal{S}$ the set of all k-tuples of indices from $1, \dots, N$. The number of elements in $\mathcal{S}$ is $\binom{N}{k}$. The result obtained in [2] is the following

1 Gagliardo type inequality

Theorem 1 *Given* $\binom{N}{k}$ *functions* $F_\sigma(x_\sigma)$ *with* $\sigma \in \mathcal{S}$, *and* $\lambda = \binom{N-1}{k-1}$ *satisfying* $F_\sigma \in L^\lambda(\mathbb{R}^\sigma, d\gamma_\sigma)$. *we define*

$$F(x) = \prod_{\sigma\in\mathcal{S}} F_\sigma(x_\sigma).$$

Then $F \in L^1(\mathbb{R}^N, d\gamma)$ *and*

$$\left(\int_{\mathbb{R}^N} |F(x)| d\gamma\right)^\lambda \leq \prod_{\sigma\in\mathcal{S}} \int_{\mathbb{R}^\sigma} |F_\sigma(x_\sigma)|^\lambda d\gamma_\sigma,$$

In [2] we established a connection between the well-known Fubini theorem and its analogue in idempotent analysis, giving a proof which shows how to pass from usual setting to the idempotent setting, and it is based on Gagliardo inequality in Gaussian space.

We discuss this result, related questions and further developments.

References

1. R. A. Adams, Sobolev Spaces, Academic Press (1975).
2. A. Avantaggiati, P. Loreti, Gagliardo type Inequality and Application, Percorsi incrociati (in ricordo di Vittorio Cafagna) Rubbettino Editore 2010.
3. E. Gagliardo, Proprietà di alcune classi di funzioni in più variabili, Ricerche Mat. 7 1958 102–137.
4. G. L. Litvinov and V. P. Maslov, Idempotent mathematics: correspondence principle and applications, Russian Mathematical Surveys 51 (1996), no. 6.

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The Galileo invariance of diffusion scattering of waves in the phase space

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Abstract In the paper, we discuss the studies of mathematical models of diffusion scattering of waves in the phase space, and relation of these models with quantum mechanics. In the previous works it is shown that in these models of classical scattering process of waves, the quantum mechanical description arises as the asymptotics after a small time. In the paper it is shown that the proposed models of diffusion scattering of waves possess the property of gauge invariance. This implies that they are described similarly in all inertial coordinate systems, i. e., they are invariant under the Galileo transformations.

1 Introduction

In the present paper we discuss the research on construction of models of diffusion scattering of waves in phase space. I have been studying this subject during last years [1–4]. In these models the quantum description of processes arises as an approximate one, asymptotical for large values of certain coefficients of the model.

In the papers mentioned above one makes an attempt to construct a model of quantum observables on the base of wave functions on the phase space. Note that in quantum mechanics, the wave function depends either only on coordinates or only on momenta, while in the present approach one considers wave functions depending both on coordinates and on momenta. This model is based also on the following observation. In quantum mechanics, the phase of the wave function of a particle (the natural hidden parameter) changes in time even for stationary states with very high velocity (if one takes into account the stationary energy). This velocity is such that a transfer of the particle with even small (non-relativistic) velocities can cause considerable changes in the phase of wave function because of the relativistic effect of more slow inner processes of a moving particle. Already taking into account this effect leads to non-commutativity of the action of coordinate and momentum shifts on the wave function. Note once again that in the proposed model one considers wave functions on the phase (and not configuration) space, and one assumes that the particle is in a diffusion process causing random shifts of the wave both by coordinates and by momenta. It is shown that the classical model of scattering of the wave, taking into account the assumptions described above, yields to arising quantum effects.

In the paper it is shown that the proposed models of diffusion scattering of waves possess the property of gauge invariance. This implies that they are described similarly in all inertial coordinate systems, i. e., they are invariant under the Galileo transformations.

2 The results obtained earlier

In the paper [4] we consider the Kramers equation [5], [6] of the form

$$\frac{\partial f}{\partial t} = \sum_{j=1}^{n}\left(\frac{\partial V}{\partial x_j}\frac{\partial f}{\partial p_j} - \frac{p_j}{m}\frac{\partial f}{\partial x_j}\right) + \gamma\sum_{j=1}^{n}\frac{\partial}{\partial p_j}\left(p_j f + kTm\frac{\partial f}{\partial p_j}\right), \tag{1}$$

where $f(x,p,t)$ is the probability density of a particle in the phase space at the moment of time t; m is the mass of the particle; $V(x)$ is the potential function of external forces acting on the particle; $\gamma = \beta/m$ is the resistance coefficient of the medium in which the particle moves, per unit of its mass; k is the Boltzmann constant; T is the temperature of the medium.

Then we consider the modified Kramers equation for the wave function $\varphi(x,p,t)$ of the form

$$\frac{\partial \varphi}{\partial t} = A\varphi + \gamma B\varphi, \tag{2}$$

where
$$A\varphi = \sum_{j=1}^{n}\left(\frac{\partial V}{\partial x_j}\frac{\partial \varphi}{\partial p_j} - \frac{p_j}{m}\frac{\partial \varphi}{\partial x_j}\right) - \frac{i}{\hbar}\left(mc^2 + V - \sum_{j=1}^{n}\frac{p_j^2}{2m}\right)\varphi \tag{3}$$

and
$$B\varphi = \sum_{j=1}^{n}\frac{\partial}{\partial p_j}\left(\left(p_j + i\hbar\frac{\partial}{\partial x_j}\right)\varphi + kTm\frac{\partial \varphi}{\partial p_j}\right).$$

Equation (2) is obtained from the Kramers equation (1) by adding to the right hand side of the summand of the form $-i/\hbar(mc^2 + V - p^2/(2m))\varphi$, and the replacement, in the diffusion operator, of multiplication of the function φ by p_j by the action of the operator $(p_j + i\hbar\partial/\partial x_j)$ on the function φ.

Adding the summand $-i/\hbar(mc^2 + V - p^2/(2m))\varphi$ is related with the additional physical requirement that the wave function at the point (x, p) oscillates harmonically with frequency $1/\hbar(mc^2 + V - p^2/(2m))$ in time.

The requirement of harmonic oscillating of the wave function φ at the point (x, p) with the large frequency $1/\hbar(mc^2 + V - p^2/(2m))$, in the case when mc^2 is much greater than V, leads to the fact that the shift of the wave function with respect to the coordinate x_j with conservation of the proper time at the point (x, p) yields the phase shift in the oscillation of the function φ. And the operator of infinitely small shift $\partial/\partial x_j$ is changed by the operator $\partial/\partial x_j - ip_j/\hbar$. (For a more detailed explanation, see [3].) Respectively, if we multiply this operator by $i\hbar$, then we obtain the operator $p_j + i\hbar\partial/\partial x_j$, used in the modified diffusion operator B.

For equation (2), in [4] we obtain results similar to that of the paper [3]. It is shown that also in this case, the process described by equation (2), for large $\gamma = \beta/m$ passes several stages. During the first rapid stage, the wave function goes to a "stationary" state. At the second, slow stage, the wave function evolves in the subspace of "stationary" states subject to the Schrodinger equation. Besides that, it is shown that at the third stage, the dissipation of the process leads to decoherence of the wave function, and any superposition of states comes to one of eigenstates of the Hamilton operator.

In the paper [4], it is shown also that if, on the contrary, the medium resistance per unit of mass of the particle $\gamma = \beta/m$ is small, and in equation (2) one can neglect the summand with the factor γ, then in the considered model, the density of the probability distribution $\rho = |\varphi|^2$ satisfies the standard Liouville equation

$$\frac{\partial \rho}{\partial t} = \sum_{j=1}^{n} \left(\frac{\partial V}{\partial x_j} \frac{\partial \rho}{\partial p_j} - \frac{p_j}{m} \frac{\partial \rho}{\partial x_j} \right), \tag{4}$$

as in classical statistical mechanics.

3 Gauge transformations

In this section we introduce and discuss the notion of gauge invariance for equation (2).

According to the approach exposed in [4], the density of probability distribution $\rho(x,p,t)$ of a quantum particle whose state at the moment of time t is given by the wave function $\varphi(x,p,t)$, is proportional to $|\varphi|^2 = \varphi(x,p,t)\varphi^*(x,p,t)$. This implies that the replacement of a wave function φ by the wave function of the form $\exp(ig/\hbar)\varphi$, where $g = g(x,p,t)$ is an arbitrary real valued function, does not change the density of the probability distribution $\rho(x,p,t)$. Such a transform of wave function is usually called a gauge transform.

Let us look how equation (2) changes under this gauge transform. To this end, let us write out equation (2) in a more general form. Let us write in it, instead of the differentiation operators $\partial/\partial p_j$ of the function φ, the operator $D_j^p = \partial/\partial p_j + iB_j/\hbar$, instead of the operators $\partial/\partial x_j - ip_j/\hbar$, the operator $D_j^x = \partial/\partial x_j + iA_j/\hbar$, and instead of the operator $\partial/\partial t + iH/\hbar$, where $H = mc^2+p^2/(2m)+V$, let us write the operator $D_0^x = \partial/\partial t + iA_0/\hbar$, where A_j, A_0, B_j are functions of x, p, and t for $j = 1, ..., n$. In these notations, equation (2) will take the form

$$D_0^x\varphi = \sum_{j=1}^{n}\left(\frac{\partial H}{\partial x_j}D_j^p\varphi - \frac{\partial H}{\partial p_j}D_j^x\varphi\right) + \gamma\sum_{j=1}^{n}D_j^p\left(i\hbar D_j^x\varphi + kTmD_j^p\varphi\right). \qquad (5)$$

By a gauge transform of equation (5) we call the following transform of the function φ and the potentials A_j, A_0, B_j, for $j = 1, ..., n$:

$$\varphi \longmapsto \varphi' = \exp(-\frac{i}{\hbar}g)\varphi; \qquad (6)$$

$$A_0 \longmapsto A_0' = A_0 + \frac{\partial g}{\partial t};$$

$$A_j \longmapsto A_j' = A_j + \frac{\partial g}{\partial x_j}, \text{ where } j = 1, ..., n;$$

$$B_j \longmapsto B_j' = B_j + \frac{\partial g}{\partial p_j}, \text{ where } j = 1, ..., n. \qquad (7)$$

It is not difficult to see that after the substitution (6) into equation (5), replacement (7), and dividing both parts of the obtained equality by $\exp(-(i/\hbar)g)$, the form of equation (5) will not change.

Geometrically, gauge transformation corresponds to transfer to another trivialization of a complex line bundle over the phase space, in which a form of linear connection is chosen, defining parallel transport of the vectors of the bundle along trajectories in the phase space.

In the particular case for equation (2), the potentials read

$$A_0 = H(x,p) = E + V; \quad A_j = -p_j; \quad B_j = 0 \text{ for } j = 1, ..., n.$$

Understanding the physical sense of the potentials in the general case for equation (5), requires separate investigation. For the Dirac equation, potentials of gauge invariance are usually related with the potentials of electromagnetic field.

4 The Galileo invariance

In this section we study the change of equation (2) under the transfer to a coordinate system moving uniformly with respect to the initial coordinate system, with the velocity u. The diffusion equation (1) is not invariant with respect to Galileo transforms under transfer to new inertial coordinate system moving with constant velocity u with respect to the old one.

The aim of this section is to study invariance of equation (2) for a free particle ($V = 0$) with respect to Galileo transforms, with gauge transforms of the wave function.

By definition of Galileo transforms, the new coordinate system is expressed through the old one by the following formulas:

$$\begin{aligned} &t' = t; \quad x' = x - ut; \quad p' = p - mu; \\ &E' = \frac{p'^2}{2m} = \frac{(p-mu)^2}{2m} = \frac{p^2}{2m} - pu + \frac{mu^2}{2} = E - pu + \frac{mu^2}{2}. \end{aligned} \tag{8}$$

Respectively, the old coordinates are expressed through the new ones by the following formulas:

$$\begin{aligned} &t = t'; \quad x = x' + ut; \quad p = p' + mu; \\ &E = \frac{p^2}{2m} = \frac{(p'+mu)^2}{2m} = \frac{p'^2}{2m} + p'u + \frac{mu^2}{2} = E' + p'u + \frac{mu^2}{2}. \end{aligned} \tag{9}$$

Substituting these expressions into equation (2), with the use of relations (3) and (4), we obtain:

$$\begin{aligned} \frac{\partial\varphi}{\partial t'} - \sum_{j=1}^{n} \frac{\partial\varphi}{\partial x'_j} u_j = \sum_{j=1}^{n} &\left(\frac{\partial V}{\partial x'_j} \frac{\partial\varphi}{\partial p'_j} - \frac{p'_j + mu_j}{m} \left(\frac{\partial}{\partial x'_j} - i\frac{p'_j + mu_j}{\hbar} \right) \varphi \right) \\ &- \frac{i}{\hbar} \left(E' + p'u + \frac{mu^2}{2} + V \right) \varphi \\ &+ \gamma \sum_{j=1}^{n} \frac{\partial}{\partial p'_j} \left(\left(p'_j + mu_j + i\hbar \frac{\partial}{\partial x'_j} \right) \varphi + kTm \frac{\partial\varphi}{\partial p'_j} \right), \end{aligned}$$

whence, after simple algebraic transformations, we obtain:

$$\frac{\partial\varphi}{\partial t'} = \sum_{j=1}^{n}\left(\frac{\partial V}{\partial x'_j}\frac{\partial\varphi}{\partial p'_j} - \frac{p'_j}{m}\left(\frac{\partial}{\partial x'_j} - i\frac{p'_j + mu_j}{\hbar}\right)\varphi\right)$$
$$-\frac{i}{\hbar}\left(E' - \frac{mu^2}{2} + V\right)\varphi$$
$$+\gamma\sum_{j=1}^{n}\frac{\partial}{\partial p'_j}\left(\left(p'_j + mu_j + i\hbar\frac{\partial}{\partial x'_j}\right)\varphi + kTm\frac{\partial\varphi}{\partial p'_j}\right).$$

If in the obtained equation one makes the substitution $\varphi = \exp((i/\hbar)g)\varphi'$, where $g = mux' + mu^2t'/2$, then (after the gauge transform) we obtain the equation

$$\frac{\partial\varphi'}{\partial t'} = \sum_{j=1}^{n}\left(\frac{\partial V}{\partial x'_j}\frac{\partial\varphi'}{\partial p'_j} - \frac{p'_j}{m}\left(\frac{\partial}{\partial x'_j} - i\frac{p'_j}{\hbar}\right)\varphi'\right)$$
$$-\frac{i}{\hbar}\left(E' + V\right)\varphi'$$
$$+\gamma\sum_{j=1}^{n}\frac{\partial}{\partial p'_j}\left(\left(p'_j + i\hbar\frac{\partial}{\partial x'_j}\right)\varphi' + kTm\frac{\partial\varphi'}{\partial p'_j}\right),$$

which coincides with equation (2). Thus, we have proved the Galileo invariance of equation (2).

References

1. Beniaminov, E.M.: A Method for Justification of the View of Observables in Quantum Mechanics and Probability Distributions in Phase Space. http://arxiv.org/abs/quant-ph/0106112 (2001)
2. Beniaminov, E.M.: Diffusion processes in phase spaces and quantum mechanics. Doklady Mathematics (Proceedings of the Russian Academy of Sciences) 76, 771–774 (2007)
3. Beniaminov, E.M.: Quantization as asymptotics of a diffusion process in phase space. Proc. Intern. Geom. Center 2(4), 7-50 (2009) (in Russian; English translation: http://arXiv.org/abs/0812.5116v1)
4. Beniaminov, E.M.: Quantum Mechanics as Asymptotics of Solutions of Generalized Kramers Equation. Electronic Journal of Theoretical Physics (EJTP) 8, No. 25 195-210 (2011)
5. Kramers, H. A.: Brownian motion in a field of force and the diffusion model of chemical reactions. Physica. 7, 284-304. (1940)
6. Van Kampen, N.G.: Stochastic Processes in Physics and Chemistry. North Holland, Amsterdam (1981)

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General transience bounds in tropical linear algebra via Nachtigall decomposition

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Abstract We present general transience bounds in tropical linear algebra based on Nachtigall's matrix decomposition. Our approach is also applicable to reducible matrices. The core technical novelty are general bounds on the transient of the maximum of two eventually periodic sequences. Our proof is algebraic in nature, in contrast to the existing purely graph-theoretic approaches.

1. Introduction

Tropical linear systems describe the behavior of transportation systems, manufacturing plants, network synchronizers, as well as certain distributed algorithms for resource allocation and routing. It is known that the sequence of tropical matrix powers, and hence every linear system, becomes periodic after an initial *transient*. In applications it is of interest to have upper bounds on the transient, to which we contribute with this work.

We use the Nachtigall decomposition [5] of square matrices in tropical algebra to show new transience bounds for sequences of matrix powers. The Nachtigall decomposition is a representation of the sequence of matrix powers of a square matrix as a maximum of a bounded number of sequences of bounded transients and bounded periods. Transience bounds for the sequence of matrix powers have been given, amongst others, by Hartmann and Arguelles [4] and Charron-Bost et al. [1]. Their proofs are purely graph-theoretic. They consider the edge-weighted graph described by the matrix as an adjacency matrix and argue about existence of walks of certain weights. We, too, use this graph interpretation of a matrix in two supplementary results (Lemma 2 and Lemma 6). However, the rest of our proof is algebraic. Because our proof is based on the Nachtigall decomposition, which is also applicable to reducible matrices, so is our proof. To the best of our knowledge, we are the first to give transience bounds for reducible matrices. An example by Even and Rajsbaum [3, Fig. 2] shows that our new bounds are asymptotically optimal.

2. Preliminaries

We consider the max-plus semiring on the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. That is, we consider the addition $(x, y) \mapsto \max(x, y)$ and the multiplication $(x, y) \mapsto x + y$. The semiring's zero element is $-\infty$ and its unit is 0. The matrix multiplication of two matrices A and B of compatible size satisfies $(AB)_{i,j} = \max_k(A_{i,k} + B_{k,j})$.

We call a sequence $f : \mathbb{N} \to \overline{\mathbb{R}}$ *eventually periodic* if there exist numbers p, T, and ϱ such that:

$$\forall n \geqslant T : \; f(n + p) = f(n) + p \cdot \varrho \tag{1}$$

In this case we call p a *period*, T a *transient*, and ϱ a *ratio* of the sequence f. It is easy to see that the ratio is unique and finite if the sequence is not eventually constantly infinite. For every period p, there exists a unique minimal transient T that satisfies (1). The next fundamental lemma shows that these minimal transients do, in fact, not depend on p:

Lemma 1 *Let $f : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic. Let p and $\hat{p}$ be periods of f with respective minimal transients T_p and $\widehat{T_p}$. Then $T_p = \widehat{T_p}$.*

We will henceforth call this unique minimal transient *the* transient of f. Also, we will call the minimal period *the* period of f.

Cohen et al. [2] established that the entrywise sequences of matrix powers A^n of a square matrix are eventually periodic. More generally, we say that a sequence of matrices is eventually periodic if every entrywise sequence is eventually periodic. Period and transience of a matrix sequence is a period and transience for all entrywise sequences.

The tropical *convolution* $f \otimes g$ of two sequences f and g is defined as

$$(f \otimes g)(n) = \max_{n_1+n_2=n} \big(f(n_1) + g(n_2)\big) \; . \tag{2}$$

To a square matrix A naturally corresponds an edge-weighted digraph $G(A)$. The weight $p(W)$ of a walk W in $G(A)$ is the sum of the weights of its edges. The matrix powers of A satisfy the correspondence

$$\left(A^n\right)_{i,j} = \max\left\{p(W) \mid W \text{ has length } n \text{ and is from } i \text{ to } j\right\} \ . \tag{3}$$

3. Nachtigall Decomposition

Nachtigall [5] introduced a representation of the sequence of matrix powers of an $N \times N$ square matrix as the maximum of at most N matrix sequences whose transients are at most $3N^2$. He showed that this representation can be computed efficiently. However, no results on the transient of the original matrix were obtained. The core of the representation is a decomposition of the original matrix into components corresponding to cycles in the matrix' digraph, The matrix sequences are defined as convolutions corresponding to this decomposition. The following lemma shows the utility of *maximum* mean cycles for the decomposition.

Lemma 2 ([5, Lemma 3.2]) *If A is an $N \times N$ matrix and k is a node of a maximum mean cycle C in $G(A)$, then both sequences $\left(A^n\right)_{i,k}$ and $\left(A^n\right)_{k,i}$ are eventually periodic with period $\ell(C)$, ratio $p(C)/\ell(C)$, and transient at most $\ell(C) \cdot (N-1)$.*

Given an $N \times N$ matrix A and a set $I \subseteq \{1, \ldots, N\}$ of indices, we define the *deletion* of I in A as the matrix B whose entries satisfy $B_{i,j} = -\infty$ if $i \in I$ or $j \in I$, and $B_{i,j} = A_{i,j}$ otherwise.

The following lemmas are used to prove the upper bound on the transient of each matrix sequence in the Nachtigall decomposition.

Lemma 3 *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with common period p, common ratio ϱ, and respective transients T_f and T_g. Then the sequence $\max(f, g)$ is eventually periodic with period p, ratio ϱ, and transient at most $\max(T_f, T_g)$.*

Lemma 4 ([5, Lemma 6.1]) *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with common period p, common ratio ϱ, and respective transients T_f and T_g. Then the convolution $f \otimes g$ is eventually periodic with period p, ratio ϱ, and transient at most $T_f + T_g + p - 1$.*

We now state an improvement of Nachtigall's theorem using essentially the same arguments as the original version. The improvement lies in an upper bound of $2N^2 - N$ on the sequences' transients instead of $3N^2$.

Theorem 1 (Nachtigall decomposition [5, Theorem 3.3]) *Let A be an $N \times N$ matrix. Then there exist eventually periodic matrix sequences $A_1(n), A_2(n), \ldots, A_N(n)$ with periods at most N and transients at most $2N^2 - N$ such that for all n:*

$$A^n = \max\big(A_1(n), A_2(n), \ldots, A_N(n)\big) \tag{4}$$

The proof proceeds by induction on N. The case $N = 1$ is trivial.

If $G(A)$ does not contain a cycle, then $(A^n)_{i,j} = -\infty$ for all $n \geqslant N$, so the transient of A is at most N and the theorem's statement is trivially fulfilled when choosing all matrix sequences $A_m(n)$ equal to A^n.

We hence suppose that $G(A)$ contains a cycle. By the definition of the matrix multiplication, whenever $n = n_1 + n_2$, we have:

$$(A^n)_{i,j} = \max_k \big((A^{n_1})_{i,k} + (A^{n_2})_{k,j}\big) \tag{5}$$

Let C be a maximum mean cycle in $G(A)$. Denote by B the deletion of the set of C's nodes in A. It follows from the definition of deletion and from the graph interpretation (3) that

$$(A^n)_{i,j} = \max\Big(\max_{k \text{ in } C} \big((A^{n_1})_{i,k} + (A^{n_2})_{k,j}\big)\ ,\ (B^n)_{i,j}\Big)\ . \tag{6}$$

In particular, (6) continues to hold when forming the maximum over all n_1 and n_2 such that $n = n_1 + n_2$. By writing $A_{i,j}(n) = (A^n)_{i,j}$ and recalling the definition (2) of convolution, we can hence write

$$(A^n)_{i,j} = \max\Big(\max_{k \text{ in } C} \big(A_{i,k} \otimes A_{k,j}\big)(n)\ ,\ (B^n)_{i,j}\Big)\ . \tag{7}$$

Lemmas 2, 3, and 4 imply that the transient of the inner maximum in (7) is at most

$$2 \cdot \ell(C) \cdot (N-1) + \ell(C) - 1 \leqslant 2N^2 - N - 1 \tag{8}$$

and its period is at most $\ell(C) \leqslant N$. Choose the matrix sequence $A_1(n)$ equal to this inner maximum, i.e., $A_1(n) = \max_{k \text{ in } C} \big(A_{i,k} \otimes A_{k,j}\big)(n)$.

By induction hypothesis, there exist matrix sequences $A_2(n),\ldots,A_N(n)$ with periods at most N and transients at most $2N^2 - N$ such that $B^n = \max\big(A_2(n), \ldots, A_N(n)\big)$. This then concludes the proof's inductive step.

4. Transience Bounds

Note that Theorem 1 does *not* imply that the transient of any sequence of matrix powers is at most $2N^2 - N$. The reason for this is that Lemma 3 is not applicable to the maximum in the Nachtigall decomposition because the involved sequences can have different ratios.

The following lemma is our main technical novelty and provides a tool for bounding the transient of a maximum of two eventually periodic sequences if their ratios are not equal.

Lemma 5 *Let $f, g : \mathbb{N} \to \overline{\mathbb{R}}$ be eventually periodic with the same period p, respective ratios ϱ_f and ϱ_g, and respective transients T_f and T_g. Assume that $\varrho_f \geqslant \varrho_g$ and that for all $n \geqslant S$ we have $g(n) = -\infty$ whenever $f(n) = -\infty$. Then the sequence* $\max(f, g)$ *is eventually periodic with period p, ratio* $\max(\varrho_f, \varrho_g)$*, and transient at most*

$$\max(T_f, T_g, S) + \frac{\Gamma}{|\varrho_f - \varrho_g|}$$

where

$$\Gamma = \max\left\{ f(m) - g(m) \mid S \leqslant m < S + p\,,\, g(m) \neq -\infty \right\} \ .$$

In the rest of this section, we provide general transience bounds for sequences of matrix powers. We do this by applying Lemma 5 entrywise to the maximum in (4). Evidently, given a Nachtigall decomposition (which is not unique), one can use Lemma 5 to bound the transient of the matrix powers. In this section, we proceed to use the existence of a Nachtigall decomposition to prove general transience bounds independent of an explicit decomposition.

We first assume that the graph $G(A)$ is strongly connected. In this case, all sequences $(A^n)_{i,j}$ are eventually periodic with ratio equal to the maximum cycle mean in $G(A)$. For every pair (i, j) of indices, the ratio of the sequence $\big(A_m(n)\big)_{i,j}$ is greater or equal to that of $\big(A_{m+1}(n)\big)_{i,j}$. We want to apply Lemma 5 to every pair of (i) a sequence of maximum ratio and (ii) a sequence of another ratio. Afterwards, we apply Lemma 3 to the resulting maxima (which all have the same ratio). But to effectively apply Lemma 5, we have to bound its parameter S for every pair.

We show that the parameter S is at most $N(N+1)$ with a graph-theoretic argument: The *exploration penalty* of graph G is the least integer *ep* such that there exists a closed walk of length n at every node in G for all multiples n of G's cyclicity[2] that satisfy $n \geqslant ep$.

[2] The *cyclicity* of a strongly connected graph is the greatest common divisor of its cycle lengths.

Lemma 6 ([1, Theorem 3]) *The exploration penalty of a graph with N nodes is at most $N(N-1)$.*

It is well-known that the difference of lengths of two walks from i to j in a strongly connected graph is always a multiple of the graph's cyclicity. In each step of the Nachtigall recursion, there exists a path from i to j via a node k of C of length at most $2N$. Hence Lemma 6 implies that if there exists *any* path from i to j of length $n \geqslant 2N + N(N-1) = N(N+1)$, then there exists also a path from i to j via k of length n. This shows that S can be chosen to be at most $N(N+1)$.

A common period p of a pair of sequences is the least common multiple of the two periods. Because in the Nachtigall decomposition the periods are at most N, there exists a common period less or equal to $N(N-1)$. Hence $S+p$ can be bounded by $2N^2$.

We thus arrive at the following theorem bounding the transient of the sequence of matrix powers. Denote by $\|A\|$ the difference of the maximum and the minimum finite entry in matrix A.

Theorem 2 *Let A be an $N \times N$ matrix such that $G(A)$ is strongly connected. Denote by λ the maximum cycle mean in $G(A)$ and λ' the second largest cycle mean weight. Then the sequence of matrix powers A^n has transient at most*

$$2N^2 + \frac{\|A\| 2N^2}{\lambda - \lambda'} \ .$$

Hartmann and Arguelles [4, Theorem 10] arrived at a bound of $\|A\|2N^2/(\lambda - \lambda^0)$ on the transient, where λ^0 is a parameter of the max-balanced graph of $G(A)$. Their bound is always smaller than ours in Theorem 2. However, in the worst case, both are asympotically in the same order of growth. Also, our technique is different than that of Harmann and Arguelles, and can also be used to provide sharper bounds if information on the Nachtigall decomposition is available.

Our technique is also applicable to the general case where $G(A)$ is not necessarily strongly connected. In the general case, the ratio $\lambda_{i,j}$ of the sequence of entries $(A^n)_{i,j}$ is equal to the maximum mean of cycles reachable from i and from which j is reachable. Denote by $\lambda'_{i,j}$ the second largest mean of cycles reachable from i and from which j is reachable. By analogous arguments, Theorem 2 continues to hold in the general case if we replace the denominator $\lambda - \lambda'$ by $\min_{i,j}(\lambda_{i,j} - \lambda'_{i,j})$.

References

1. Bernadette Charron-Bost, Matthias Függer, and Thomas Nowak. On the transience of linear max-plus dynamical systems. Under submission, 2011. `arXiv:1111.4600v1 [cs.DM]`
2. Guy Cohen, Didier Dubois, Jean-Pierre Quadrat, and Michel Viot. Analyse du comportement périodique de systèmes de production par la théorie des dioïdes. INRIA RR 191, 1983.
3. Shimon Even and Sergio Rajsbaum. The use of a synchronizer yields maximum computation rate in distributed systems. *Theory of Computing Systems* 30(5):447–474, 1997.
4. Mark Hartmann and Cristina Arguelles. Transience bounds for long walks. *Mathematics of Operations Research* 24(2):414–439, 1999.
5. Karl Nachtigall. Powers of matrices over an extremal algebra with applications to periodic graphs. *Mathematical Methods of Operations Research* 46:87–102, 1997.

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Planar flows and quadratic relations over semirings

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1 Introduction

In this work, acting in spirit of Lindström's construction [7], we consider a wide class of functions which take values in an arbitrary commutative semiring and are generated by flows (systems of paths) in a planar acyclic directed graph. Functions of this sort satisfy plenty of "stable" (or "universal") quadratic relations, extending well-known quadratic relations for minors of matrices (in particular, Plücker's and Dodgson's ones) and their tropical analogues. We develop a combinatorial method to completely characterize the set of such "stable" relations. In particular, applying this method to Gessel–Viennot's model, one can describe quadratic relations on Schur functions (related to semi-standard Young tableaux). The full version of this work is to appear in J. Algebraic Combinatorics (DOI 10.1007/s10801-012-0344-6); see also Arxiv:1102.2578v2[math.CO].

We start with specifying terminology and notation, and with backgrounds.

1.1 Commutative semirings

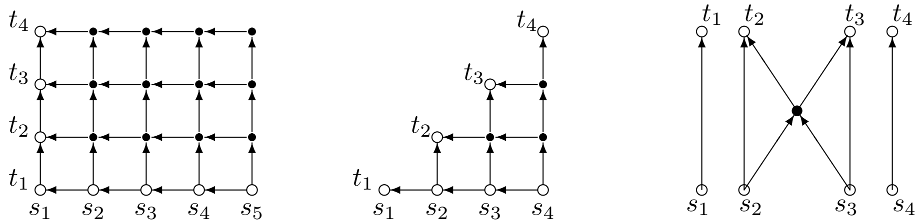
In order to embrace both algebraic and tropical cases (and more), we will deal with functions taking values in an arbitrary *commutative semiring* (briefly, *CS*), a set $\mathfrak{S}$ equipped with two associative and commutative binary operations $\oplus$ (addition) and $\odot$ (multiplication) satisfying the distributive law $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. When needed, we additionally assume that $\mathfrak{S}$ contains neutral elements $\underline{0}$ (for addition) and/or $\underline{1}$ (for multiplication). Important special cases are:

(i) a commutative ring (when $\underline{0} \in \mathfrak{S}$ and each element has an additive inverse);

(ii) a CS with division (when $\underline{1} \in \mathfrak{S}$ and each element has a multiplicative inverse); e.g., the set $\mathbb{R}_{>0}$ of positive reals (with $\oplus = +$ and $\odot = \cdot$), and the tropicalization $\mathfrak{L}_{\max}$ of a totally ordered abelian group $\mathfrak{L}$ (with $\oplus = \max$ and $\odot = +$); the most popular case of the latter is the real tropical semiring $\mathbb{R}_{\max}$.

1.2 Planar flows

By a *planar network* we mean a finite directed planar *acyclic* graph $G = (V, E)$ in which two subsets $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_{n'}\}$ of vertices are distinguished, called *sources* and *sinks*, respectively. We assume that the sources and sinks, also called *terminals*, lie on the *boundary* O of a compact convex region in the plane, and the remaining part of G lies inside O. The terminals appear in O in the *clockwise cyclic order* $s_n, \dots, s_1, t_1, \dots, t_{n'}$ (with possibly $s_1 = t_1$ or $s_n = t_{n'}$). Three examples are illustrated in the picture.



Let $\mathcal{E}^{n,n'}$ denote the set of pairs $(I \subseteq [n], I' \subseteq [n'])$ with equal sizes: $|I| = |I'|$ (where $[k] := \{1, 2, \dots, k\}$). By an $(I|I')$-*flow* we mean a collection ϕ of $|I|$ *pairwise (vertex) disjoint directed paths* in G going from the source set $S_I := \{s_i : i \in I\}$ to the sink set $T_{I'} := \{t_j : j \in I'\}$. The set of $(I|I')$-flows is denoted by $\Phi_{I|I'} = \Phi^G_{I|I'}$.

Each vertex $v \in V$ is endowed with a *weight* $w(v) \in \mathfrak{S}$, where $\mathfrak{S}$ is a CS (alternatively, one can consider a weighting on the edges, which does not affect our results in essence). This gives rise to the function $f = f_{G,w}$ on $\mathcal{E}^{n,n'}$ defined by

$$f(I|I') := \bigoplus\nolimits_{\phi \in \Phi_{I|I'}} w(\phi), \qquad (I, I') \in \mathcal{E}^{n,n'}, \tag{1}$$

where $w(\phi)$ denotes the weight $\odot(w(v) : v \in V_\phi)$ of a flow ϕ, and V_ϕ is the set of vertices occurring in ϕ. We call f a *flow-generated function*, or an *FG-function* for short, and say that f is determined by G, w. The set of such functions over all corresponding G and w (with $n, n', \mathfrak{S}$ fixed) is denoted by $\mathbf{FG}_{n,n'}(\mathfrak{S})$.

Remark 1. When $\mathfrak{S} = \mathbb{R}$, (1) is specified as $f(I|I') := \sum_{\phi\in\Phi_{I|I'}}(\prod_{v\in V_\phi} w(v))$, and when $\mathfrak{S} = \mathbb{R}_{\max}$, (1) turns into $f(I|I') := \max_{\phi\in\Phi_{I|I'}}(\sum_{v\in V_\phi} w(v))$. In the former (latter) case, we refer to f as an *algebraic* (resp. *tropical*) *FG-function.*

Remark 2. Note that an $(I|I')$-flow in G may not exist, making $f(I|I')$ undefined if $\mathfrak{S}$ does not contain $\underline{0}$. To overcome this trouble, we formally extend $\mathfrak{S}$, when needed, by adding an "extra neutral" element $*$, setting $* \oplus a = a$ and $* \odot a = *$ for all $a \in \mathfrak{S}$. In the extended semiring, one defines $f(I|I') := *$ in case $\Phi_{I|I'} = \emptyset$.

When an $(I|I')$-flow ϕ enters the first $|I| =: k$ sinks (i.e. $I' = [k]$), we say that ϕ is a *flag flow* for I. Accordingly, notation $\Phi_{I|\,[k]}$ is abbreviated to Φ_I, and $f(I|\,[k])$ to $f(I)$. When we are interested in the flag case only, f is regarded as a function on the set $2^{[n]}$ of subsets of $[n]$.

1.3 Lindström's lemma

Assume that weights w of vertices of G belong to a commutative ring and consider the $n' \times n$ matrix M whose entries m_{ji} are defined as $\sum_{\phi\in\Phi_{\{i\}|\{j\}}}(\prod_{v\in V_\phi} w(v))$ (cf. Remark 1). For $(I, I') \in \mathcal{E}^{n,n'}$, let $f_M(I|I')$ denote the minor of M with the column set I and the row set I'. A remarkable property shown by Lindström [7] is that $f_M = f_{G,w}$.

(Note that the class of matrices whose minor functions are flow-generated is large. In particular, it has been shown that any totally nonnegative matrix (a real matrix whose all minors are nonnegative) is such; see [1]. The question whether this class contains all matrices over any commutative ring is still open, but we can show that it contains any matrix over a *field*; see Arxiv:1102.2578v2[math.CO].)

1.4 Quadratic relations

Minors of (real or complex) matrices obey many quadratic relations. Most popular among them are quadratic relations on flag minors, or *Plücker relations* (which, in particular, describe flag manifolds and Grassmannians embedded in corresponding projective spaces). Therefore, by Lindström's lemma, similar relations should be valid for any FG-function $f = f_{G,w}$ when $\mathfrak{S} = \mathbb{R}$ or $\mathbb{C}$ (or even an arbitrary commutative ring). Below are two examples.

(i) The simplest example of Plücker relations (in the flag case) involve triples: for any three elements $i<j<k$ in $[n]$ and any subset $X\subseteq[n]-\{i,j,k\}$,

$$f(Xik)f(Xj)=f(Xij)f(Xk)+f(Xjk)f(Xi), \qquad (2)$$

where for brevity we write $Xi'\dots j'$ for $X\cup\{i',\dots,j'\}$ (and as before, $f(I)$ stands for $f(I|\,[|I|]))$. This is called the *AP3-relation* (abbreviating "algebraic Plücker relation with triples").

(ii) The simplest relation in the non-flag case arises from Dodgson's condensation formula for matrices [3]: for elements $i<k$ of $[n]$ and elements $i'<k'$ of $[n']$ and for $X\subseteq[n]-\{i,k\}$ and $X'\subseteq[n']-\{i',k'\}$,

$$f(iX|\,i'X')\,f(Xk|X'k')=f(iXk|\,i'X'k')\,f(X|X')+f(iX|X'k')\,f(Xk|\,i'X'). \qquad (3)$$

The "tropical counterpart" of (2) is the *TP3-relation*, viewed as

$$f(Xik)+f(Xj)=\max\{f(Xij)+f(Xk),f(Xjk)+f(Xi)\}. \qquad (4)$$

This is valid for any tropical FG-function f; see [2] (where the case $\mathfrak{S}=\mathbb{R}_{\max}$ is considered, but the argument is extended straightforwardly to any $\mathcal{L}_{\max}$).

In general, the quadratic relations of our interest involve FG-functions on $\mathcal{E}^{n,n'}$ over an arbitrary CS $\mathfrak{S}$ and can be expressed as

$$\bigoplus\nolimits_{(A,A')\in\mathcal{A}}\left(f(XA|X'A')\odot f(X\overline{A}|X'\overline{A}\,')\right) = \bigoplus\nolimits_{(B,B')\in\mathcal{B}}\left(f(XB|X'B')\odot f(X\overline{B}|X'\overline{B}\,')\right). \qquad (5)$$

Here: X,Y (resp. X',Y') are disjoint subsets of $[n]$ (resp. $[n']$); we write KL for $K\cup L$; the complement $Y-C$ of $C\subseteq Y$ is denoted by $\overline{C}$, and the complement $Y'-C'$ of $C'\subseteq Y'$ by $\overline{C}\,'$. The families $\mathcal{A},\mathcal{B}$ consist of certain pairs $(C\subseteq Y, C'\subseteq Y')$, admitting multiple ones. (The sizes of sets above are assumed to be *agreeable*: they should satisfy $|X|+|C|=|X'|+|C'|$ and $|X|+|\overline{C}|=|X'|+|\overline{C}'|$, or, equivalently, $2|X|+|Y|=2|X'|+|Y'|$ and $|Y|-2|C|=|Y'|-2|C'|$.)

In fact, an instance of (5) represents a variety of relations of "the same type", which does not depend on X,Y,X',Y' and is specified by two patterns $\mathcal{A}_0$ and $\mathcal{B}_0$. More precisely, letting $m:=|Y|$ and $m':=|Y'|$, take the order preserving maps $\gamma:[m]\to Y$ and $\gamma':[m']\to Y'$ (i.e. $\gamma(i)<\gamma(j)$ for $i<j$, and similarly for γ'). Then the *pattern* $\mathcal{A}_0$ (inducing $\mathcal{A}$) consists of pairs $(A_0\subseteq[m],\,A'_0\subseteq[m'])$ so that $\mathcal{A}=\{(\gamma(A_0),\gamma'(A'_0))\colon (A_0,A'_0)\in\mathcal{A}_0)$, and the pattern $\mathcal{B}_0$ (inducing $\mathcal{B}$) is defined similarly. We write $\mathcal{A}=\gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B}=\gamma_{Y,Y'}(\mathcal{B}_0)$.

It should be noted that in the flag case, the sets X', Y', as well as A', B' in (5), are determined uniquely. For this reason, we omit them in the above expressions and think of $\mathcal{A}, \mathcal{B}$ (resp. $\mathcal{A}_0, \mathcal{B}_0$) as consisting of subsets of Y (resp. $[m]$).

Examples. Relation (3) deals with $Y = \{i, k\}$, $Y' = \{i', k'\}$, $[m] = \{1, 2\}$, $[m'] = \{1', 2'\}$, $\mathcal{A} = \{i|i'\}$, $\mathcal{B} = \{ik|i'k',\ i|k'\}$, $\mathcal{A}_0 = \{1|1'\}$, and $\mathcal{B}_0 = \{12|1'2',\ 1|2'\}$. In turn, Plücker's type relations (2) and (4) concern $Y = \{i, j, k\}$, $m = 3$, $\mathcal{A} = \{ik\}$, $\mathcal{B} = \{ij,\ jk\}$, $\mathcal{A}_0 = \{13\}$, and $\mathcal{B}_0 = \{12,\ 23\}$.

Definition. When (5) holds for fixed $\mathcal{A}_0, \mathcal{B}_0$ as above and any corresponding $\mathfrak{S}, G, w, X, Y, X', Y'$ and the families $\mathcal{A} := \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_{Y,Y'}(\mathcal{B}_0)$, we call (5) a *stable quadratic relation*, or an *SQ-relation*, and say that this relation is induced by the patterns $\mathcal{A}_0, \mathcal{B}_0$.

Our goal is to give a relatively simple combinatorial method of characterizing the patterns $\mathcal{A}_0, \mathcal{B}_0$ inducing SQ-relations. In fact, our method generalizes a flow rearranging approach used in [2] for proving the TP3-relation for tropical FG-functions. It consists in reducing the task to a certain combinatorial problem on *matchings*, and as a consequence, provides an “efficient” procedure to recognize whether or not a given pair $\mathcal{A}, \mathcal{B}$ yields an SQ-relation. It should be noted that our approach is close in essence to a lattice paths method elaborated in Fulmek and Kleber [5] and Fulmek [4] to generate quadratic identities on Schur functions.

2 Balanced families and the main result

Consider (agreeable) $X, Y, X', Y', \mathcal{A}, \mathcal{B}, \mathcal{A}_0, \mathcal{B}_0$ as above. It will be convenient for us to think that the elements of Y and Y' are placed, respectively, on the lower half and on the upper half of a circumference O in the plane, in the increasing order from left to right. Also, considering a member (C, C') of $\mathcal{A} \cup \mathcal{B}$, we call the elements of C and C' *white*, and the elements of their complements $\overline{C} = Y - C$ and $\overline{C}' = Y - C'$ *black*. For members of patterns $\mathcal{A}_0$ and $\mathcal{B}_0$, white/black colorings on $[m] \sqcup [m']$ are defined similarly (where $\sqcup$ denotes the disjoint union).

Let M be a *perfect matching* on $Y \sqcup Y'$, i.e. M is a partition of $Y \sqcup Y'$ into 2-element subsets, or *couples*. We say that M is *feasible* for (C, C') (as above) if:

(2.6) (i) For a couple $\pi \in M$, if either $\pi \subseteq Y$ or $\pi \subseteq Y'$, then the elements of π have different colors;

(ii) If one element of $\pi \in M$ belongs to Y and the other to Y', then these elements have the same color;

(iii) M is planar, in the sense that the chords of O connecting the couples in M are pairwise not intersecting.

Let $\mathcal{M}(C, C')$ denote the set of feasible matchings for (C, C'). We define $\mathcal{M}(\mathcal{A})$ to be the family being the union of sets $\mathcal{M}(C, C')$ (respecting multiplicities) over all $(C, C') \in \mathcal{A}$. Analogous families are defined for $\mathcal{B}$ and for $\mathcal{A}_0, \mathcal{B}_0$ (concerning matchings on $[m] \sqcup [m']$).

Definition. Families $\mathcal{A}, \mathcal{B}$ are called *balanced* if $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{B})$ (regarding $\mathcal{M}(\cdot)$ as a multi-set).

(Clearly $\mathcal{A}, \mathcal{B}$ are balanced if and only if so are the patterns $\mathcal{A}_0, \mathcal{B}_0$.)

Our main result is the following

Theorem 1 (5) *is an SQ-relation if and only if* $\mathcal{A}, \mathcal{B}$ *are balanced.*

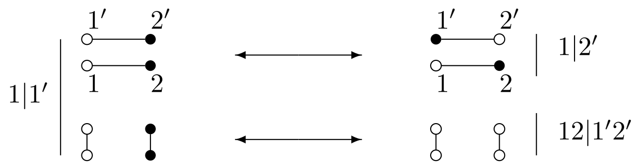
A sketch of the proof of this theorem will be outlined in Sections 4 and 5.

3 Examples of stable quadratic relations

In this section we illustrate Theorem 1 with several simple examples (for more examples, see Arxiv:1102.2578v2[math.CO]). According to this theorem, once we are able to show that one or another pair of patterns $\mathcal{A}_0, \mathcal{B}_0$ is balanced, we can declare that (5) holds for any corresponding $X, Y, X', Y', \mathcal{A}, \mathcal{B}$.

3.1

Let $m = m' = 2$. Consider the patterns $\mathcal{A}_0 = \{1|1'\}$ and $\mathcal{B}_0 = \{1|2', 12|1'2'\}$ for the intervals $[m] = \{1, 2\}$ and $[m'] = \{1', 2'\}$. One can see that the only member $1|1'$ of $\mathcal{A}_0$ admits two feasible matchings, namely, $\mathcal{M}(1|1') = \{\{12, 1'2'\}, \{11', 22'\}\}$, whereas each member of $\mathcal{B}_0$ has exactly one feasible matching, namely, $\mathcal{M}(1|2') = \{\{12, 1'2'\}\}$ and $\mathcal{M}(12|1'2') = \{\{\{11', 22'\}\}$. This implies that $\mathcal{A}_0, \mathcal{B}_0$ are balanced. The corresponding feasible matchings and bijection are illustrated in the picture (where the white/black partitions and matchings on $[m] \sqcup [m']$ are indicated by using two-level diagrams).



This gives rise to the SQ-relation extending Dodgson's condensation formula (3) (by taking $Y = (i < k)$, $Y' = (i' < k')$, $\mathcal{A} = \{i|i'\}$, and $\mathcal{B} = \{i|k',\ ik|i'k'\}$).

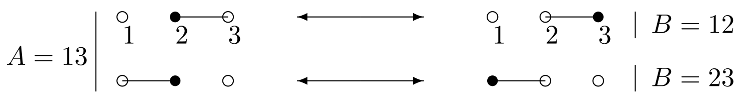
The next two examples concern SQ-relations of Plücker's type (the flag case). Here all members of patterns $\mathcal{A}_0, \mathcal{B}_0$ are subsets C of $[m]$ (as before, we say that the elements of C are *white*, and the ones of $\overline{C} := [m] - C$ are *black*). One can check that these subsets have the same cardinality p; one may assume, w.l.o.g., that $p \geq m - p =: q$. Furthermore, instead of perfect matchings on $[m] \sqcup [m']$ occurring in the general case, we now should consider matchings M of cardinality q on $[m]$. Such an M is called *feasible* for a (white) subset $C \subseteq [m]$ of size p if

(i) the elements of each couple in M have different colors; and

(ii) there are no $i < j < k < \ell$ such that $ik, j\ell \in M$ (i.e. M is nested), and there are no $i < j < k$ such that $ik \in M$ and $j \in C - \cup(\pi \in M)$.

3.2

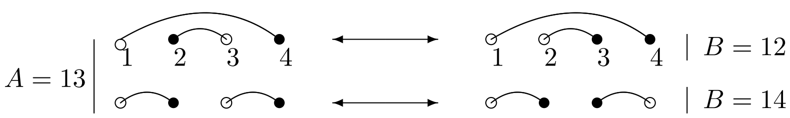
When $m = 3$ and $p = 2$, there are three p-element subsets in $[m]$, namely, $12, 13, 23$. Each of 12 and 23 admits only one feasible matching, namely, $\mathcal{M}(12) = \{\{23\}\}$ and $\mathcal{M}(23) = \{\{12\}\}$, whereas 13 has two feasible matchings: $\mathcal{M}(13) = \{\{12\}, \{23\}\}$. Hence the patterns $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 23\}$ are balanced. The feasible matchings and bijection are illustrated in the picture.



This gives rise to the SQ-relation "on triples" extending (2) and (4) (by taking $Y = (i < j < k)$, $\mathcal{A} = \{ik\}$, and $\mathcal{B} = \{ij,\ jk\}$).

3.3

For $m = 4$ and $p = 2$, take $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 14\}$. Each of 12 and 14 admits a unique feasible matching: $\mathcal{M}(12) = \{\{14, 23\}\}$ and $\mathcal{M}(14) = \{\{12, 34\}\}$, whereas $\mathcal{M}(13)$ consists of two feasible matchings: just those $\{14, 23\}$ and $\{12, 34\}$. Hence $\mathcal{A}_0, \mathcal{B}_0$ are balanced. See the picture.



This implies the following SQ-relation: for $i<j<k<\ell$ and $X\subseteq[n]-\{i,j,k,\ell\}$,

$$f(Xik)\odot f(Xj\ell)=(f(Xij)\odot f(Xk\ell))\oplus(f(Xi\ell)\odot f(Xjk)).$$

4 A sketch of proof of "if" part in the theorem.

Consider corresponding $G,w,\mathfrak{S},X,Y,X',Y',\mathcal{A},\mathcal{B}$. We have to show that if $\mathcal{A},\mathcal{B}$ are balanced, then (5) is valid.

First of all one easily shows that it suffices to examine only those planar networks G (with n sources and n' sinks) that satisfy the following condition:

(C) the source set S and sink set T are disjoint, and each vertex has either at most one entering edge, or at most one leaving edge, or both.

Below we refer to an arbitrary, not necessarily directed, path P in G as a *route*, referring to its edges as *forward* and *backward* ones, depending on their orientation in P. A route P is called *simple* if all vertices in it are distinct. A closed route with distinct vertices is called a *circuit*.

Our approach is based on examining certain pairs of flows in G and rearranging them to form other pairs. Fix $(A,A')\in\mathcal{A}$ and consider an $(XA|X'A')$-flow ϕ and a $(X\overline{A}|X\overline{A}')$-flow ϕ' in G. The pair (ϕ,ϕ') is called a *double flow* for (A,A'), and the set of such double flows is denoted by $\mathcal{D}(A,A')$. We use two lemmas; their proofs are rather simple and rely on condition (C). Here we write $C\triangle D$ for the symmetric difference $(C-D)\cup(D-C)$ of sets C,D, and regard a flow as edge set.

Lemma 1 *$\phi\triangle\phi'$ is partitioned into (the edge sets of) pairwise disjoint circuits $C_1,\dots,C_d$ and simple routes $P_1,\dots,P_p$, where $p=\frac{1}{2}(m+m')$, and each P_i connects either S_A and $S_{\overline{A}}$, or S_A and $T_{A'}$, or $S_{\overline{A}}$ and $T_{\overline{A}'}$, or $T_{A'}$ and $T_{\overline{A}'}$. In each circuit or route, the edges of ϕ and the edges of ϕ' have opposite directions.*

The next lemma explains how to rearrange a double flow (ϕ,ϕ') for (A,A') so as to obtain a double flow for another (useful) pair $(B\subseteq Y,\ B'\subseteq Y')$. Define $\mathcal{P}(\phi,\phi'):=\{P_1,\dots,P_p\}$. For a route P in $\mathcal{P}(\phi,\phi')$, let $\pi(P)$ denote the pair of elements in $Y\sqcup Y'$ corresponding to the end vertices of P. By Lemma 1, $\pi(P)$ belongs to one of $A\times\overline{A}$, $A\times A'$, $A'\times\overline{A}'$, $\overline{A}\times\overline{A}'$. Moreover, the set

$$M(\phi,\phi'):=\{\pi(P):P\in\mathcal{P}(\phi,\phi')\}$$

is a perfect matching on $Y\sqcup Y'$.

Lemma 2 *Choose an arbitrary subset $M_0\subseteq M(\phi,\phi')$. Define $Z:=\cup(\pi\in M_0)$, $B:=A\triangle(Z\cap Y)$, and $B':=A'\triangle(Z\cap Y')$. Let U be the set of edges of routes*

$P \in \mathcal{P}(\phi, \phi')$ with $\pi(P) \in M_0$. Then $\psi := \phi \triangle U$ gives an $(XB|X'B')$-flow, and $\psi' := \phi' \triangle U$ gives an $(X\overline{B}|X'\overline{B}')$-flow. Also $\psi \sqcup \psi' = \phi \sqcup \phi'$.

Obviously, $M(\psi, \psi') = M(\phi, \phi')$ and $\mathcal{P}(\psi, \psi') = \mathcal{P}(\phi, \phi')$, and the transformation of ψ, ψ' by use of the routes $P \in \mathcal{P}(\psi, \psi')$ with $\pi(P) \in M_0$ returns ϕ, ϕ'.

Now consider the FG-function $f = f_{G,w}$ on $\mathcal{E}^{n,n'}$. The summand concerning $(A, A') \in \mathcal{A}$ in the L.H.S. of (5) can be expressed via double flows as follows:

$$
\begin{aligned}
f(XA|X'A') \odot f(X\overline{A}|X'\overline{A}') \\
&= \left(\bigoplus_{\phi \in \Phi_{XA|X'A'}} w(\phi)\right) \odot \left(\bigoplus_{\phi' \in \Phi_{X\overline{A}|X'\overline{A}'}} w(\phi')\right) \\
&= \bigoplus_{(\phi,\phi') \in \mathcal{D}(A,A')} w(\phi) \odot w(\phi') \\
&= \bigoplus_{M \in \mathcal{M}(A,A')} \bigoplus_{(\phi,\phi') \in \mathcal{D}(A,A')\,:\, M(\phi,\phi')=M} w(\phi) \odot w(\phi').
\end{aligned} \tag{7}
$$

The summand concerning $(B, B') \in \mathcal{B}$ in the L.H.S. of (5) is expressed similarly.

Finally, for $(A, A') \in \mathcal{A}$ and $M \in \mathcal{M}(A, A')$, consider $(\phi, \phi') \in \mathcal{D}(A, A')$ such that $M(\phi, \phi') = M$ (if it exists). Since $\mathcal{A}, \mathcal{B}$ are balanced, (A, A', M) is bijective to some (B, B', M) such that $(B, B') \in \mathcal{B}$ and $M \in \mathcal{M}(B, B')$. Since M is a feasible matching for both (A, A') and (B, B'), it follows from (6)(i),(ii) that (B, B') is obtained from (A, A') by "recoloring" w.r.t. some $M_0 \subseteq M$. Then the transformation of (ϕ, ϕ') by use of the routes $P \in \mathcal{P}(\phi, \phi')$ with $\pi(P) \in M_0$ (as described in Lemma 2), results in a double flow (ψ, ψ') for (B, B') such that $\psi \sqcup \psi' = \phi \sqcup \phi'$, implying $w(\psi) \odot w(\psi') = w(\phi) \odot w(\phi')$. Moreover, $(\phi, \phi') \mapsto (\psi, \psi')$ gives a bijection between all double flows for (A, A', M) and those for (B, B', M). Now (5) follows by considering the last term in (7).

5 Necessity of the balancedness

Part "only if" of Theorem 1 says that if patterns $\mathcal{A}_0, \mathcal{B}_0$ are not balanced, then there exist corresponding $G, w, \mathfrak{S}, X, Y, X', Y'$ for which (5) with $\mathcal{A} = \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y,Y'}(\mathcal{B}_0)$ is violated. (Hereinafter X, Y are disjoint subsets of $[n]$, X', Y' are disjoint subsets of $[n']$, and $X, Y, X', Y', \mathcal{A}_0, \mathcal{B}_0$ should be *agreeable*, i.e. there hold $m + 2|X| = m' + 2|X'|$ and $m - 2|C| = m' - 2|C'|$ for all $(C, C') \in \mathcal{A}_0 \cup \mathcal{B}_0$, where $m := |Y|$, $m' := |Y'|$.) We can show a sharper result, saying that if the patterns are not balanced, then (5) is violated for *any* choice of X, Y, X', Y' and for $\mathfrak{S} := \mathbb{Z}_+$.

Proposition 1 *Suppose that patterns $\mathcal{A}_0, \mathcal{B}_0$ are not balanced. Fix (agreeable) X, Y, X', Y'. Then there exists, and can be explicitly constructed, a planar network $G = (V, E)$ such that (5) is false for $f = f_{G,w}$, where $w(v) = 1$ for all $v \in V$.*

The idea of the proof is roughly as follows. Since $\mathcal{A}_0, \mathcal{B}_0$ are not balanced, there exists a planar perfect matching M on $Y \sqcup Y'$ such that

$$|\mathcal{A}_M| \neq |\mathcal{B}_M|,$$

where $\mathcal{A}_M$ is the set of members of $\mathcal{A}$ having M as a feasible matching, and similarly for $\mathcal{B}$. We succeed to construct a planar network G (depending on X, Y, X', Y', M) with the following properties: for any pair $(C \subseteq Y,\ C' \subseteq Y')$,

(P1) If $M \in \mathcal{M}(C, C')$, then G has a unique $(XC|X'C')$-flow and a unique $(X\overline{C}|X'\overline{C}')$-flow, i.e. $|\Phi_{XC|X'C'}| = |\Phi_{X\overline{C}|X'\overline{C}'}| = 1$;
(P2) If $M \notin \mathcal{M}(C, C')$, then at least one of $\Phi_{XC|X'C'}$ and $\Phi_{X\overline{C}|X'\overline{C}'}$ is empty.

Take the function $f = f_{G,w}$ for $w \equiv 1$. By (P1) and (P2), for a pair (C, C'), each of the values $f(XC|X'C')$ and $f(X\overline{C}|X'\overline{C}')$ is equal to 1 if $M \in \mathcal{M}(C, C')$, and at least one of them is 0 otherwise. This implies that the values in the L.H.S. and R.H.S. of (5) are exactly $|\mathcal{A}_M|$ and $|\mathcal{B}_M|$, respectively. Thus, these values are different and (5) is violated.

6 Applications to Schur functions

It is known that Schur functions (polynomials) are expressed as minors of a certain matrix, by Jacobi–Trudi's formula. Therefore, these functions satisfy many quadratic relations. [4, 5] and some other works (see a discussion in [4]) explain how to obtain quadratic relations for ordinary and skew Schur functions by use of a lattice paths method based on the Gessel–Viennot interpretation of semistandard Young tableaux [6]. This lattice path method is, in fact, a specialization to a particular planar network of the flow approach that we described in Sections 1,2. Below we give a brief discussion on this subject.

Recall that a *partition* of length r is an r-tuple λ of weakly decreasing nonnegative integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. The *Ferrers diagram* of λ is meant to be the array F_λ of cells with r left-aligned rows containing λ_i cells in ith row. (We assume that the row indices grow from the bottom to the top.) For $N \in \mathbb{N}$, an N-*semistandard Young tableau* of shape λ is a filling T of F_λ with natural numbers not exceeding N so that the numbers weakly increase in each row and strictly increase in each column. We associate to T the monomial x^T to be the product of variables $x_1, \ldots, x_N$, each x_k being taken with the degree equal to the number of occurrences of k in T. Then the *Schur function* for λ and N is the polynomial

$$s_\lambda = s_\lambda(x_1, \ldots, x_N) := \sum\nolimits_T x^T,$$

where the sum is over all N-semistandard Young tableaux T of shape λ. Besides, one often considers a *skew* Schur function $s_{\lambda/\mu}$, where μ is a partition of length r with $\mu_i \le \lambda_i$; it is defined in a similar way w.r.t. the skew Ferrers diagram $F_{\lambda/\mu}$ obtained by removing from F_λ the cells of F_μ. When needed, an "ordinary" diagram F_λ is regarded as $F_{\lambda/\mu}$ with $\mu = (0, \ldots, 0)$, and similarly for tableaux.

There is a one-to-one correspondence between the partitions λ of length r and the r-element subsets A_λ of the set $\mathbb{Z}_{>0}$ of positive integers, namely:

$$\lambda = (\lambda_1 \ge \ldots \ge \lambda_r) \Longleftrightarrow A_\lambda := \{\lambda_r + 1, \lambda_{r-1} + 2, \ldots, \lambda_1 + r\}. \tag{8}$$

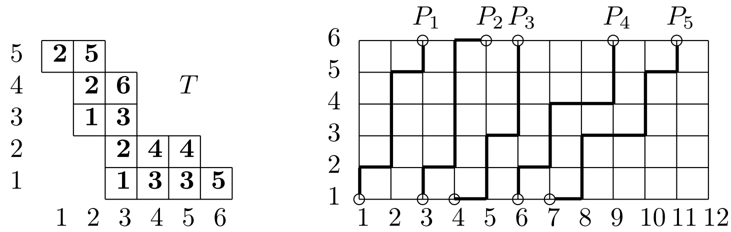
The graph of our interest is the directed square grid $\Gamma = \Gamma(N)$ whose vertices are the points (i, j) for $i \in \mathbb{Z}_{>0}$ and $j \in [N]$ and whose edges e are directed up or to the right, i.e. $e = ((i, j), (i, j+1))$ or $((i, j), (i+1, j))$ (it suffices to take a finite truncation of this grid). The vertices $s_i := (i, 1)$ and $t_i := (i, N)$ are regarded as the sources and sinks in Γ, respectively, and we assign to each horizontal edge e at level h the weight to be the indeterminate x_h:

$$w(e) := x_h \qquad \text{for} \quad e = ((i, h), (i + 1, h)), \;\; i \in \mathbb{Z}_{>0}, \;\; h = 1, \ldots, N, \tag{9}$$

and assign weight 1 to each vertical edge. Now using the Gessel–Viennot model [6] (in a slightly different form), one can associate to an N-semistandard skew Young tableau T with shape λ/μ the system $\mathcal{P}_T = (P_1, \ldots, P_r)$ of directed paths in Γ, where for $k = 1, \ldots, r$:

(6.10) P_k is related to $(r + 1 - k)$th row of T: it goes from the source $s_{k+\mu_{r+1-k}}$ to the sink $t_{k+\lambda_{r+1-k}}$, and for $h = 1, \ldots, N$, the number of horizontal edges of P_k at level h equals the number of occurrences of h in kth row of T.

So the sources occurring in $\mathcal{P}_T$ are the s_i for $i \in A_\mu$, and the sinks are the t_j for $j \in A_\lambda$. Observe that the semistandardness of T implies that these paths are pairwise disjoint, i.e. $\mathcal{P}_T$ is an $(A_\mu|A_\lambda)$-flow in Γ. One can see the converse as well: if $\mathcal{P}$ is an $(A_\mu|A_\lambda)$-flow in Γ, then the filling T of $F_{\lambda/\mu}$ determined, in a due way, by the horizontal edges of paths in $\mathcal{P}$ is just a semistandard skew Young tableau, and one has $\mathcal{P}_T = \mathcal{P}$. This gives a nice bijection between corresponding flows and tableaux. The next picture illustrates an example of a semistandard Young tableau T with $N = 6$, $r = 5$, $\lambda = (6, 5, 3, 3, 2)$ and $\mu = (2, 2, 1, 1, 0)$, and its corresponding flow $\mathcal{P}_T = (P_1, \ldots, P_5)$.



Note that when T is "ordinary" (i.e. $\mu = \mathbf{0}$), the sources used in $\mathcal{P}_T$ are $s_1, s_2, \ldots, s_r$; in other words, $\mathcal{P}_T$ is a *co-flag flow* (it becomes a flag flow if we reverse the edges of Γ and swap the sources and sinks).

The above bijection between the N-semistandard skew Young tableaux with shape λ/μ and the $(A_\mu|A_\lambda)$-flows in $\Gamma = \Gamma(N)$ implies that (ordinary or skew) Schur functions are "values" of the flow-generated function $f_{\Gamma,w}$ for the weighting w as in (9). (It leads to no confusion that the weights are given on the horizontal edges of Γ and belong to a polynomial ring.) This enables us to exhibit quadratic relations on Schur functions, by properly translating SQ-relations on FG-functions.

References

1. F. Brenti, Combinatorics and total positivity, *J. Combin. Theory*, Ser. A, **71** (1995) 175–218.
2. V. Danilov, A. Karzanov, and G. Koshevoy, Tropical Plücker functions and their bases, in: *Tropical and Idempotent Mathematics* (ed. G.L. Litvinov and S.N. Sergeev), *Contemporary Mathematics* **495** (2009) 127–158.
3. C.L. Dodgson, Condensation of determinants, *Proc. of the Royal Soc. of London* **15** (1866) 150–155.
4. M. Fulmek, Bijective proofs for Schur function identities, *ArXiv*:0909.5334v1[math.CO], 2009.
5. M. Fulmek and M. Kleber, Bijective proofs for Schur function identities which imply Dodgson's condensation formula and Plücker relations, *Electron. J. Combin.* **8** (1): Research Paper 16, 2001, 22 pp.
6. I.M. Gessel and X. Viennot, Determinants, paths, and plane partitions, *Preprint*, 1989.
7. B. Lindström, On the vector representation of induced matroids, *Bull. London Math. Soc.* **5** (1973) 85–90.

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Tropical Plücker functions and Kashiwara crystals of types A, B, and C

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1 Introduction

Kashiwara [K90] introduced the fundamental notion of a *crystal* in representation theory. This is an edge-colored directed graph in which each connected monochromatic subgraph is a finite path and there are certain interrelations on the lengths of such paths, described in terms of a Cartan matrix M; this matrix characterizes the *type* of a crystal. An important class of crystals is formed by the crystals of representations, or *regular* crystals; these are associated to irreducible highest weight integrable modules (representations) over the quantum enveloping algebra related to M. There are several models to characterize the regular crystals for a variety of types; e.g., via generalized Young tableaux [KN94], Littelmann's path model [Lt95], MV-polytopes [K].

Here we propose a new model of crystals for the Cartan matrix of type A. In this model, the set of vertices of the crystal is the set of integer-valued tropical Plücker functions on the Boolean cube $2^{[n+1]}$. To decide if a pair of functions f and g is connected by an edge of some color i, we have to restrict f and g to a surface (the union of certain 2-faces in the Boolean cube which correspond to some rhombus tiling) adopted to i.

This model is symmetric on the colors (see Section 5). This allows us to obtain regular crystals for the Cartan matrices of Dynkin B_n- and C_n-types as symmetric extracts from crystals of A_{2n-1}- and A_{2n}-types, respectively. Note that among the above-mentioned models, Littelmann's path model and Kamnitzer's MV-polytope model are also symmetric, and this property was used in [H] and [NS] for construction crystals of B- and C-types as symmetric extracts from A-types; a direct combinatorial proof, based on the so-called crossing model, is given in [DKK12].

There are other advantages of our model. Firstly, it is not too intricate and provides a new viewpoint on Young tableaux of B and C types. Secondly, using our model, we obtain an explicit description of the principal lattice of crystals of types A, B and C. The principal lattice in A-type crystals was introduced and studied in [DKK08].

In Section 2 we recall some basic facts on tropical Plücker functions (TP-functions). In Section 3 we define a structure of ПЇS free crystal on the set of TP-functions. In Section 4 we consider 'bounded' subcrystals (intervals) in a connected free crystal. In the last Section 5, we explain how crystals of B_m- and C_m-types can be derived from symmetric TP-functions on $2^{[2m]}$ and $2^{[2m+1]}$, respectively.

2 Tropical Plücker functions

1. For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A real-valued function f on the set of subsets of $[n]$, that is a function on the Boolean cube $2^{[n]}$, is said to be a *tropical Plücker function*, or a *TP-function*, if it satisfies the *TP3-relation*

$$f(Aik) + f(Aj) = \max\{f(Aij) + f(Ak),\ f(Ai) + f(Ajk)\}, \tag{1}$$

for any triple $i < j < k$ in $[n]$ and subset $A \subseteq [n] - \{i, j, k\}$, where for brevity we write $Ai' \ldots j'$ instead of $A \cup \{i', \ldots, j'\}$.

The set of integer-valued TP-functions on $2^{[n]}$ is denoted by TP_n (the set of real-valued functions is denoted by $TP_n(\mathbb{R})$).

2. The set TP_n is a subset of the space $\mathbb{R}^{2^{[n]}}$ of all functions on $2^{[n]}$. $TP_n(\mathbb{R})$ is stable under multiplication on positive numbers. It is not stable under summation, however, the polyhedral conic complex $TP_n(\mathbb{R})$ contains a lineal of *principal* TP-functions of the dimension $2n$. The lineal is constituted from sums of affine functions and functions which depend only on cardinalities of sets.

3. **Definition.** A subset $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : TP_n(\mathbb{R}) \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on B, and moreover, values on B can be chosen arbitrarily.

Such bases do exist and, for our aims here, we consider bases of the form of spectra of rhombus tiling diagrams. Let us recall the notion of a tiling diagram (for details, see [DKK10]).

Tiling diagrams live within a zonogon, which is defined as follows. In the upper half-plane, take n non-colinear vectors $\xi_1 = (a_1, 1), \dots, \xi_n = (a_n, 1)$ so that $a_1 < a_2 < \dots < a_n$. Then the set $Z = Z_n := \{\lambda_1\xi_1 + \dots + \lambda_n\xi_n : 0 \le \lambda_i \le 1,\ i = 1, \dots, n\}$ is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i : 1 \le \lambda \le 1\}$, $i = 1, \dots, n$. It is the image of the solid cube $[0,1]^{[n]}$ into the plane $\mathbb{R}^2$ by a linear projection π, defined by $\pi(x) := x_1\xi_1 + \dots + x_n\xi_n$. The boundary of Z consists of two parts: the *left boundary*, $lbd(Z)$, formed by the points (vertices) $z_i^\ell := \xi_1 + \dots + \xi_i$ $(i = 0, \dots, n)$, and the *right boundary*, $rbd(Z)$, formed by the points $z_i^r := \xi_{i+1} + \dots + \xi_n$ $(i = 0, \dots, n)$. The points $z_0^\ell = z_n^r$ and $z_n^\ell = z_0^r$ are the minimal vertex and the maximal vertex of Z correspondingly.

A subset $X \subseteq [n]$ is identified with the corresponding vertex of the n-cube and with the point $\sum_{i \in X} \xi_i$ in the zonogon Z.

By a *tile* (or a *rhombus*) we mean a parallelogram τ of the form $X + \{\lambda\xi_i + \lambda'\xi_j : 0 \le \lambda, \lambda' \le 1\}$, where $X \subset [n]$ and $1 \le i < j \le n$; we also call it an *ij-tile* at X and denote by $\tau(X; i, j)$. According to a natural visualization of τ, its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively. Also we say that a point (subset) $Y \subseteq [n]$ is of *height* $|Y|$.

4. **Definition.** A *tiling diagram*, or a *tiling* for short, is a collection T of tiles $\tau(X; i, j)$ which forms a polyhedral decomposition of Z_n.

A *vertex* (an *edge*) of a tiling T is a vertex (an edge) of some rhombus of T. Thus the set of vertices of T defines a collection of corresponding subsets in $2^{[n]}$. Such a collection of subsets is the *spectrum* of the tiling T, $Sp(T)$. Note that boundary vertices of Z_n, $\emptyset$, $\{1\}$, $\{1,2\}$,..., $\{1,2,...,n\}$, $\{2,...,n\}$,..., $\{n\}$, belong to the spectrum of every tiling.

5. For any tiling T, the spectrum $Sp(T)$ is a TP-basis (see [DKK10]). The bijection

$$TP_n(\mathbb{R}) \to \mathbb{R}^{Sp(T)}$$

is a piecewise linear map. One can consider tilings as charts of an atlas for TP_n. The transformation maps between charts take the form of sequences of TP3-relations (1); see [DKK10].

6. Let R be a tile $\tau(A;i,j)$. Then an *excess* of a function $f : 2^{[n]} \to \mathbb{R}$ in the rhombus R is said to be the amount

$$\varepsilon(f,R) = f(Ai) + f(Aj) - f(A) - f(Aij).$$

Any functions on vertices of a tiling T is defined by its values at the vertices on the right boundary of Z_n and the excesses at tiles of T.

3 Free crystal of type A

1. A digraph $K = (V(K), E(K))$ is an n *edge-colored* if $E(K) = E_1 \sqcup \ldots \sqcup E_n$, where any edge $e \in E_i$ has a color 1. An edge-colored digraph K is a *pre-crystal* if two following axioms are satisfied. The first axiom requires that, for any color i, the subgraph $(V(K), E_i)$ is a disjoin union of strings (finite or infinite). A translation along an edge of a color i is understood as an action of the operation $\mathbf{i}$ on the set $V(K)$. Namely, if an edge (v,u) has color i, then $\mathbf{i}(v) = u$, and we say that the operation i *acts* at v. If, for a vertex v there is no edge of color i emanating from v, then we say that $\mathbf{i}$ *does not act* at v.

The reversing edges of the graph K defines the reverse operations $\mathbf{i}^{-1}$. These operations are inverse in the following sense: if $\mathbf{i}$ acts at v and $w = \mathbf{i}v$, then $\mathbf{i}^{-1}$ acts at w and $v = \mathbf{i}^{-1}w$.

Example. A *commutative pre-crystal* is an Abelian group $\mathbb{Z}^{[n]}$ on which an operator $\mathbf{i}$ sends x to $x + 1_i$, where 1_i is the i-th basis vector.

Let K and K' be two n-colored pre-crystals. A *morphism* $K \to K'$ is a mapping $\varphi : V(K) \to V(K')$ which commutes with the actions of operations $\mathbf{i}$, that is if $\mathbf{i}$ does act at v in the pre-crystal K, then $\mathbf{i}$ acts at $\varphi(v)$ and there holds $\varphi(\mathbf{i}v) = \mathbf{i}\varphi(v)$.

The second axiom of pre-crystals requires the existence of a *weight map*, a morphism $wt : K \to \mathbb{Z}^{[n]}$.

2. Crystals are associated to (generalized) Cartan matrices. Let $M = (m_{ij})$, $i,j \in [n]$ be a (generalized) Cartan matrix, that is $m_{ij} \in \mathbb{Z}$, $m_{ii} = 2$, and $m_{ij} \leq 0$ for $i \neq j$. For a pre-crystal to be a crystal, more axioms, which relate the weight map and Cartan data, have to be satisfied. However, according to 2-color reduction theorem [KKMMNN], a 'bounded' pre-crystal is a crystal of an integrable M-module iff the restriction of K to any pair of colors i, j is a crystal of the corresponding $M|ij$-module. For simply- and doubly-laced cases, it suffices to know crystals of three types $A_1 + A_1$, A_2 and B_2. A structure of such crystals are studied in depth, for example, in [DKK07, DKK09].

3. Let us define a structure of pre-crystal with n colors on the vertex set being integer-valued TP-functions $TP = TP_{n+1}(\mathbb{Z})$ on $2^{[n+1]}$ (as we will see later such a pre-crystal is an A_n-crystal indeed). We need several notions.

Let us call a tile τ of a tiling T a *left rhombus* if it shares 2 edges with the left boundary of the zonogon Z_{n+1}. Specifically, τ is a left-rhombus at the height h if $b(\tau) = [h-1]$,$l(\tau) = [h]$, and $t(\tau) = [h+1]$. Denote such a rhombus tile by LR_h. Analogously a right rhombus RR_h at the height h is defined.

We say that a tiling T (in the zonogon $Z = Z_{n+1}$) *is fitted* to the color i ($i = 1, \dots, n$), if T contains the left rhombus LR_i.For any color i, there exists a tiling which is fitted to i

4. Now all is ready to define a crystal operation $\mathbf{i}$ ($i = 1, \dots, n$) at a TP-function $f \in TP_{n+1}$. Pick a tiling T, which is fitted to the color i. Then the function $\mathbf{i}f$ is defined (see 2.5) by the rule

$$(\mathbf{i}f)(v) = \begin{cases} f(v) + 1, \text{ if } v = [i], \\ f(v) \qquad \text{otherwise.} \end{cases}$$

In other words, in the chart T the function $\mathbf{i}f$ differs from f in the single vertex $[i]$ of T. However, the functions f and $\mathbf{i}f$ may be different at some other vertices of the Boolean cube. Nevertheless, they coincide at the vertices of the right boundary $rbd(Z_{n+1})$.

5. **Theorem**. *The operations* $\mathbf{i}$ $(i = 1, ..., n)$ *are correctly defined and they endow the set* $TP = TP_{n+1}$ *with a structure of an* A_n*-crystal.*

According to 2-color reduction theorem [KKMMNN], we have to consider the case $n = 3$. In such a case one can establish an explicit bijection with the crossing model for A_2-crystal from [DKK07].

6. The crystal operations $\mathbf{i}$ commute with addition of any (integer-valued) principal TP-function. That is, for any TP-function f and any principal TP-function p, there holds

$$\mathbf{i}(f + p) = \mathbf{i}f + p.$$

7. As we remarked above the crystal actions preserve values of TP-functions at the vertices at $rbd(Z_{n+1})$. Thus, the values at these vertices are $n+2$ 'integrals', and any connected component of crystal TP_{n+1} is specified by $x \in \mathbb{Z}^{n+2}$, a list of values at the vertices of $rbd(Z)$. Denote by $K[x]$ such a crystal. Since, for different x, the crystals $K[x]$ are isomorphic. We consider a crystal $K = K[0]$ with the vertices being integer-valued TP-functions which are equal to zero at any vertex of $rbd(Z_{n+1})$.

8. Consider the subset P of principal TP-functions which belong to K. That is the set of principal TP-functions which equal 0 at the vertices of $rbd(Z)$. Because of 2.8, we may specify the excesses of such a function for rhombuses at each height. Thus, we get an isomorphism between P and the Abelian group $\mathbb{Z}^n$. The basis of the lattice P is constituted from the principal functions $p_1, \ldots, p_n$, where the i-th function p_i is specified by the following conditions:

1. equal 0 at each vertex of $rbd(Z)$;
2. for any height $j \neq i$, and any rhombus R at the height j, $\varepsilon(p_i, R) = 0$;
3. for any rhombus R at the height i, $\varepsilon(p_i, R) = 1$.

4 Subcrystals of K

1. Since the operations $\mathbf{i}$ and $\mathbf{i}^{-1}$ act at each vertex of K, the crystal K is a *free* crystal, that is all monochromatic strings are infinite. Here we interested in subsets of K which are A_n-type crystals.

Let us define a (partial) order $\preceq$ on the vertices of K. For functions $f, g \in K$, we write $f \preceq g$ if there exists a word $\mathbf{w}$ in the alphabet $\{\mathbf{1}, \ldots, \mathbf{n}\}$ such that there holds $g = \mathbf{w}f$. In other words, $f \preceq g$ if there exists a path in the digraph K emanated from f and ended in g.

For a principal vertex $p \in K$, we denote by K_p (and K^p) the set $\{f \in K,\ p \preceq f\}$ (and $\{f \in K,\ f \preceq p\}$). The vertex p is a single source in the poset K_p (the single sink in the poset K^p). We have

$$K^p = p + K^0 \quad \text{and} \quad K_q = q + K_0.$$

Because of this, we are interested in the sets K^0 and K_0.

2. Here is an alternative definition of the set K_0. Recall, that a function $f : 2^{[n+1]} \to \mathbb{R}$ is *submodular*, if, for any rhombus R, there holds $\varepsilon(f, R) \geq 0$. It is not difficult to prove that *a TP-function f is submodular iff, for some tiling T and every tile $\tau \in T$, there holds $\varepsilon(f, \tau) \geq 0$.*

3. **Theorem**. *The set K_0 is the set of integer-valued submodular TP-functions which equal to 0 at $rbd(Z_{n+1})$.*

4. The intersection K_0 and the lattice of principal TP-functions $P \cong \mathbb{Z}^n$ is the semi-group $\mathbb{Z}^n_+$. In fact, because of (3.8), every principal function p_i, $i = 1, \ldots, n$, is submodular. Moreover, only non-negative linear combinations of these functions are submodular.

5. We get the following corollary from Theorem 4.3

Corollary. *The crystal K is connected.*

If fact, let us show that any function of K is of the form $\mathbf{w}\,0$, where $\mathbf{w}$ is a word in the alphabet $\{\mathbf{1}^{\pm 1}, \ldots, \mathbf{n}^{\pm 1}\}$. Specifically, for any $f \in K$, we can find a principal function $p \in P$ such that the TP-function $f + p$ is submodular. Then, according to Theorem 4.3, there exists a word $\mathbf{w}$ in the alphabet $\{\mathbf{1}, \ldots, \mathbf{n}\}$ such that there holds $f+p = \mathbf{w}0$. Hence $f = \mathbf{w}(-p)$. Since, due to Theorem 4.3, there exists a word $\mathbf{v}$ such that $p = \mathbf{v}0$, we have have $-p = \mathbf{v}^{-1}0$. All this implies $f = \mathbf{w}\mathbf{v}^{-1}0$.

6. Here we give a description of the set K^0 using excesses.

Theorem. *A TP-function $f \in K$ belongs to K^0 iff, for every right rhombus RR_i, there holds $\varepsilon(f, RR_i) \leq 0$.*

Let $p \in P \cap K$ be a principal function. Then *a function $f \in K$ belongs to K^p iff, for any $i = 1, \ldots, n$, there holds $\varepsilon(f, RR_i) \leq \varepsilon(p, RR_i)$.*

7. The crystals of the form of an interval, $K^p_q = K^p \cap K_q$, where $p, q \in P$ and $q \preceq p$, correspond to finite-dimensional integrable modules. Wlog, we may consider $q = 0$. The graph K^p_0 is connected, has finitely many vertices, a single source 0 and a single sink p.

Because of Theorem 4.3 and 4.6, a function $f \in K$ belongs to the crystal $K^p_0 = K_0 \cap K^p$ (with some $p \in P$) iff

1. For any rhombus R, there holds $\varepsilon(f, R) \geq 0$;
2. for any right rhombus RR_i, $i = 1, \ldots, n$, there holds $\varepsilon(f, RR_i) \leq \varepsilon(p, RR_i)$.

The intersection of a crystal K^p_0 and the principal lattice P is constituted from the functions of the form $\sum_i c_i p_i$, where $0 \leq c_i \leq \varepsilon(p, RR_i)$, that is an integer parallelepiped. For a tuple $c = (c_1, ..., c_n) \in \mathbb{Z}^n_{\geq 0}$, we denote by $K(c)$ the crystal K^p_0, where $p = \sum_i c_{n+1-i} p_i$.

8. **Proposition**. 1) *c_i is equal to the maximal number α such that a function $\mathbf{i}^{\alpha}0$ belongs to $K(c)$.*

2) *c_{n-i+1} is equal to the maximal number β such that a function $\mathbf{i}^{-\beta}p$ belongs to $K(c)$.*

5 Symmetric extracts from symmetric A-crystals

1. The inversion $\sigma(i) = n + 2 - i$ of the set $[n + 1]$ can be considered as an inversion of the Dynkin diagram A_n. Consider the inversion γ of the Boolean cube $2^{[n+1]}$, defined by $\gamma(A) = \sigma([n + 1] - A)$.

Consider an extension of the inversion γ to the zonogon $Z = Z_{n+1}$. For this, consider symmetric (w.r.t. σ) vectors $\xi_i = (x_i, 1)$, that is $\xi_{n+2-i} = (-x_i, 1)$. Denote by γ the symmetry of the plane w.r.t. the horizontal line $y = (n+2)/2$. This symmetry sends Z to itself: if a point $v \in Z$ corresponds to a subset $A \subset [n+1]$, then the point $\gamma(v)$ corresponds to the subset $\gamma(A)$. This symmetry extends to the space of functions on the Boolean cube. Namely, let $f : 2^{[n+1]} \to \mathbb{R}$ be a function on the Boolean cube. Then the function $\gamma^* f : 2^{[n+1]} \to \mathbb{R}$ sends a set A to $f(\gamma(A))$. Obviously, f is a TP-function if and only if $\gamma^* f$ is a TP-function.

Denote by $\widetilde{TP}$ the set of symmetric TP-functions, $\gamma^* f = f$. We are going to endow the set $\widetilde{TP}$ with a crystal structure. This depends on the parity of n.

2. Let n be odd, $n = 2m-1$, $m \geq 1$. In this case there are plenty of symmetric tilings. A tiling T is symmetric if its set of vertices and edges is stable under the symmetry γ, i.e. $\gamma T = T$.

In this case, any symmetric TP-function defines a symmetric function on any symmetric tiling, and vice versa. Consider m operations $\widetilde{\mathbf{1}}, \dots, \widetilde{\mathbf{m}}$ on $\widetilde{TP}$, where $\widetilde{\mathbf{1}} = \mathbf{1n} = \mathbf{n1}$, ..., $\widetilde{\mathbf{m-1}} = (\mathbf{m-1})(\mathbf{m+1})$, $\widetilde{\mathbf{m}} = \mathbf{m}$, where $\mathbf{1}, \dots, \mathbf{n}$ are the crystal operations on TP. These operations can be defined as follows. For an operation $\widetilde{\mathbf{i}}$, we consider a symmetric tiling T fitted to the color i. Then, by the symmetry, T also is fitted to the color $n+1-i = 2m-i$. The operation $\widetilde{\mathbf{i}}$ on the vertices of T is defined by the rule

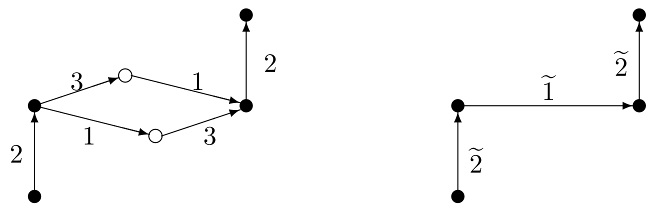
$$(\widetilde{\mathbf{i}}f)(v) = \begin{cases} f(v)+1, & \text{if } v = [i] \text{ or } [2m-i] \\ f(v) & \text{otherwise.} \end{cases}$$

Note that for $i = m$, we have $[m] = [2m-m]$, and the operation $\widetilde{\mathbf{m}}$ increases by 1 the value of a function at the symmetric vertex $[m]$.

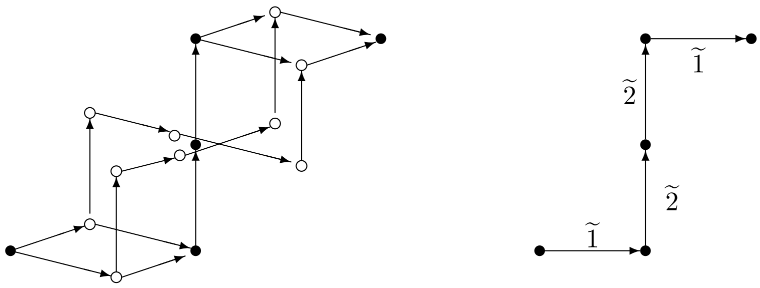
3. **Theorem**. *$\widetilde{TP}$ is a free B_m-crystal.*

4. Considering symmetric functions in the subcrystals K_0, K^0 and $K(c)$, with symmetric c ($c_{\sigma(i)} = c_i$, $i = 1, \dots, m$) of type A_{2m-1}, we obtain B_m-subcrystals in the free B_m-crystal $\widetilde{TP}$. The symmetric part of $\widetilde{K}(c)$ of $K(c)$ is an interval in the poset $\widetilde{K}$ consisting of symmetric TP-functions between the principal vertices 0 and $p = \sum_{i=1}^{2m+1} c_i p_i$.

Let us consider simplest examples for $n = 3$. The A_3-crystal $K(0,1,0)$ is drawn in the left part of the picture below. The symmetric vertices are indicated by bold circles, and the extracted B_2-crystal $\widetilde{K}(0,1)$ is depicted in the right part.

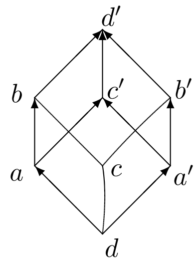


The A_3-crystal $K(1,0,1)$ is drawn in the left part of the next picture, and the extracted B_2-crystal $\widetilde{K}(1,0)$ in the right part.



5. Now let n be even, $n=2m$. In this case there are no symmetric tilings, but there exist symmetric *hexagonal-rhombus tilings*, or *HR-tilings* for short. Tiles of an HR-tiling are rhombi or hexagons (where a hexagon is the zonogon Z_3). An HR-tiling is symmetric if it contains, for each rhombus R, the symmetric rhombus $\gamma(R)$, and for each hexagon H, the symmetric hexagon $\gamma((H)$.

Consider the case $n=2$. There is a unique HR-tiling, the zonogon Z_3 itself. Below we illustrate Z_3 and its two rhombus tilings, we denoted values of a function at the vertices of the cube by letters a, a', b, b', c, c', d and d'.



A symmetric function is specified by the following conditions: $a=b$, $a'=b'$, and $c=c'$. TP3-relation (1) reads as $c+c'=\max(a+b', b+a')$ and boils down to the equality $2c=a+a'$. Because of this, the values of a symmetric TP-function on the boundary of Z_3 determine the whole function.

In this case, a symmetric TP-function in K is a triple a, $b=a$, $c=a/2$.

Apply the sequence of operations **1221** to a symmetric TP-function corresponding to a, $b = a$ and $c = a/2$. The result is a symmetric function corresponding to $\tilde{a} = a + 2$, $\tilde{b} = a + 2$, and $\tilde{c} = c + 1$. We can apply the sequence of crystal operations **2112** to the same symmetric TP-function. The result is again $\tilde{a} = a + 2$, $\tilde{b} = a + 2$, and $\tilde{c} = c + 1$. Thus, the symmetric extraction of A_2 endowed with the operation **1221** is an A_1-crystal.

6. Any symmetric TP-function on the Boolean cube $2^{[2m+1]}$ defines a symmetric function on the vertices of any HR-tiling, and vice versa. As before, we denote by $\widetilde{TP}$ the set of symmetric TP- functions. Define the operations on $\widetilde{TP}$ as follows. The operations $\widehat{\mathbf{1}}, \ldots, \widehat{\mathbf{m-1}}$ are defined as in (5.2): $\widehat{\mathbf{1}} := \mathbf{1}(\mathbf{2m})$, $\ldots$, $\widehat{\mathbf{1}} := (\mathbf{m-1})(\mathbf{m+2})$. The operation $\widehat{\mathbf{m}}$ is defined as the sequence $\mathbf{m}(\mathbf{m+1})(\mathbf{m+1})\mathbf{m} = (\mathbf{m+1})\mathbf{mm}(\mathbf{m+1})$.

In terms of symmetric HR-tilings, these operations are expressed as follows. For $1 \leq i < m$, take a symmetric HR-tiling which is fitted to the color i. Then the ith operation is defined on a vertex v of the tiling by the rule

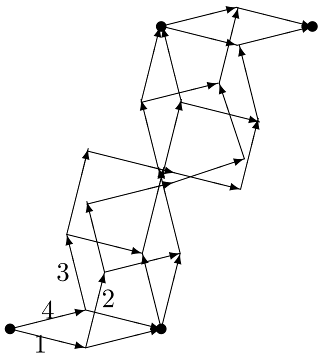
$$(\widetilde{\mathbf{i}}f)(v) = \begin{cases} f(v) + 1, \text{ if } v = [i] \text{ or } [2m - i] \\ f(v) \qquad \text{otherwise.} \end{cases} \tag{2}$$

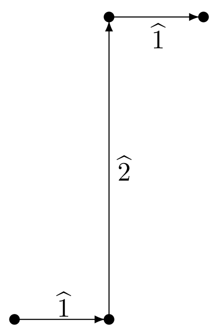
For $i = m$, we take a symmetric HR-tiling which has a hexagon containing the vertices $[m-1]$,$[m]$,$[m+1]$,$[m+2]$. Then the operation $\widetilde{\mathbf{m}}$ is defined by the rule

$$(\widetilde{\mathbf{m}}f)(v) = \begin{cases} f(v) + 2, \text{ if } v = [m] \text{ or } v = [m+1] \\ f(v) \qquad \text{otherwise.} \end{cases} \tag{3}$$

7. **Theorem**. *The set of symmetric function $\widetilde{TP}$ endowed with operations (2) and (3) is a free C_m-crystal.*

8. Analogous to part 5.4, one can define the C_m-subcrystals $\widetilde{K^0}$, $\widetilde{K_0}$, and $\widetilde{K}(c)$. The next picture illustrates the extract $\widetilde{K}(1,0)$ from the crystal $K(1,0,0,1)$.





Supported by RFBR grant 10-01-9311-CNRSL_ a.

References

[DKK07] V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy, Combinatorics of regular A_2-crystals, *J. Algebra*, 310 (2007) 218–234. (*ArXiv:math*.RT/0604333)

[DKK08] V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy, The crossing model for regular A_n-crystals, *J. Algebra*, 320 (2008) 3398–3424. (*ArXiv:math*.RT/0612360)

[DKK09] V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy, B_2-crystals: Axioms, structure, models. *JCTA*, 116 (2009) 265–289. (*ArXiv:math*.RT/0611641)

[DKK10] V. Danilov, A. Karzanov and G. Koshevoy, Plücker environments, wiring and tiling diagrams, and weakly separated set-systems, *Adv. Math.*, 224 (2010) 1–44. (*ArXiv*:0902.3362v3[math.CO])

[DKK12] V.I. Danilov, A.V. Karzanov, and G.A. Koshevoy, On the structure of regular crystals of types A, B, C. *ArXiv:1201.4549v2.*

[H] J. Hong, Mirković-Vilonen cycles and polytopes for a symmetric pair. *Representation Theory*, v. 13 (2009) 19-32.

[K] J. Kamnitzer, Mirković-Vilonen cycles and polytopes. *Ann. of Math.* 171 (2010), 731-777.

[KKMMNN] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki, Affine crystals and vertex models, *International J. of Modern Physics A*, 7, Suppl. 1A (1992) 449–484.

[K90] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, *Comm. Math. Phys.*, 133 (2) (1990) 249–260.

[KN94] M. Kashiwara and T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, *J. Algebra*, 165 (1994) 295-345.

[Lt95] P. Littelmann, Paths and root operators in representation theory, *Ann. of Math.*, 142 (3) (1995) 499–525.

[NS] S. Naito and D. Sagaki, Lakshmibai-Seshadri paths fixed by a diagram authomorphism. *J. Algebra*, 245 (2001) 395–412.

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Discriminant of system of equations

A. Esterov

1. Introduction

What polynomial in the coefficients of a system of algebraic equations should be called its discriminant? We prove a package of facts that provide a possible answer. Let us call a system typical, if the homeomorphic type of its set of solutions does not change as we perturb its (non-zero) coefficients. The set of all atypical systems turns out to be a hypersurface in the space of all systems of k equations in $n \geq k-1$ variables, whose monomials are contained in k given finite sets. This hypersurface B is the union of two well-known strata: the set of all systems that have a singular solution (this stratum is conventionally called the discriminant) and the set of all systems, whose principal part is degenerate (they can be regarded as systems with a singular solution at infinity). None of these two strata is a hypersurface in general, and codimensions of their components have not been fully understood yet (e.g. dual defect toric varieties are not classified), so the purity of dimension of their union seems somewhat surprising. We deduce it from a similar tropical purity fact of independent interest: the stable intersection of a tropical fan with a boundary of a polytope in the ambient space has pure codimension one in this tropical fan.

A generic system of equations in an irreducible component B_i of the hypersurface B always differs from a typical system by the Euler characteristic of its set of solutions. Regarding the difference of these two Euler characteristics as the multiplicity of B_i, we turn B into an effective divisor, whose equation we call the Euler discriminant of a system of equations by the following reasons. Firstly, it vanishes exactly at those systems that have a singular solution (possibly at infinity). Secondly, despite its topological definition, it admits a simple linear-algebraic formula for its computation, and a positive formula for its Newton polytope. Thirdly, it interpolates many classical objects and inherits many of their nice properties: for $k = n+1$, it is the sparse resultant (defined by vanishing on consistent systems of equations); for $k = 1$, it is the principal A-determinant (defined as the sparse resultant of the polynomial and its partial derivatives); as we specialize the indeterminate coefficients of our system to be polynomials of a new parameter, the Euler discriminant turns out to be preserved under this base change, similarly to discriminants of deformations. This allows, for example, to specialize our results to generic polynomial maps: the bifurcation set of a dominant polynomial map, whose components are generic linear combinations of given monomials, is always a hypersurface, and a generic atypical fiber of such a map differs from a typical one by its Euler characteristic.

2. Degenerate systems

For a finite set $H \subset \mathbb{Z}^n$, we study the space $\mathbb{C}[H]$ of all *Laurent polynomials* $h(x) = \sum_{a\in H} c_a x^a$, where x^a stands for the monomial $x_1^{a_1} \ldots x_n^{a_n}$, the coefficient c_a is a complex number, and the polynomial h is considered as a function $(\mathbb{C} \setminus 0)^n \to \mathbb{C}$. For a linear function $v : \mathbb{Z}^n \to \mathbb{Z}$, denote the intersection of H with the boundary of the affine half-space $H + \{v < 0\}$ by H^v, and the highest v-degree component $\sum_{a\in H^v} c_a x^a$ by h^v (if $v = 0$, then we set $H^0 = H$ and $h^0 = h$).

In what follows, we denote a collection of finite sets $A_0, \ldots, A_k$ in $\mathbb{Z}^n$ by A, the space $\mathbb{C}[A_0] \oplus \ldots \oplus \mathbb{C}[A_k]$ by $\mathbb{C}[A]$, consider its element $f = (f_0, \ldots, f_k) \in \mathbb{C}[A]$ as a map $(\mathbb{C} \setminus 0)^n \to \mathbb{C}^{k+1}$, and denote $(f_0^v, \ldots, f_k^v)$ by f^v.

Theorem 1 *Assume that $A_0+\ldots+A_k$ is not contained in an affine hyperplane. The following three conditions are equivalent for the system of equations $f = 0$:*
1) There exists an arbitrarily small $\tilde f \in \mathbb{C}[A]$, such that the sets $\{f = 0\}$ and $\{f + \tilde f = 0\}$ are not diffeomorphic.
2) There exists an arbitrarily small $\tilde f \in \mathbb{C}[A]$, such that the sets $\{f = 0\}$ and $\{f + \tilde f = 0\}$ have different Euler characteristic.
3) There exists a linear function $v : \mathbb{Z}^n \to \mathbb{Z}$ such that the differentials $df_0^v, \ldots, df_k^v$ are linearly dependent at some point of the set $\{f^v = 0\}$.

The assumption on $A_0 + \ldots + A_k$ cannot be dropped, because, otherwise, the Euler characteristic of $\{f = 0\}$ equals 0 for every $f \in \mathbb{C}[A]$ by homogeneity considerations.

Definition 1 A system $f \in \mathbb{C}[A]$ is said to be *degenerate*, if it satisfies any of the three conditions above.

Condition 3 was first introduced in [K76] for $k = 0$ and in [Kh77] for arbitrary k. Condition 2 will play the role of tameness on a complex torus for our purpose (cf. the definition in [Br88]), although it is not equivalent to tameness at all. Its similarity to tameness admits further development: e. g. for non-degenerate f and a generic local system L on $(\mathbb{C} \setminus 0)^n$, so that $H((\mathbb{C} \setminus 0)^n, L) = 0$, the twisted homology $H(\{f = 0\}, L)$ vanish except for the middle dimension.

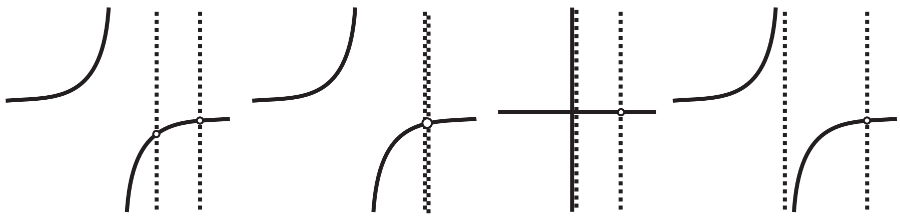
For example, if $k = n$, then B is the resultant set (i.e. the set of all consistent systems of equations in $\mathbb{C}[A]$, see [St94]); if $A_0 = \ldots = A_k$ is the set of vertices of the standard n-dimensional simplex, then $f_0, \ldots, f_k$ are linear, and B is defined by vanishing of the product of the maximal minors for the matrix of coefficients of $f_0, \ldots, f_k$.

Definition 2 The collection A is said to be *relevant*, if the dimension of the convex hull of $A_{i_0} + \ldots + A_{i_p}$ is at least p for every sequence $0 \le i_0 < \ldots < i_p \le k$, and equals n for $p = k$.

Theorem 2 *If A is relevant, then the set B of all degenerate systems in $\mathbb{C}[A]$ is a non-empty hypersurface.*

The assumption of relevance cannot be dropped, because, otherwise, the set of consistent systems has codimension greater than 1 (see [St94]).

The similar question of whether the A-discriminant $\{f \in \mathbb{C}[A] \,|\, f = 0$ is not regular$\}$ is a hypersurface is well known for $k = 0$ as the problem of classification of dual defect polytopes, and is still open (see [CC07], [D06], [?], [MT], [E10], etc). Moreover, for $k > 0$, the A-discriminant may be not of pure dimension: e.g. for $A_0 = \{0, 1\} \times \{0, 1\}$ and $A_1 = \{0, 1, 2\} \times \{0\}$, there is a codimension 1 component, to which $f = \Big(a + bx + cy + dxy,\, r(x - p)^2\Big)$ belongs, and a codimension 2 component, to which $f = \Big(a(x - b)(y - c),\, r(x - b)(x - p)\Big)$ belongs. "Fortunately", the latter one is swallowed up by the codimension 1 stratum of B, to which $f = \Big(a(x - b)(y - c) + d,\, r(x - b)(x - p)\Big)$ belongs because of its singularity at infinity. The generic configuration of $f_0 = 0$ (in solid lines) and $f_1 = 0$ (in dotted lines) is shown on the picture below, followed by the configurations of the three mentioned degenerations.



3. Euler discriminant

For $f \in \mathbb{C}[A] \setminus B$ and generic $\tilde{f}$ in an irreducible component $B_i \subset B$, denote the difference of the Euler characteristics $e\{\tilde{f} = 0\} - e\{f = 0\}$ by e_i.

Proposition 1 *If A is relevant, then $e_i > 0$ for $n-k$ even and $e_i < 0$ for $n-k$ odd.*

Definition 3 If A is relevant, then the equation of the effective divisor $(-1)^{n-k} \sum_i e_i B_i$ is called the *A-Euler discriminant* and is denoted by $E_A = E_{A_0,\ldots,A_k}$.

By Proposition 1, E_A is a non-constant polynomial on $\mathbb{C}[A]$, defined up to multiplication by a non-zero constant. By Theorem 1, the equation $E_A = 0$ describes all degenerate systems of equations in $\mathbb{C}[A]$. Despite its topological definition, this polynomial is unexpectadly easy to study algebraically. In particular, its Newton polytope is as follows.

Recall that a coherent triangulation T of a finite subset $H \subset \mathbb{Z}^N$ is a set of N-dimensional simplices, such that

1) their vertices are contained in H,

2) the union of them is the convex hull of H,

3) the intersection of any two of them is their common face (maybe empty),

4) they are the domains of linearity of a convex piecewise-linear function (this property is called *coherence* or *convexity*).

Let A_* be the union of the sets $\{e_i\} \times A_i$ in $\mathbb{Z}^{k+1} \times \mathbb{Z}^n$, where $e_0, \ldots, e_k$ is the standard basis of $\mathbb{Z}^{k+1}$. Coefficients of polynomials in $\mathbb{C}[A]$ form a natural coordinate system $(c_{a,i})_{a \in A_i,\, i=0\ldots,k}$ on it, so that $c_{a,i}(f)$ is the coefficient of the monomial x^a in the i-th polynomial of the tuple $f \in \mathbb{C}[A]$. For a simplex S with vertices in $A_{\{0,\ldots,k\}}$, let c_S be the product of all $c_{a,i}$, such that (e_i, a) is a vertex of S, and, for every $j \neq i$, the set $\{e_j\} \times \mathbb{Z}^n$ contains more than one vertex of S.

Proposition 2 *The set of monomials*

$$\{\prod_{S\in T} c_S^{\mathrm{Vol}\, S} \,|\, T \textit{ is a coherent triangulation of } A_{\{0,\ldots,k\}}\}$$

is the set of vertices for the Newton polytope of the A-Euler discriminant E_A (the Newton polytope is in the natural coordinate system $(c_{a,i})$, and Vol *stands for the integer volume, normalized by the condition* Vol(*standard simplex*) $= 1$*).*

This Newton polytope is a natural generalization of the well known secondary polytope for $k = 0$.

4. Application to topology of polynomial maps

There is a more invariant definition of the Euler discriminant, explaining its behaviour under the specialization of the indeterminate coefficients of $f \in \mathbb{C}[A]$. In particular, specializing $(c_{a,i})$ to generic constants for $a \neq 0$, one can deduce the following.

Definition 4 The *bifurcation set* B_p of a morphism of algebraic varieties $p : Q \to M$ is the complement to the maximal open set $S \subset M$, such that the restriction of p to the preimage $p^{-1}(S)$ is a locally trivial fibration.

Corollary 1 *If A is relevant, the tuple $g \in \mathbb{C}[A]$ is nondegenerate, and every k of the $k+1$ polynomials in g also form a nondegenerate tuple, then the bifurcation set of the map $g : (\mathbb{C} \setminus 0)^n \to \mathbb{C}^{k+1}$ is a hypersurface, and a generic atypical fiber of g differs from a typical one by its Euler characteristic.*

One part of Corollary 1 addresses the question of purity of the bifurcation locus, which is trivial for $k = 0$ and is classical for $k = n - 1$: see [J93] for the purity of the Jelonek set.

Another part of this statement addresses the question of distinguishing atypical fibers of polynomial maps by their discrete invariants. This question, in contrast to the first one, is trivial for $k = n - 1$ and is classical for $k = 0$: see e. g. [HL84] for the case of two variables, [SieTib95], [Par95], and [ALM00] for polynomials with isolated singularities at infinity, [NZ90] for non-degenerate polynomials. This question was also addressed for $k = n-2$ (see e. g. [HT08]),but less is known for arbitrary k (see e. g. [Gaf99]). Note that the most common setting for these studies is the assumption of isolated singularities at infinity, which is neither weaker nor stronger than the one in Corollary 1. Polynomials with isolated singularities at infinity may be degenerate (e.g. $(x-y)^2 + (x-y) + c$), and, vice versa, nondegenerate polynomials, whose Newton polytopes are not simple, may have non-isolated singularities in any smooth compactification (e.g. $xyz(x+y)(z+1)+c$). It is an interesting problem of toric singularity theory to unify these two settings.

Many of the aforementioned works are also concerned with estimating the degree of the bifurcation set in terms of the degree of the mapping (see also [OT95] for $k=0, n=2$, [J03] and [JK03] for $k=0$, and [J03'] for the general case). For nondegenerate maps, the precise answer regarding the degree can be deduced from Proposition 2.

5. Tropical counterpart

The aforementioned facts regarding the Euler discriminant can be reduced to a similar statemnent about tropical complexes (Proposition 3 below). A *polyhedral complex* of dimension k in $\mathbb{R}^n$ is a locally finite union of closed convex rational k-dimensional polytopes $P_i \in \mathbb{R}^n$. The *stable intersection* of sets P and Q in $\mathbb{R}^n$ is the set of all points $x \in P \cap Q$, such that

$$\forall\varepsilon\, \exists\delta\, :\, v \in \mathbb{R}^n, |v| < \delta \;\Rightarrow\; \mathrm{dist}(x, (P+v)\cap Q) < \varepsilon.$$

This operation is denoted by $\wedge$, is commutative, but not associative (different brackets in $\{x=0\} \wedge \{y=0\} \wedge \{y \leqslant |x|\} \subset \mathbb{R}^2$ lead to different answers), and its result may be of unexpected dimension (like $\{z = |x| + |y|\} \wedge \{z = 0\} = \{0\} \subset \mathbb{R}^3$). To avoid these issues, we should restrict our consideration to *tropical complexes*.

A point x of a k-dimensional polyhedral complex P is said to be *smooth*, if its transposed copy $P - x$ coincides with a vector subspace in a neighborhood of $0 \in \mathbb{R}^n$. This vector subspace is called the *tangent space at* x and is denoted by T_xP.

Definition 5 A closed polyhedral complex $P \subset \mathbb{R}^n$ is said to be *tropical*, if it admits a positive locally constant non-zero function w : {smooth points of P} $\to \mathbb{R}$, such that, for every rational subspace $L \subset \mathbb{R}^n$ of complementary dimension, the *tropical intersection number* of its transposed copy $L - x$ and P

$$\sum_{p \in P \cap (L-x)} w(p) \left| \frac{\mathbb{Z}^n}{(\mathbb{Z}^n \cap L) + (\mathbb{Z}^n \cap T_pP)} \right|$$

does not depend on x (this sum makes sense for almost all $x \in \mathbb{R}^n$).

Proposition 3 *Let P be a polyhedron in $\mathbb{R}^n$, open in its affine span, and let T be a tropical complex in $\mathbb{R}^n$ such that* $\dim P + \dim T = n + k$. *Then*
1) The stable intersection $S = T \wedge P$ is k-dimensional or empty.
2) The intersection of the closure of S with the relative boundary ∂P is $(k-1)$-dimensional or empty.
3) It is empty if and only if every connected component of S is contained in an affine subspace that is contained in P.

Both statements remain valid, if we define S as the conventional intersection $T \cap P$, and claim the dimension of the intersections in (1) and (2) to be greater or equal than what we have in the stable case. The proof of this refinement follows the same lines as the proof of Proposition 3.

We now explain in what sense it is the tropical version of purity results for bifurcation sets over $\mathbb{C}$. Let T be a p-dimensional tropical complex in $\mathbb{R}^q \times \mathbb{R}^p$. A point $x \in T$ is said to be *regular* for the projection $T \to \mathbb{R}^p$, if a generic fiber of this projection has at most one point in a small neighborhood of x. The *tropical Jelonek set* of the projection $T \to \mathbb{R}^p$ is the set of images of all points $x \in T$ that are not regular for this projection. A p-dimensional tropical complex $T \subset \mathbb{R}^{p+q}$ is said to be *regular*, if every its point x admints a projection $\mathbb{R}^{p+q} \to \mathbb{R}^p$, for which the point x is regular.

Conjecture 1 (Tropical Jelonek theorem) If a p-dimensional tropical complex is regular, then the tropical Jelonek set of every its projection to $\mathbb{R}^p$ is a polyhedral (not necessarily tropical) complex of pure codimension 1.

Let P be a convex polyhedron in $\mathbb{R}^n$, represent it as $\{x \mid h(x) = c\}$, where $c \in \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous piecewise linear function, whose restriction to every ray from the origin is linear. Let $T \in \mathbb{R}^1 \times \mathbb{R}^n$ be the corner locus of the function $\max(|y|, h - c)$ on $\mathbb{R}^1 \times \mathbb{R}^n$, where y is the standard coordinate on $\mathbb{R}^1$. Then the tropical Jelonek theorem for the projection $T \to \mathbb{R}^n$ is exactly Proposition 3.

References

[ALM00] Artal Bartolo, E.; Luengo, I.; Melle Hernández, A.; Milnor number at infinity, topology and Newton boundary of a polynomial function. Math. Z. 233 (2000), no. 4, 679–696.

[Br88] S. Broughton; Milnor numbers and the topology of polynomial hypersurfaces. Inv. Math. 92 (1988) 217–241.

[CC07] R. Curran, E. Cattani; Restriction of A-Discriminants and Dual Defect Toric Varieties. J. Symb. Comput. 42 (2007), 115–135.

[D06] S. Di Rocco; Projective duality of toric manifolds and defect polytopes. Proc. of the London Math. Soc. (3) 93 (2006), no. 1, 85–104.

[E10] A. Esterov; Newton polyhedra of discriminants of projections. Discrete Comput. Geom., 44 (2010), no. 1, 96–148, arXiv:0810.4996.

[Gaf99] T. Gaffney, Fibers of polynomial mappings at infinity and a generalized Malgrange condition, Compositio Math. 119 (1999) 157–167.

[GKZ94] I. M. Gelfand, M. M. Kapranov, A.V.Zelevinsky; Discriminants, Resultants, and Miltidimensional Determinants. Birkhäuser, 1994.

[J93] Z. Jelonek; The set of points at which the polynomial mapping is not proper. Ann. Polon. Math. 58, 259–266 (1993) MR1244397 (94i:14018)

[J03] Jelonek, Z.; On bifurcation points of a complex polynomial. Proc. Amer. Math. Soc. 131 (2003), no. 5, 1361–1367.

[J03'] Jelonek, Z.; On the generalized critical values of a polynomial mapping. Manuscripta Math. 110 (2003), no. 2, 145–157.

[JK03] Jelonek, Z., Kurdyka, K.; On asymptotic critical values of a complex polynomial. J. Reine Angew. Math. 565 (2003), 1–11.

[Kh77] A. G. Khovanskii, Newton polyhedra and the genus of complete intersections. Func. Anal. Appl., 12 (1978), 38–46.

[K76] A. G. Kouchnirenko; Polyédres de Newton et nombres de Milnor. Inv. Math. 32(1) (1976),. 1–32.

[MT] Y. Matsui K. Takeuchi; A geometric degree formula for A-discriminants and Euler obstructions of toric varieties. Adv. Math. 226 (2011), 2040–2064.

[NZ90] Némethi, A., Zaharia, A.; On the bifurcation set of a polynomial function and Newton boundary. Publ. Res. Inst. Math. Sci. 26 (1990), no. 4, 681–689.

[OT95] L. Thanh, M. Oka; Estimation of the number of the critical values at infinity of a polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$. Publ. Res. Inst. Math. Sci. 31 (1995), no. 4, 577–598.

[Par95] Parusinski, A.; On the bifurcation set of complex polynomial with isolated singularities at infinity. Compositio Math. 97 (1995), no. 3, 369–384.

[SieTib95] D. Siersma, M. Tibar; Singularities at infinity and their vanishing cycles. Duke Math. Journal 80, 3 (1995), 771–783.

[St94] B. Sturmfels; On the Newton polytope of the resultant. J. Algebraic Combin. 3 (1994), no. 2, 207–236.

[HL84] Ha Huy Vui, Lê Dũng Tráng; Sur la topologie des polynômes complexes. Acta Math. Vietnam. 9 (1984), no. 1, 21–32 (1985).

[HT08] Ha Huy Vui, Nguyen Tat Thang; On the topology of polynomial functions on algebraic surfaces in $\mathbb{C}^n$. Singularities II, 61–67, Contemp. Math., 475, Amer. Math. Soc., Providence, RI, 2008.

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Complexity of tropical and min-plus linear prevarieties

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Abstract In the tropical algebra, a vector x is a solution to a polynomial $g_1(x) \oplus g_2(x) \oplus \ldots \oplus g_k(x)$, where $g_i(x)$'s are tropical monomials, if the minimum in $\min_i(g_i(x))$ is attained at least twice. In the min-plus algebra, solutions of systems of equations of the form $g_1(x) \oplus \ldots \oplus g_k(x) = h_1(x) \oplus \ldots \oplus h_l(x)$ are studied. In this paper we consider computational problems related to tropical linear system. We show that the solvability problem (both over $\mathbb{Z}$ and $\mathbb{Z} \cup \{\infty\}$) and the problem of deciding the equivalence of two linear systems (both over $\mathbb{Z}$ and $\mathbb{Z} \cup \{\infty\}$) are equivalent under polynomial-time reductions to mean payoff games and also equivalent to analogous problems in min-plus algebra. In particular, all these problems belong to the complexity class $\mathsf{NP} \cap \mathsf{coNP}$. We also show that computing the dimension of the solution space of a tropical linear system and of a min-plus linear system is NP-complete. We extend some of our results to the systems of min-plus linear inequalities.

1. Introduction

Let K be either $\mathbb{Z}$, or $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. By the *tropical linear system* associated with a matrix $A \in K^{m \times n}$ we understand the system of expressions $\min_{1 \le j \le n}\{a_{ij} + x_j\}$, $1 \le i \le m$, or, in other words, the vector $A \odot x$ for $x = (x_1, \ldots, x_n)$. We say that $x \neq (\infty, \ldots, \infty)$ is a solution to the tropical linear system if for every row $1 \le i \le m$ there are two columns $1 \le k < l \le n$ such that $a_{ik} + x_k = a_{il} + x_l = \min_{1 \le j \le n}\{a_{ij} + x_j\}$. Following the notation of [7], the set of solutions of a tropical linear system is called *tropical linear prevariety.* It follows from the analysis of [7] that this set is a union of polyhedra of possibly different dimensions. The *dimension* of a tropical prevariety is the largest dimension of polyhedra contained in it.

Min-plus linear system associated with a pair of matrices $A, B \in K^{m \times n}$ is the system $\min_{1 \le j \le n}\{a_{ij} + x_j\} = \min_{1 \le j \le n}\{b_{ij} + x_j\},\ 1 \le i \le m$.

Min-plus linear system of inequalities associated with a pair of matrices $A, B \in K^{m \times n}$ is the system $\min_{1 \le j \le n}\{a_{ij} + x_j\} \le \min_{1 \le j \le n}\{b_{ij} + x_j\},\ 1 \le i \le m$.

In this paper we are interested in computational aspects of tropical and min-plus algebras. The most basic motivation comes from linear algebra and, more specifically, systems of linear equations. In the case of classical algebra, the Gaussian elimination solves linear systems in polynomial time. In the case of tropical semiring, things turn out to be more complicated and no polynomial time algorithm is known neither for tropical linear systems, nor for min-plus linear systems. For the tropical case it is known, however, that the problem is in the complexity class $\mathsf{NP} \cap \mathsf{coNP}$, there are also pseudopolynomial algorithms [1,4], i.e., with complexity being polynomial in the size of a system and in absolute values of its coefficients, and it is also known that the problem reduces to the well known and long standing problem *mean payoff games* [1]. The same is known for the solvability problem for min-plus linear systems, and in addition it was proven by Bezem et al. [2] that the problem is polynomial-time equivalent to mean payoff games.

Another complexity aspect of min-plus algebra related to our consideration is the solvability problem of min-plus systems of linear inequalities. It is known that the solvability problem for these systems is equivalent to mean payoff games [1].

The first result of our paper is that the solvability problem for tropical linear systems is also equivalent to mean payoff games. Thus on one hand we characterize the complexity of solvability problem of tropical linear systems and on the other hand give a new reformulation of mean payoff games. In particular, our result means that the solvability problem for tropical linear systems is equivalent to the solvability problem for min-plus linear systems, establishing a closer connection between two problems of linear algebra over the min-plus semiring.

Next we study other problems related to tropical linear systems: the problem of equivalence of two given tropical linear systems and the problem of computing the dimension of a tropical prevariety. The former problem turns out to be also equivalent to mean payoff games. The same statement for min-plus linear systems is also true and follows easily from known results. The dimension problem of the tropical prevariety turns out to be NP-complete. We prove the analogous result for the case of min-plus linear systems and min-plus systems of inequalities.

These results are obtained for both $\mathbb{Z}$ and $\mathbb{Z}_\infty$ domains (there is no obvious translation between these two cases).

The proofs of our results can be found on arXiv [5].

2. Preliminaries

Next we recall the definition of mean payoff games. In an instance of a mean payoff game we are given a directed graph $G = (V, E)$, whose vertices are divided into two disjoint sets $V = V_1 \sqcup V_2$, some fixed initial node $v \in V_1$ and a function $w\colon E \to \mathbb{Z}$ assigning weights to the edges of G. In the beginning of the game a token is placed in the initial vertex v. At each turn of the game, one of the two players moves the token to another node of the graph. If the token is currently in some node $u \in V_1$, then the first player can move it to any node w such that $(u, w) \in E$. If $u \in V_2$, then the second player can move the token to any node w such that $(u, w) \in E$. The game is infinite and the process of the game can be described by the sequence of nodes $v_0, v_1, v_2, \dots$ which the token visits. Note that $v_0 = v$. The first player wins the game if

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(v_{i-1}, v_i) > 0. \quad (1)$$

The mean payoff game problem is to decide whether the first player has a winning strategy. For more information on mean payoff games see survey [6].

In this paper we consider the following problems.

TropSolv. Given an integer matrix $A \in \mathbb{Z}^{m\times n}$, decide whether the corresponding tropical system is solvable.

TropEquiv. Given two integer matrices $A \in \mathbb{Z}^{m\times n}$, $B \in \mathbb{Z}^{k\times n}$, decide whether the corresponding tropical systems over the same set of variables are equivalent.

TropDim. Given an integer matrix $A \in \mathbb{Z}^{m\times n}$ and a number $k \in \mathbb{N}$, decide whether the dimension of the tropical prevariety corresponding to the tropical system is at least k.

For all problems above there are also variants of them over $\mathbb{Z}_\infty$, we denote them by the subscript ∞. For local dimension of tropical prevariety (that is the dimension of the neighborhood of some point) over $\mathbb{Z}_\infty$ in a point with some infinite coordinates we consider just the dimension over finite coordinates only.

3. Tropical linear systems and mean payoff games

First we prove the following theorem.

Theorem 1 *The problems* TROPSOLV *and* TROPSOLV_∞ *are polynomially equivalent to mean payoff games.*

In particular, it follows that the problems TROPSOLV and TROPSOLV_∞ are polynomial time equivalent to each other. The resulting proof of equivalence of these two purely tropical problems rather unnaturally goes through mean payoff games. We also give a direct proof of this equivalence.

Next we prove a result on equivalence problem.

Theorem 2 *The problems* TROPEQUIV*,* TROPEQUIV_∞ *are polynomial time equivalent to mean payoff games.*

Analogous results for min-plus linear systems follow from known results [1,2].

For both min-plus and tropical linear systems we also give direct combinatorial proofs of equivalence between solvability and equivalence problems.

4. Dimension and the tropical rank

Now we proceed to the dimension of tropical prevarieties. First we study the relation to ranks of the matrices. There are many notions of "rank" in tropical algebra. For instance, Develin et al. [3] studied *Barvinok rank*, *Kapranov rank* and *tropical rank*. For them there is a relation

$$\textit{tropical rank}(A) \leq \textit{Kapranov rank}(A) \leq \textit{Barvinok rank}(A), \qquad (2)$$

for any matrix A, and all inequalities can be strict in (2) [3].

We show the following result.

Lemma 1 *For any matrix* $A \in \mathbf{R}^{m\times n}$ *we have*

$$n - \textit{tropical dimension}(A) \leq \textit{tropical rank}(A),$$

and the inequality can be both tight and strict.

This lemma together with (2) shows that there is a relation between the tropical dimension and ranks of the tropical matrix, but this relation is not enough for computational needs.

5. Combinatorial characterization of the dimension of the tropical prevariety

For our characterization we will need the following definition.

Definition 1 Let A be a matrix of size $m \times n$. We associate with it a table of stars A^* of the same size $m \times n$, where we put $*$ to the entry (i, j) iff $a_{ij} = \min_k a_{ik}$ and we leave all other entries empty.

Table A^* captures properties of the tropical system A essential to us. For example, the vector $x = (x_1, \ldots, x_n)$ is a solution to the system A iff there are at least two stars in every row of the table $(\{a_{ij} + x_j\}_{ij})^*$.

Next we give a combinatorial characterization of local dimension (at a given point) of a tropical prevariety in terms of the table A^*.

Definition 2 The block triangular form of size d of the matrix A is a partition of the set of rows of A into sets $R_1, R_2, \ldots, R_d$ (some of the sets R_i might be empty) and a partition of the set of columns of A into nonempty sets $C_1, \ldots, C_d$ with the following properties:

1. for every i each row in R_i has at least two stars in columns C_i in A^*;
2. if $1 \leq i < j \leq d$ then rows in R_i have no stars in columns C_j in A^*.

Theorem 3 *For a solution x of the tropical linear system A the local dimension of the system A in point x is equal to the maximal d such that there is a block triangular form of the matrix $\{a_{ij} + x_j\}_{ij}$ of size d.*

The analogous fact is true for the tropical linear systems over $\mathbb{Z}_\infty$.

We also prove analogous results for min-plus linear systems of equations and inequalities adapting the notion of block-triangular form properly.

6. Computing the dimension of tropical and min-plus linear prevarieties is NP-complete

Theorem 4 TropDim *and* TropDim$_\infty$ *are* NP*-complete.*

We prove analogous results for min-plus linear systems of equations and inequalities.

Acknowledgements. The first author is grateful to Max-Planck Institut für Mathematik, Bonn for its hospitality during the work on this paper.

The question on the complexity of equivalence of min-plus linear prevarieties was posed by Vladimir Voevodsky, who encouraged the authors to study relations between tropical and min-plus linear prevarieties.

References

1. M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. *International Journal of Algebra and Computation*, 22(1), 2012.
2. M. Bezem, R. Nieuwenhuis, and E. Rodríguez-Carbonell. Hard problems in max-algebra, control theory, hypergraphs and other areas. *Inf. Process. Lett.*, 110(4):133–138, 2010.
3. M. Develin, F. Santos, and B. Sturmfels. On the rank of a tropical matrix. *Combinatorial and Computational Geometry*, 52:213–242, 2005.
4. D. Grigoriev. Complexity of solving tropical linear systems. *Preprint MPI für Mathematik, Bonn*, 2010-60, 2010. To appear in Computational Complexity.
5. D. Grigoriev and V. V. Podolskii. Complexity of tropical and min-plus linear prevarieties. *CoRR*, abs/1204.4578, 2012.
6. H. Klauck. Algorithms for parity games. In E. Gradel, W. Thomas, and T. Wilke, editors, *Automata Logics, and Infinite Games*, volume 2500 of *Lecture Notes in Computer Science*, pages 553–563. Springer Berlin / Heidelberg, 2002.
7. J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. *Idempotent Mathematics and Mathematical Physics, Contemporary Mathematics*, 377:289–317, 2003.

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The asymptotic rank of semi-groups of tropical matrices

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Abstract As it is now well known, there are several notions of rank of a tropical matrix, corresponding to several equivalent definitions of the rank in usual linear algebra. However, the minimum of the ranks of matrices in a closed semigroup does not depend on the choosen notion. Let us call asymptotic rank of a semigroup be the minimum rank of elements in its closure. The asymptotic rank of a matrix is the asymptotic rank of the semigroup of its powers. We give a polynomial algorithm to check if the asymptotic rank of a finitely generated semigroup of matrices is the size d of the matrices. As a byproduct, the algorithm produces a product of at most d generators whose asymptotic rank if less than d whenever the asymptotic rank of the semigroup is less than d. On the other hand, the asymptotic rank is 0 iff the set of generators is mortal, and mortality is known to be an NP-hard problem.

1 Introduction

Tropical mathematics is a mathematics carried out over idempotent semirings, in particular over the ***tropical semiring*** $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$, the real numbers together with formal element $-\infty$, equipped with the operations of maximum and summation

$$a + b := \max\{a, b\} \qquad a \cdot b := a \underset{\text{sum}}{+} b,$$

addition and multiplication respectively. The unit element of the semiring $\mathbb{T}$ is 0 while $-\infty$ serves as the zero element of $\mathbb{T}$. We equip $\mathbb{T}^\times := \mathbb{R}$ with the Euclidean topology, and assume that $\mathbb{T}$ is homeomorphic to $[0, \infty)$.

We use the standard algebraic notation $\cdot$ and $+$ for the semiring operations, addition and multiplication respectively. Accordingly, a^m means the tropical product $a \cdots a$ with a repeated m times. In the usual way, for short, we write ab for the product $a \cdot b$. Note also that x^{-1} stands for $-x \in \mathbb{R}$.

Recalling that $(\mathbb{T}, + \ , \cdot\ )$ is a semiring, then in the usual way, we have the multiplicative monoid $M_d(\mathbb{T})$ of $n \times n$ matrices with entries in $\mathbb{T}$, whose multiplication is induced from the operations of $\mathbb{T}$ as in the familiar matrix construction. The ***unit*** element I of $M_d(\mathbb{T})$, is the matrix with 0 on the main diagonal and whose off-diagonal entries are $-\infty$; the ***zero*** matrix is $Z = (-\infty)I$. Therefore, the monoid $M_d(\mathbb{T})$ by itself is also a semiring. The entries of a matrix A, are denoted by A_{ij}.

As it is now well known, there are several notions of rank of a tropical matrix, that do not coincide as in the usual algebra. A complete survey can be found in [2].

As was already noticed by several authors ([1], [9]) in some nice cases those notions do coincide. Here, we prove that it is the case for the limit point of the projective powers of a matrix A: Theorem 1 gives a formula for this common rank that only depends of the critical graph of the iterated matrix.

This common rank is called the ***asymptotic rank*** of Matrix A and denoted $\operatorname{asrk}(A)$. It is the minimum of the rank of matrices in the closed semigroup generated by A.

The main aim of the present work is to generalize this Formula to a finitely generated semigroup, whose ***asymptotic rank*** is defined to be minimum of the rank of matrices in the closure of the semigroup.

Before stating precise results, let us recall some necessary background from max-plus theory.

The ***weighted digraph*** $\mathcal{G}(A) := (V, E)$ associated with a $d \times d$ matrix A, is defined to have node set $V = \{1, \dots, d\}$, and an arc (i, j) from i to j (of ***weight*** $w((i, j)) = A_{i,j}$) whenever $A_{i,j} \neq -\infty$. In this view, reordering of rows and columns of A is equivalent to relabeling of vertices of $\mathcal{G}(A)$.

A ***walk*** is a sequence of arcs $(i_1, j_1), \dots, (i_m, j_m)$, with $j_k = i_{k+1}$ for every $k = 1, \dots, m-1$. The ***length*** of a walk γ is the number of its arcs. The ***weight*** of a walk γ is defined to be the tropical product of the weights of all the arcs (i_k, j_k) composing γ, counting repeated arcs. Its ***average weight*** is its weight devided by its length.

The ***max-plus spectral radius*** $\rho(A)$ of a matrix A is the maximum of the average weights of the eelementary circuits.

A walk is ***elementary*** if each vertex appears at most once. A walk that starts and ends at the same vertex is called a ***circuit***; an arc $\rho := (i, i)$ is called a ***self-loop***, or ***loop*** for short.

A directed graph is called ***strongly connected*** if there is a walk from each vertex in the graph to every other vertex. The maximal strongly connected subgraphs of a given graph are called its ***strongly connected components*** .

The ***tropical determinant*** is defined to be the permanent

$$\mathrm{Perm}(A) := \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}, \tag{1}$$

where S_n is the set of all the permutations on $N = \{1, \ldots, n\}$.

A matrix $A \in M_d(\mathbb{T})$ is said to be ***singular*** if there exist at least two different permutations that attain simultaneously the evaluation of $\mathrm{Perm}(A)$, that is

$$\mathrm{Perm}(A) = a_{1\sigma(1)} \cdots a_{n\sigma(n)} = a_{1\tau(1)} \cdots a_{n\tau(n)},$$

for some $\sigma \neq \tau$ in S_n, otherwise A is called ***nonsingular*** matrix. If A is nonsingular, we denote by τ_A the maximizing permutation.
The term nonsingular is also known in the early literature as ***strongly regular*** [4]. A square matrix $A \in M_d(\mathbb{T})$ is nonsingular iff it has tropical rank d.

1.1 Statements

As we already mentioned, there are several notions of rank of a tropical matrix, that do not coincide as in the usual algebra. A complete survey can be found in [2]. Here, we only need to know that the smallest rank is the so called ***tropical rank***, defined as the size of the largest nonsingular minor, and that the largest ranks are the ***row and column ranks***, defined as the smallest number of generators of the tropical span of the rows (resp. columns) of the matrix. (see [2][Theorem 8.6]) Moreover, we know that the sequence of ranks of $(A^n)_{n\in\mathbb{N}}$ are nonincreasing for those three notions.

Theorem 1 (Formula for the asymptotic rank) *For any notion of rank, the asymptotic rank of the semigroup generated by a matrix* $A \in M_d(\mathbb{T})$ *is the* *sum of the cyclicities of the strongly connected components of the critical graph of A.*

If all strongly connected components of $\mathcal{G}(A)$ intersect $\mathcal{G}_c(A)$, then the asymptotic rank is the ultimate rank.

The semigroup generated by A thus has asymptotic rank d iff the critical graph of A is a union of disjoint elementary circuits containing all nodes of $\mathcal{G}(A)$. Let us notice that such graphs are exactly the graph of permutations of the set V of vertices of $\mathcal{G}(A)$, the strongly connected components being the cycles of the permutation.

As a corollary, we have

Corollary 1 (Asymptotic rank, one generator) *Let A be a matrix in $M_d(\mathbb{T})$. The semigroup generated by A has asymptotic rank d iff the critical graph of A is the graph of a permutation. That being the case, $\rho(A) = \frac{1}{d}\operatorname{Perm}(A)$ and $\mathcal{G}_c(A)$ is the graph of τ_A.*

Theorem 2 (Asymptotic rank, finitely many generators) *Let S be a finite set of matrices in $M_d(\mathbb{T})$, all with asymptotic rank d and set $M = \max_{A\in S}\rho(A)^{-1}A$. The asymptotic rank of the semigroup generated by S is d iff*

1. *$\rho(M) = 0$*
2. *for any $A \in S$ and any arc (i,j) of $\mathcal{G}_c(M)$, if $A_{ij} = M_{ij}$, then (i,j) is an arc of $\mathcal{G}_c(A)$.*

2 Semigroup of powers

In this section, we prove Theorem 1. It mainly follows from the so called *ultimate expansion* of [8][Theorem 5.6], which state that for n large enough, we have :

$$A^n = \max_{1\le i\le k} \rho_i^n C_i S_i^n R_i$$

where the matrices C_i, S_i, R_i are build inductively so that

1. $\rho(A) = \rho_1 > \rho_2 \cdots \rho_k$,
2. all $(C_i S_i^n R_i)$ are periodic,
3. S_1 is the restriction of A to $\mathcal{G}_c(A)$
4. the columns of C_1 (resp. the rows of R_1) are the columns (resp. the rows) of matrix $((\rho(A)^{-1}A)^\gamma)^*$ with indices in $\mathcal{G}_c(A)$, where γ is the cyclicity of $\mathcal{G}_c(A)$ and $*$ denotes the Kleene star operation.
5. the finite entries of $C_i S_i^n R_i$ for $i > 1$ correspond to $-\infty$-entries of $C_1 S_1^n R_1$.

Moreover, the ultimate expansion has only one term if all strongly connected components of $\mathcal{G}(A)$ intersect $\mathcal{G}_c(A)$ and this ensures the last statement of Theorem 1.

C_1, S_1 and R_1 will be called just C, S and R. We see that $\rho(A)^{-\gamma n-l}A^{\gamma n+l}$ tends to CS^lR and that the nonsingular minors of CS^lR are nonsingular minors of $A^{\gamma n+l}$ for large n. This implies that the asymptotic rank of A with respect to the tropical rank is exactly the minimum of the tropical ranks of the CS^lR. But the sequence of ranks of the CS^lR is both nonincreasing and periodic, so it is constant.

Let us show that the tropical rank of $CS^\gamma R$ is larger than the sum s of the cyclicities of the strongly connected components of $\mathcal{G}_c(A)$. First, let us notice that $P := CS^\gamma R_{ij}$ is the maximum of walks from i to j on $\mathcal{G}(\rho(A)^{-1}A)$ that contain a node of $\mathcal{G}_c(A)$. Specifically, $P_{ii} = 0$ for any $i \in \mathcal{G}_c(A)$. Now take a node of $\mathcal{G}_c(A)$ in each of its cyclicity classes (the strongly connected components of $\mathcal{G}_c(A^\gamma)$). The set of those nodes has size s and the minor it defines is nonsingular by definition of $\mathcal{G}_c(A)$ and γ.

To complete the proof, let us show that the column rank of $CS^\gamma R$ is less than s. This follows from the max-plus spectral theory applied to A^γ. See for instance [5] that say that the eigenspace of A^γ is the span of the columns of C and that the size of its generating sets is the number of strongly connected components of A^γ, that is s. The row rank is of course the same because $\mathcal{G}_c(A)$ and $\mathcal{G}_c({}^tA)$ have the same strongly connected components .

3 Finitely generated semigroup

In this section, we prove Theorem 2 .

Let S be a finite set of $d \times d$ matrices with asymptotic rank d. Without loss of generality, we assume they have spectral radius 0 thus $M = \max_{A \in S} A$. We denote by $< S >$ the semigroup generated by S.

By definition $\rho(M) \geq \rho(A) = 0$ and if $\rho(M) > 0$, then there is an elementary circuit $(i_1, \cdots, i_k, i_{k+1} = i_1)$ with positive weight w in $\mathcal{G}(M)$. For each l, we choose $A(l) \in S$ s.t. $A(l)_{i_l i_{l+1}} = M_{i_l i_{l+1}}$ and define $P := (A(1) \cdots A(k))$, so that $\rho(P) \geq P_{i_1, i_1} = w > 0$. Since the circuit is elementary, product P of matrices in S with positive spectral radius and this product can not have asymptotic rank d, according to Corollary 1 and the additivity of the permanent of non singular matrices.

Otherwise, $\rho(M) = 0$, and, since the critical graph of any matrix $A \in S$ contains all nodes of $\{1, \cdots, d\}$, so does $\mathcal{G}_c(M)$ and thus there is $V \in \mathbb{R}^n$ s.t. $V = MV \geq AV$ for any $A \in S$. Let us define $\overline{B}_{ij} := B_{ij} + V_j - V_i$, for any $B \in M_d(\mathbb{T})$ and remark a few things :

1. $\overline{S} = \{\overline{A} | A \in S\}$ is a set of nonpositive matrices.
2. If D denotes the diagonal matrix whose diagonal entries are those of V, then $\overline{B} = D^{-1}BD$, so that $\overline{AB} = \overline{A}\,\overline{B}$ and $< \overline{S} >= \overline{< S >}$.
3. Any circuit has the same weight with respect to B or to $\overline{B}$, so that $\rho(\overline{B}) = \rho(B)$, $\mathcal{G}_c(\overline{B}) = \mathcal{G}_c(B)$ and $\text{asrk}(\overline{B}) = \text{asrk}(B) = d$.
4. For any $A \in S$, since $\text{asrk}(\overline{A}) = d$, there is a unique permutation τ_A s.t. for any i, $\overline{A}_{i\tau_A(i)} = 0$. This implies that for any $P \in < S >$ there is a permutation τ (the product of the permutation τ_A of its factors) s.t. for any i, $\overline{P}_{i\tau_P(i)} = 0$.
5. Since any $P \in < \overline{S} >$ is nonpositive, it satisfies $\rho(P) = 0$ and its critical circuits are the ones with only zeros on their arcs. The circuits corresponding to orbits of τ_P are critical and $\text{asrk}(P) = d$ iff there are no other critical circuits.
6. For any $A \in S$, and any i, $(\overline{A}0)_i = \max_j \overline{A}_{i,j} \leq 0$ but $(\overline{A}0)_i \geq \overline{A}_{i,\tau_A(i)} = 0$, so that $\overline{A}0 = 0$. Equivalentely, $AV = V$.

On our way, we have proved the following.

Proposition 1 *Let S be a finite set of matrices in $M_d(\mathbb{T})$, all with asymptotic rank d and spectral radius* 0 *set $M = \max_{A \in S} A$. Then, we have*

1. *If $\rho(M) > 0$, then there exist a product P of at most d matrices in S with positive spectral radius and this product does not have asymptotic rank d.*
2. *Otherwise, $\rho(M) = 0$, and there is a finite vector V that is a common fixed point of all elements of S.*

Assume condition (2) of Theorem 2 is not satisfied. Then, there is an elementary critical circuit $(i_1, i_2, \cdots i_k, i_{k+1} = i_1)$ in $\mathcal{G}(M)$, and a $B \in \overline{S}$ with $(i_1, i_2) = (a, b)$ such that $B_{ab} = M_{ab}$, that is $\overline{B}_{ab} = 0$, but $b \neq \tau_B(a)$. For each l, we choose $A(l) \in S$ s.t. $A(l)_{i_l i_{l+1}} = M_{i_l i_{l+1}}$ and define $P := (A(2) \cdots A(k))$. By definition $B_{ab} + P_{ba} = 0$, i.e. $B_{ab} = P_{ba} = 0$.

Thus, there is a loop on b in the critical graph of PB. Let us prove there is another circuit going through b.

Let c be $\tau_B(a)$. By definition of a, c is different from b and $B_{ac} = 0$ so that the arc from b to c in the graph of PB has weight 0.

On the other hand, there is a $k \geq 1$ s.t. $(\tau_B \tau_P)^k(c) = c$, so that $\tau_P(\tau_B \tau_P)^{k-1}(c) = \tau_B^{-1}(c) = a$, $(P(BP)^{k-1})_{ca} = 0 = B_{ab}$ and there is a circuit from c to b in the graph of PB with only arcs of weight 0. Finally, we built a product PB of at most d matrices, whose critical graph contains two different circuits going through node b : the loop on b and one circuit going through c, thus $\mathrm{asrk}(PB) < n$.

Conversely, assume condition (2) is satisfied and take $A(1), \cdots, A(k) \in \overline{S}$, $P = A(1) \cdots A(k)$. Let $c = i_1 \cdots i_t$ be a critical circuit of $\mathcal{G}(P)$. Each arc of c has weight 0 relatively to $\overline{P}$ and for any $m \leq t$, there is a walk $p_m = i_1^m \cdots i_{k+1}^m$ from i_m to i_{m+1} s.t. for any p, $\overline{A(p)}_{i_p^m i_{p+1}^m} = 0$. In particular every arc (i_p^m, i_{p+1}^m) is in the graph of $\max_{A \in S} \overline{A} = \overline{M}$ which has the same arcs as $\mathcal{G}(M)$. The concatenation of the p_m is a closed circuit on $\mathcal{G}(M)$ with weight 0. It can be split into critical circuits. The condition implies that all arcs on those circuits are critical w.r.t the matrices that give them the same weight as M. This means that for any m and p, (i_p^m, i_{p+1}^m) is in $\mathcal{G}_c(A(p))$, that is $i_{p+1}^m = \tau_{A(p)}(i_p^m)$. By composition, $i_{m+1} = \tau_P(i_m)$ and c is a circuit of τ_P. Theorem 1 ensures that $\mathrm{asrk}(P) = n$.

At this stage, we proved that the conditions (1) and (2) of Theorem 2 imply that all elements in $< S >$ have asymptotic rank d. It does not automatically means that $\mathrm{asrk}(< S >) = d$, because there could be a limit of elements of $< S >$ along a sequence that is not a sequence of powers and that converges to a matrix with lower rank. There is no such sequence, because $< S >$ is projectively finite. This follows from the Max-Plus Burnside Theorem [6] applied to $< \overline{S} >$.

4 Complexity

First, let us recall that computing the tropical rank of a square matrix $A \in M_d(\mathbb{T})$ is NP-hard [7], but checking if this rank is d is polynomial.

The computability of this asymptotic rank is still an open question. (Except if we assume that the entries are finite and rational, in which case the semigroup is projectively bounded, see [6]). Moreover, checking if the asymptotic rank is 0 means checking if the semigroup contains the zero matrix and this is an NP-hard problem known as mortality problem (see [3]). On the other hand Theorem 2 implies that it can be checked in a polynomial number of operations if the asymptotic rank is d.

References

1. M. Akian, R. Bapat, and S. Gaubert. *Handbook of Linear Algebra*, chapter Max-plus algebra. Chapman and Hall, 2009.

2. Marianne Akian, Stephane Gaubert, and Alexander Guterman. Linear independence over tropical semirings and beyond. In G.L. Litvinov and S.N. Sergeev, editors, *Proceedings of the International Conference on Tropical and Idempotent Mathematics*, number 495 in Contemporary Mathematics. AMS, 2009.
3. V. D. Blondel, S. Gaubert, and J. N. Tsitsiklis. Approximating the spectral radius of sets of matrices in the max-algebra is NP-hard. *IEEE Trans. Automat. Control*, 45(9):1762–1765, 2000.
4. Peter Butkovič. Strong regularity of matrices—a survey of results. *Discrete Appl. Math.*, 48(1):45–68, 1994.
5. P. Butkovič, R. A. Cuninghame-Green, and S. Gaubert. Reducible spectral theory with applications to the robustness of matrices in max-algebra. *SIAM J. Matrix Anal. Appl.*, 31(3):1412–1431, December 2009.
6. S. Gaubert. On the Burnside problem for semigroups of matrices in the (max, +) algebra. *Semigroup forum*, 52:271–292, 1996.
7. Ki Hang Kim and Fred W. Roush. Factorization of polynomials in one variable over the tropical semiring. Technical report, arXiv, 2005.
8. Sergei Sergeev and Hans Schneider. CSR expansions of matrix powers in max algebra. *Trans. Amer. Math. Soc.*, 2012.
9. Marianne Johnson, Zur Izhakian, and Mark Kambites. Pure dimension and projectivity of tropical polytopes. Technical report, arXiv, 2011.

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Time slicing approximation and Stationary phase method for Path integral with Brownian-bridge-type action.

O.V. Gulinsky

R.P.Feynman proposed a heuristic (Lagrangian) formulation of quantum mechanics (see [1]) by expressing the fundamental solution for the Schrödinger equation with the help of the path integral with path phaser $\exp\{\imath\nu S(\gamma)\}$, where $S(\gamma) = \int L(t, \dot{\gamma}(t), \gamma(t))dt$ is an action with a classical Lagrangian $L(t, \dot{\gamma}(t), \gamma(t))$, weighting a path γ. His approach to quantum mechanics is based on the hypothesis that all possible paths of a particle moving from a source to a detector should be considered as equally probable. The nearly classical paths are not weighted more heavily than paths that are far from classical, that is the different individual paths in the summation do not have different amplitude. As a consequence the integral in finite dimensional time slicing approximation is defined w.r.t the product Lebesgue measure. This fact induces serious difficulties for mathematically rigorous justification of the integral. R.Feynman suggested that the path integral can be considered as integration over complex-valued "measure" $\exp\{\imath\nu S(\gamma)\}\mathcal{D}[\gamma]$, however it was proved [2] that it is not the case. Nevertheless for some special cases and classes of integrable functions there are several rigorous approaches to the problem (see e.g. [3], [4], [5], [6], [7], [8], [17]).

In this paper we try to provide a well defined stochastic model, based on a "measurement" (interaction) scheme, that permits to restrict the domain of integration. The model leads to the standard integration (mathematical expectation instead of oscillatory integrals in [5], [6], [7], [8], [17]) over stochastic trajectories which are asymptotically close in distributions to the trajectories of a Brownian bridge (hence Gaussian). It means that the phase function $S(\gamma)$ in $\exp\{\imath\nu S(\gamma)\}$

transforms to a *complex-valued* phase function (see (22)) with the classical action $S(\gamma)$ as real part and a quadratic form, defined (in finite dimension) by (16), as imaginary part. (We do not pretend on a physical interpretation of the scheme. For details about the principle of quantum measurements consult [9]. In [10] stochastic Schrödinger equations or Belavkin equations are obtained based on a quantum filtering theory. See also Remark 1 below and the reference therein)

An important feature of our approach is the possibility of applying the stationary phase method to obtain a semiclassical approximation of the integral. The result (see (23)) reads as follows: similar to a standard stationary phase approximation, the phaser (as envisaged by R.Feynman) is determined by the classical action evaluated along the classical path between the two endpoints (imaginary part of the phase function vanishes in the stationary point) while the amplitude prefactor is modified by the Brownian bridge distribution. As a consequence, a properly scaled semiclassical approximation can be considered as a regularization procedure for the estimation of the path integral. In this regard it is worth mentioning that there is a group of methods, popular in chemical community ("Thawed Gaussian Approximation", "Frozen Gaussian approximation", the Herman-Kluk expression) based on propagating semiclassical Gaussian wave packets with *complex-valued* phase function, i.e. approximate solution of Schrödinger equations which are sufficiently concentrated in space and in frequency around the classical Hamiltonian phase-space flow. However this flow (a family of canonical transformation of the classical phase space) is the solution of the Cauchy problem for Hamilton's equations. Therefore a correction of the classical action S in the exponent with respect to the endpoint is needed (a quadratic form as imaginary part in fact) and various form of prefactors (the Herman-Kluk is the best known) are used (see [12] for a rigorous formulation in terms of Fourier integral operators and references therein).

R.Feynman and F.Vernon [13](see also [1], [14]) generalized the path integral approach to studying the evolution of an open quantum system coupled with a quantum environment. This heuristic approach (called the Feynman-Vernon influence functional method) leads to double paths integral over forward and backward paths coupled through the influence functional. We mention here that it is also possible to extend our model, using *complex-valued* empirical process and Brownian bridge, to construct an analog of the Feynman-Vernon approach (not a mathematical justification, for a rigorous formulation of Feynman-Vernon method in the spirit of [6] see [15]).

1 Time slicing approximation

In constructing the time slicing approximation by piecewise classical paths we adopt some notations and results from [8], [17], [18]. Let

$$L(t,\dot{x},x)=\frac{1}{2}|\dot{x}|^2-V(t,x) \tag{1}$$

be the Lagrangian with a smooth time dependent potential $V(t,x)$ on the configuration space $\mathbb{R}^d$. A continuous map γ from the interval $[s,s']$ to $\mathbb{R}^d$ is the classical path if it is a solution with the boundary condition $\gamma(s)=y$ and $\gamma(s')=x$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$ of the Euler equation

$$\frac{d^2}{dt^2}\gamma(t)+\partial_x V(t,\gamma(t))=0. \tag{2}$$

The action $S(\gamma)$ of a path γ is the functional

$$S(\gamma)=\int_s^{s'} L\Big(t,\frac{d}{dt}\gamma(t),\gamma(t)\Big)dt. \tag{3}$$

If $\gamma^{cl}=\gamma^{cl}(t,x,y)$ is the classical path then $S(\gamma^{cl})$ is a function of (s',s,x,y). Let $\Delta: 0=T_0<T_1<\cdots<T_J<T_{J+1}=T$ be any division of the interval $[0,T]$, $t_j=T_j-T_{j-1}$, $|\Delta|=\max_j\{t_j\}$ and x_j, $j=0,1,\ldots,J,J+1$ be arbitrary points of the configuration space $\mathbb{R}^d$. Then *the piecewise classical* path $\gamma_\Delta=\gamma_\Delta(t,x_{J+1},x_J,\ldots,x_1,x_0)$ with vertices $(x_{J+1},x_J,\ldots,x_1,x_0)\in\mathbb{R}^{d(J+2)}$ is a broken path defined by the Euler equation

$$\frac{d^2}{dt^2}\gamma_\Delta(t)+\partial_x V(t,\gamma_\Delta(t))=0,\quad \text{for}\quad T_{j-1}<t<T_j, \tag{4}$$

and boundary conditions

$$\gamma_\Delta(T_j)=x_j,\quad \text{for}\quad j=0,1,\ldots,J,J+1. \tag{5}$$

For the simplicity of notations in what follows $d=1$. Throughout this paper we assume that for any nonnegative m there exits a positive constant υ_m such that

$$\max_{|\alpha|=m}\sup_{(t,x)\in[0,T]\times\mathbb{R}}|\partial_x^\alpha V(t,x)|\le\upsilon_m(1+x)^{\max(2-m,0)}. \tag{6}$$

Under this condition the solution γ^{cl} for (2) exists uniquely provided $|s-s'|\le\delta$ with sufficiently small δ (see e.g [18], Section 2). We always assume that $T\le\delta$ and δ is sufficiently small. The set $\Gamma(\Delta)$ of all piecewise classical paths γ_Δ associated with the division Δ makes a smooth manifold and the set of all piecewise classical paths is a dense subset of the Sobolev space $\mathbf{H}^1([0,T];\mathbb{R})$

(see [18]). Let $a(\gamma)$ be a functional defined on $\mathbf{H}^1([0,T];\mathbb{R})$ then the functional $a(\gamma_\Delta)$ is the restriction of $a(\gamma)$ to $\Gamma(\Delta)$ and can be written as a function $a_\Delta(x_{J+1}, x_j, \ldots, x_1, x_0)$. In particular, the action functional $S(\gamma_\Delta)$ is given by

$$S(\gamma_\Delta) = S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) = \int_0^T L\Big(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t)\Big)dt = \sum_{j=1}^{J+1} S_j(x_j, x_{j-1}), \tag{7}$$

where $S_j(x_j, x_{j-1}) = S_j(T_j, T_{j-1}, x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} L\Big(t, \frac{d}{dt}\gamma_\Delta(t), \gamma_\Delta(t)\Big)dt$.

Denote $\theta = (x_1, \ldots, x_J)$. A piecewise classical time slicing approximation to the Feynman path integral is defined by

$$I(\Delta; S, a, \nu)(x_{J+1}, x_0) = \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi\imath t_j}\right)^{1/2} \tilde{\int_{\mathbb{R}^J}} \exp\{\imath\nu S(\gamma_\Delta)\} a(\gamma_\Delta) \prod_{j=1}^{J} dx_j =$$
$$\prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi\imath t_j}\right)^{1/2} \tilde{\int_{\mathbb{R}^J}} e^{\imath\nu S_\Delta(x_{J+1},\theta,x_0)} a_\Delta(x_{J+1}, \theta, x_0) d\theta, \tag{8}$$

where $\nu = 2\pi h^{-1}$ with Planck's constant h. In general the integral in (8) does not converges absolutely and is treated as an oscillatory integral(see [19], [8]).

Let $\theta^* = \Big(x_J^*, \ldots, x_1^*\Big)$ be a critical point of the functional (7), that is θ^* is the solution for the system of equations with respect to $(x_J, \ldots, x_1)$:

$$\partial_{x_j}\Big(S_{j+1}(x_{j+1}, x_j) + S_j(x_j, x_{j-1})\Big) = 0, \quad \text{for} \quad j = 1, \ldots, J, \tag{9}$$

where x_0 and x_{J+1} are fixed. If $T < \delta$ then there exists a unique solution $\theta^*(x_{J+1}, x_0)$ satisfying (9) (see [16], Proposition 2.3 and [17], Lemma 5.1). Let $\gamma^*_\Delta = \gamma_\Delta(t, x_{J+1}, \theta^*, x_0)$ be the piecewise classical path corresponding to the critical point $\theta^*(x_{J+1}, x_0)$ and $S_\Delta(x_{J+1}, \theta^*, x_0) = S(\gamma^*_\Delta)$.

Remark 1 {see [8], Proposition 2.4} The orbit $\gamma_\Delta(t, x_{J+1}, \theta^*, x_0)$ coincides with the classical path $\gamma^{cl}(x_{J+1}, x_0)$ satisfying on $[0,T]$ the boundary condition $\gamma(0) = x_0$, $\gamma(T) = x_{J+1}$ and hence $S_\Delta(x_{J+1}, \theta^*, x_0) = S(\gamma^{cl}(x_{J+1}, x_0))$. We will use this property in our stochastic model setting.

To apply the stationary phase method in general case of an oscillatory integral (8) it is necessary to impose some restrictions on the integrand a_Δ. A natural class of integrable functions is the set of symbols $S^m_{\rho,\delta}$ (see [19]) . In [18] a more specific and complicated version of such class is proposed ("Assumption 1") and the following result is obtained (see also [17] and [16]):

For $T \leq \delta$, under assumption (6) and "Assumption 1"

$$I(\Delta; S, a, \nu)(x_{J+1}, x_0) = \left(\frac{\nu}{2\pi\imath T}\right)^{1/2} \exp\left\{\imath\nu S(\gamma^{cl}(x_{J+1}, x_0))\right\}$$

$$\times\left(\left[\frac{\prod_{j=1}^{J+1} t_j}{T} \det\Big(S''(x_{J+1}, \theta^*, x_0)\Big)\right]^{-\frac{1}{2}} a(\gamma^{cl}(x_{J+1}, x_0)) + \frac{T}{\nu} r(x_{J+1}, x_0)\right) \tag{10}$$

and for any K there exist positive constants A_K and $M(K)$ such that if $\alpha_0, \alpha_{J+1} \leq K$

$$|\partial^\alpha_{x_{J+1}} \partial^\beta_{x_0} r(x_{J+1}, x_0)| \leq C_{M(K)} A_K (1 + |x_{J+1}| + |x_0|)^m. \tag{11}$$

2 Empirical processes and Brownian Bridge

Let $\xi_1(\omega), \xi_2(\omega), \ldots$ be i.i.d random variables uniformly distributed on $[0, T]$. Define the empirical distribution

$$\hat{U}_n(t, \omega) = \frac{1}{n} \sum_{i=1}^n \chi(\xi_i(\omega) \leq t), \tag{12}$$

where χ is the indicator function. The sequence of the processes $(\hat{U}_n)$ converges pointwisely on $[0, T]$ to $U(t) = t$ with probability one by the Law of Large Numbers. Define the process

$$Y_n(t) = Y_n(t, \omega) = \frac{1}{n} \sum_{i=1}^n \chi(\xi_i(\omega) \leq t) - t = \hat{U}_n(t, \omega) - t. \tag{13}$$

The Glivenko-Cantelli theorem asserts that the uniform convergence takes place

$$\mathbb{P}\{\sup_{t\in[0,T]} |Y_n(t)| > \varepsilon\} \leq 8(n+1) \exp\{-\frac{n\varepsilon^2}{32}\} \tag{14}$$

and in particular with probability one $\lim_{n\to\infty} \sup_{t\in[0,T]} |Y_n(t)| = 0$. Let $F_n(t, y) = \mathbb{P}\{\omega : Y_n(t, \omega) \leq y\}$ be the distribution function of the random variable $Y_n(t, \omega)$. Note that the estimation (14) shows that the tail $1 - F_n(t, y)$ of the distribution $F_n(t, y)$ decreases exponentially fast.

Let $B(t)$ be a standard Brownian motion. The Brownian bridge $B^0(t)$ from 0 to 0 on [0,T] ($B^0(0) = B^0(T)$) is defined by

$$B^0(t) = B(t) - \frac{t}{T} B(T).$$

$B^0(t)$ is a Gaussian process with the mean function $m(t) = \mathbf{E}B^0(t) = \mathbf{E}[B(t) - \frac{t}{T}B(T)] = 0$, and the covariance function $c(s, t) = \mathbf{E}\Big[\Big(B(s) - \frac{s}{T}B(T)\Big)\Big(B(t) - \frac{t}{T}B(T)\Big)\Big] = \min(s, t) - \frac{st}{T}$.

Introduce the process

$$\zeta_n(t) = \sqrt{n}Y_n(t) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}\chi(\xi_i(\omega) \le t) - t\Big), \tag{15}$$

and denote $\Phi_n(t, y)$ the distribution function of the random variable $\zeta_n(t)$. The Donsker theorem (see [20]) stands that the sequence (ζ_n) converges in distribution to the Brownian bridge B^0 on $\mathrm{D}[0, T]$ equipped with the Skorohod topology.

For any division $\Delta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T$ the random variables $B^0(T_1), \ldots, B^0(T_J)$ are jointly normal with the density

$$f(x_1, \ldots, x_J) = \tag{16}$$

$$= \sqrt{\frac{T}{T - T_J}}\prod_{j=1}^{J}\frac{1}{\sqrt{2\pi(T_j - T_{j-1})}}\exp\Big\{-\frac{1}{2}\sum_{j=1}^{J}\frac{(x_j - x_{j-1})^2}{T_j - T_{j-1}} - \frac{x_J^2}{2(T - T_J)}\Big\}. \tag{17}$$

Let $F_n(y_1, \ldots, y_J)$ be the joint distribution function of random variables $Y_n(T_1), \ldots, Y_n(T_J)$. The joint distribution of random variables $\zeta_n(T_1), \ldots, \zeta_n(T_J)$ is denoted by $\Phi_n(y_1, \ldots, y_J)$. It is clear that the sequence (Φ_n) converges in distribution to the normal distribution with density (16).

3 Problem formulation

Let γ^{cl} be the classical path, defined by (2), with boundary conditions $\gamma^{cl}(T) = x_{J+1}$, $\gamma^{cl}(0) = x_0$. For a fixed division Δ, instead of the set of piecewise classical paths $\gamma_\Delta = \gamma_\Delta(t, x_{J+1}, x_J, \ldots, x_1, x_0)$ with arbitrary vertices $x_j \in \mathbb{R}, j = 0, 1, \ldots, J, J+1$ we consider (for a given realization $Y_n(t, \omega)$) piecewise classical paths γ_Δ, defined by (4),(5), with vertices

$$(x_{J+1}, \gamma^{cl}(T_1) + v(T_1)Y_n(T_J), \ldots, \gamma^{cl}(T_1) + v(T_1)Y_n(T_1), x_0)$$

where $v(t)$, $0 \le t \le T$ is a deterministic function. Let

$$S(\gamma_\Delta) = S_\Delta\Big(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)Y_n(T_J),\ \ldots, \gamma^{cl}(T_1) + v(T_1)Y_n(T_1), x_0\Big)\Big\}$$

be the action functional, defined by (7), and

$$a(\gamma_\Delta) = a_\Delta(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)Y_n(T_J), \ldots, \gamma^{cl}(T_1) + v(T_1)Y_n(T_1), x_0).$$

In this paper instead of the standard piecewise classical time slicing approximation to the Feynman path integral (8) we introduce and study the integral

$$\int_{\mathbb{R}^J} \exp\Big\{\imath n S_\Delta\Big(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)y_J, \ldots, \gamma^{cl}(T_1) + v(T_1)y_1, x_0\Big)\Big\}$$
$$\times a_\Delta(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)y_J, \ldots, \gamma^{cl}(T_1) + v(T_1)y_1, x_0\Big) dF_n(y_1, \ldots, y_J)$$
$$= \mathbf{E} a_\Delta(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)Y_n(T_J), \ldots, \gamma^{cl}(T_1) + v(T_1)Y_n(T_1), x_0)$$
$$\times \exp\Big\{\imath n S_\Delta\Big(x_{J+1}, \gamma^{cl}(T_J) + v(T_J)Y_n(T_J), \ldots, \gamma^{cl}(T_1) + v(T_1)Y_n(T_1), x_0\Big)\Big\} \tag{18}$$

Note that (18) is a standard integral which converges absolutely for a broad class of functions.

Remark 1 One can consider $\hat{U}_n(t, \omega)$ as an estimate of "quantum time" and $Y_n(t, \omega)$ as the deviation of the estimated time $\hat{U}_n(t, \omega)$ from the "physical time". The model postulates that the deviation $v(t)Y_n(t)$ from the classical path $\gamma^{cl}(t)$ in the action functional $S(x, u, z)$ at any point $0 < t < T$ is proportional to $Y_n(t)$ with a constant $v(t)$ and, up to the information coming from n observations, is determined by the distribution function $F_n(y_1, \ldots, y_J)$. (As mentioned above, we prefer to avoid interpretation debates. For the detailed discussion of a measurement model for quantum time based on the dynamical system-clock interaction we refer to [11], where conditions for a physically meaningful measurement operator were formulated and a realization of unsharp measurement is discussed.)

4 Main idea and result

For the sake of simplicity we explain the idea of the approach by fixing the simplest division $\Delta : 0 < T_1 < T_2 = T$ and $v(t) = 1$. Set $x_0 = z$, $x_2 = x$ and let γ^{cl} be the classical path with boundary conditions $\gamma^{cl}(T) = x$ and $\gamma^{cl}(0) = z$. Let γ_Δ be the *piecewise classical* path defined by (4),(5) with boundary conditions $\gamma_\Delta(T) = x$, $\gamma_\Delta(T_1) = \gamma^{cl}(T_1) + Y_n(T_1)$, $\gamma_\Delta(0) = z$, where $Y_n(t) = Y_n(t, \omega)$ defined by (13). Let

$$S(\gamma_\Delta) = S_\Delta(x, \gamma^{cl}(T_1) + Y_n(T_1), z) = S_2(x, \gamma^{cl}(T_1) + Y_n(T_1)) + S_1(\gamma^{cl}(T_1) + Y_n(T_1), z)$$

be the action functional defined by (7). The aim is to estimate the integral

$$\int_{\mathbb{R}} a_\Delta(x, \gamma^{cl}(T_1) + y, z) \exp\Big\{\imath n S_\Delta(x, \gamma^{cl}(T_1) + y, z)\Big\} dF_n(T_1, y). \tag{19}$$

Let $a_\Delta(\cdot, y, \cdot)$ be a symbol $S^m_{1,0}$ with some real m (see [19]). By the exponential inequality (14), the integral (19) converges absolutely. With the help of the centralized empirical process $\zeta_n(t)$ and its finite dimensional distribution $\Phi_n(y_1, \ldots, y_J)$, (19) can be written in the equivalent form

$$\int_{\mathbb{R}} a_\Delta\Big(x, \gamma^{cl}(T_1) + \frac{y}{\sqrt{n}}, z\Big) \exp\Big\{\imath n S_\Delta(x, \gamma^{cl}(T_1) + \frac{y}{\sqrt{n}}, z)\Big\} d\Phi_n(T_1, y). \quad (20)$$

Following the scheme of [21], we prove by applying Donsker's theorem, Edgeworth's expansion for CLT(see [22]) and the stationary phase estimate ([19], Theorem 7.7.5) that (20) can be replaced with the accuracy $O\Big(\frac{1}{\sqrt{n}}\Big)$ by

$$\frac{\sqrt{n}}{\sigma(T_1)\sqrt{2\pi}} \int_{\mathbb{R}} \varrho(u - \gamma^{cl}(T_1)) a_\Delta(x, u, z) \exp\Big\{\imath n S_\Delta(x, u, z)\Big\} \exp\Big\{ - \frac{n(u - \gamma^{cl}(T_1))^2}{2\sigma^2(T_1)}\Big\} du \quad (21)$$

where $0 \le \varrho(\cdot) \le 1$ is a smooth function with compact support $[-2, 2]$ and $\sigma^2(T_1) = \frac{T_1(T-T_1)}{T}$ is the variance of the Brownian bridge at $t = T_1$. Here $\Phi_n(T_1, y)$ is replaced by finite dimensional (J=1) distribution (16) of the Brownian bridge and the obvious change of variables is made. The function $\varrho(\cdot)$ can be introduced due to Glivenko-Cantelli estimate (14) and is needed to apply the stationary phase method. Let introduce the complex value function

$$f(x, u, z) = S_\Delta(x, u, z) + \imath\frac{(u - \gamma^{cl}(T_1))^2}{2\sigma^2(T_1)} = \Big(S_2(x, u) + S_1(u, z)\Big) + \imath\frac{(u - \gamma^{cl}(T_1))^2}{2\sigma^2(T_1)}. \quad (22)$$

Let $U^*(x, z)$ be the unique critical point of the functional $S_\Delta(x, u, z)$ (see (9) and the subsequent comments) and $S_\Delta(x, U^*(x, z), z)$ be the value of the functional at the critical point . Recall (see Remark 1) that the orbit $\gamma_\Delta(U^*(x, z))$ corresponding to the critical point $U^*(x, z)$ coincides with the classical path $\gamma^{cl}(x, z)$ satisfying on $[0, T]$ the boundary condition $\gamma(0) = z$ and $\gamma(T) = x$. So $S_\Delta(x, U^*(x, z), z) = S(\gamma^{cl}(x, z))$ and we can replace $\gamma^{cl}(T_1)$ in (22) by $U^*(x, z)$. Next, we note that $\mathrm{Im} f \ge 0$, $\mathrm{Im} f(U^*(x, z)) = 0$ and $\det[f''(x, U^*(x, z), z)] \ne 0$, hence $U^*(x, z)$ is the unique critical point of the functional $f(x, u, z)$ and we can apply to (21) the stationary phase method in the form of Theorem 7.5.5 in [19]. Combining all these arguments together and noting that by definition $\varrho(U^*(x, z) - \gamma^{cl}(T_1)) = \varrho(0) = 1$ we end up with the following result

$$\Bigg| \int_{\mathbb{R}} a_\Delta(x, \gamma^{cl}(T_1) + y, z) \exp\Big\{\imath n S_\Delta(x, \gamma^{cl}(T_1) + y, z)\Big\} dF_n(T_1, y) - \frac{1}{\sigma(T_1)\sqrt{2\pi}}$$
$$\times \exp\Big\{\imath n S(x, \gamma^{cl}(x, z), z)\Big\} \Big[\det\Big(\imath^{-1} f''(x, U^*(x, z), z)\Big)\Big]^{-\frac{1}{2}} a(\gamma^{cl}(x, z))\Bigg| \le C\frac{1}{\sqrt{n}}. \quad (23)$$

In the spirit of the quantum theory one can replace n in (23) by $\nu = 2\pi h^{-1}$ with Planck's constant h. As mentioned above the phaser in the stationary phase approximation is determined by the classical action evaluated along the classical path between the two endpoints while the Hessian $f''(x, U^*(x,z), z)$ of $f(x,u,z)$ with respect to u at the critical point $U^*(x,z)$ is modified by the Brownian bridge distribution.

References

1. Feynman R.P. and Hibbs A.R., Quantum mechanics and path integral, McGraw-Hill, New York, 1965.
2. Cameron, R. H., A family of integralsserving to connect the Wiener and Feynman integrals, J. Mathematics and Physics, **39** (1960),126-140.
3. Itô, K., Generalized uniform complex measure in Hilbertian metric spaces and its application to the Feynman path integrals, Proc. 5th Berkeley symposium on Math. Statistics and Probability, Univ. of California Press, Berkeley, **2**, part 1, (1967), 145-161.
4. Nelson, E., Feynman path integrals and Schrödinger equation, J. Math. Physics, **5**(1964), 332-343.
5. Albeverio, S., Høegh-Krohn, R, J., Oscillatory integrals and the method of stationary phase in infinitely many dimensions with application to the classical limit of quantum mechanics, I. Inv. Math., **40** (1977), 59-106.
6. Elworthy, D. and Truman, A., Feynman maps, Cameron-Martin formulae and anharmonic oscillator, Ann. Inst. H. Poincaré Phys. Théor., 41(2)(1984), 115-142.
7. Fujiwara, D., A Construction of the fundamental solution for the Schrödinger equation, J.d'Analyses Math., **35**,pp.41-96 (1979)
8. Fujiwara, D., Some Feynman path integrals as oscillatory integrals over a Sobolev manifold, in: Lecture Notes in Math., vol.1540, Springer, 1993, pp. 39-53.
9. Belavkin V.P., Nondemolition principle of quantum of quantum measurement theory, Foundation of Physics 24(5)p.685,1994 .
10. Barchielli A., and Belavkin V.P., Measurements continuous in time and a posteriori states in quantum mechanics, J. Phys. A: Math. Gen. 24 (1991), 1495-1514.
11. Belavkin V.P. and Perkins M.G, The nondemolition measurement of quantum time, Int. J. of Theor. Phys. 37(1), 219-226,1998.
12. Swart T. and Rousse V., A mathematical justification for the Herman-Kluk propagator, Comm. Math. Phys., 286(2), 820-842, 2006.
13. Feynman R.P. and Vernon F.L., The theory of a general quantum system interacting with a linear dissipative system, Ann. Phys., 24 (1963),118-173.
14. Caldiera A.O. and Leggett A.J., Path integral approach to quantum brownian motion, Physica A, 121(1983),587-616.
15. Albeverio S., Cattaneo L., Mazzucchi S., Di Persio L., A rigorous approach to the Feynman-Vernon influence functional and its application, J. Math. Phys. 48(10), 2007.
16. Fujiwara, D.,The Stationary phase method with an estimate of the reminder term on a space of large dimension, Nagoya Math. J., **124**(1991) 61-97.
17. Kumano-go, N., Feynman path integrals as analysis on path space by time slicing approximation, *Bull. Sci. Math.*, **128** (2004), 197-251.
18. Fujiwara, D. and Kumano-go, N., An Improved Remainder Estimate of Stationary Phase Method for Some Oscillatory Integrals over a Space of Large dimension, Funkcialaj Ekvacioj, 49 (2006), 59-86.
19. Hörmander, L., The analysys of Linear Partial Diffrential Operators I, Springer-Verlag, Berlin, New York, Heidelberg,Tokyo, 1983
20. Billingsley, P., Convergence of probability measures, Wiley, New-York, 1968.
21. Gulinsky, O., On stochastic setting of stationary phase method, Linear Algebra and its Applications, **435** (2011),1575-1584.
22. Bhattacharya, R. N., Rates of weak convergance and asymptotic expansions for classical central limit theorems, Ann. Math. Statist. **42**(1971), 241-259.

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New type of noncommutative geometry arising from a quantization

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This communication is devoted to the following problem: how to develop a differential calculus on the enveloping algebra $U(gl(m)_\hbar)$[1] so that for $\hbar = 0$ we get just the usual differential calculus on the commutative algebra $\mathrm{Sym}(gl(m))$. A particular case $m = 2$ will be considered in detail. Also, we shall elucidate the role playing in this construction by the so-called Reflection Equation Algebra related to the Quantum Group $U_q(sl(m))$ and exhibit applications of our approach to a quantization of certain dynamical models.

There are known a few approaches to constructing differential calculus on Noncommutative (NC) algebras. One of them due to A.Connes is based on a cocycle complex replacing the usual de Rham complex. Another approach is based on so-called universal differential forms which are defined via the classical form of the Leibniz rule but without the usual permutation relation between elements of the algebra and their differentials. As a result the family of such differential forms $\Omega(A)$ is much bigger than the usual differential algebra provided the algebra A is commutative. In our approach the differential algebra $\Omega(U(gl(m)_\hbar))$ is a deformation of the commutative algebra $\Omega(\mathrm{Sym}(gl(m)))$. Besides, we define partial derivatives in generators of the algebra $U(gl(m)_\hbar)$ which coincide with the classical ones as $\hbar \to 0$.

Let us restrict ourselves to the particular case $m = 2$ and pass to the compact form of (the complexification of) this algebra, namely, that $U(u(2)_\hbar)$. This algebra is generated by four generators t, x, y, z with the following multiplication

[1] The notation $gl(m)_\hbar$ means that a quantizing parameter $\hbar$ is introduced in the Lie bracket of the algebra $gl(m)$.

table

$$[x,\, y] = \hbar\, z, \quad [y,\, z] = \hbar\, x, \quad [z,\, x] = \hbar\, y, \quad [t,\, x] = [t,\, y] = [t,\, z] = 0.$$

Let us introduce the partial derivatives ∂_t, ∂_x, ∂_y, ∂_z via the so-called *permutation relations* between these derivatives and the above generators of the algebra $U(u(2)_\hbar)$. Namely, we put

$$\tilde\partial_t\, t - t\,\tilde\partial_t = \frac{\hbar}{2}\,\tilde\partial_t,\ \ \tilde\partial_t\, x - x\,\tilde\partial_t = -\frac{\hbar}{2}\,\partial_x,\ \ \tilde\partial_t\, y - y\,\tilde\partial_t = -\frac{\hbar}{2}\,\partial_y,\ \ \tilde\partial_t\, z - z\,\tilde\partial_t = -\frac{\hbar}{2}\,\partial_z,$$

$$\partial_x\, t - t\,\partial_x = \frac{\hbar}{2}\,\partial_x,\ \ \partial_x\, x - x\,\partial_x = \frac{\hbar}{2}\,\tilde\partial_t,\ \ \partial_x\, y - y\,\partial_x = \frac{\hbar}{2}\,\partial_z,\ \ \partial_x\, z - z\,\partial_x = -\frac{\hbar}{2}\,\partial_y,$$

$$\partial_y\, t - t\,\partial_y = \frac{\hbar}{2}\,\partial_y,\ \ \partial_y\, x - x\,\partial_y = -\frac{\hbar}{2}\,\partial_z,\ \ \partial_y\, y - y\,\partial_y = \frac{\hbar}{2}\,\tilde\partial_t,\ \ \partial_y\, z - z\,\partial_y = \frac{\hbar}{2}\,\partial_x,$$

$$\partial_z\, t - t\,\partial_z = \frac{\hbar}{2}\,\partial_z,\ \ \partial_z\, x - x\,\partial_z = \frac{\hbar}{2}\,\partial_y,\ \ \partial_z\, y - y\,\partial_z = -\frac{\hbar}{2}\,\partial_x,\ \ \partial_z\, z - z\,\partial_z = \frac{\hbar}{2}\,\tilde\partial_t,$$

where $\tilde\partial_t$ stands for the "shifted derivative" in t: $\tilde\partial_t = \partial_t + \frac{2}{\hbar}\, I$.

In order to define the action $\partial_u(t^a x^b y^c z^d),\ u \in \{t, x, y, z\}$ of a partial derivative on a monomial in the generators t, x, y, z we proceed as follows. By using the above permutation relations we transpose the derivative to the most right position and apply the counit defined by $\varepsilon(1) = 0,\ \varepsilon(\partial_u) = 0$ (and consequently, $\varepsilon(\tilde\partial_t) = \frac{2}{\hbar}$) to it. For instance, in virtue of the permutation relations we have

$$\partial_x\, yz = (y\partial_x + \frac{\hbar}{2}\partial_z)\, z = y(z\partial_x - \frac{\hbar}{2}\partial_y) + \frac{\hbar}{2}(z\partial_z + \frac{\hbar}{2}\tilde\partial_t).$$

Now, by applying the counit we conclude that $\partial_x(yz) = \frac{\hbar}{2}$. This result turns into the classical one as $\hbar = 0$.

It is not difficult to see that the partial derivatives commute with each other.

Now, we are able to define an analog of the de Rham operator on the algebra $U(u(2)_\hbar)$. Let $\bigwedge(u(2))$ be the usual skew-symmetric algebra with four generators dt, dx, dy, dz. Introduce the space of differential forms on $U(u(2)_\hbar)$ by putting

$$\Omega(U(u(2)_\hbar)) = \bigwedge(u(2)) \otimes U(u(2)_\hbar)$$

and define an analog of the de Rham operator on the algebra $U(u(2)_\hbar)$ as

$$d(f) = dt\,\partial_t(f) + dx\,\partial_x(f) + dy\,\partial_y(f) + dz\,\partial_z(f),$$

where f is an arbitrary element of this algebra. Furthermore, the operator d can be extended to the higher order differential forms in the usual way

$$d(\omega\, f) = \omega\, d(f),\ \ \omega \in \bigwedge(u(2)),\ f \in U(u(2)_\hbar).$$

Thus, the de Rham operator d is well defined and due to the commutativity of the partial derivatives it is easy to see that $d^2 = 0$, i.e. the operator d is a differential indeed.

By introducing the permutation relations

$$du \otimes a = a \otimes du, \;\; \forall\, u \in \{t,\, x,\, y,\, z\}, \;\; \forall\, a \in U(u(2)_\hbar)$$

we can introduce the structure of an associative algebra on the space $\Omega(U(u(2)_\hbar))$, but the de Rham operator d is not compatible with this structure via the classical Leibniz rule. For instance, we have $d(yz) = dx\,\frac{\hbar}{2} + dy\,z + dz\,y$.

Observe that such type differential calculus can be defined on any enveloping algebra $U(gl(m)_\hbar)$ and even on super-algebras $U(gl(m|n)_\hbar)$. In the case of the algebra $U(gl(m)_\hbar)$ generated by elements $l_i^j,\; 1 \le i, j, \le m$ the partial derivatives $\partial_{l_k^l}$ can be introduced by $\partial_{l_k^l}(l_i^j) = \delta_i^k \delta_l^j$ and extended on higher monomials via a modified Leibniz rule which can be expressed as follows. In $gl(m)$ there is a product $l_i^j \circ l_k^l = \delta_k^j l_i^l$ such that $[l_i^j, l_k^l] = l_i^j \circ l_k^l - l_k^l \circ l_i^j$. Then the result of applying the derivative to a monomial of third degree reads

$$\partial_{l_i^j}(l_a^b\, l_c^d\, l_k^l) = \partial_{l_i^j}(l_a^b)\, l_c^d\, l_k^l + l_a^b\, \partial_{l_i^j}(l_c^d)\, l_k^l + l_a^b\, l_c^d\, \partial_{l_i^j}(l_k^l) +$$

$$\hbar \left(\partial_{l_i^j}(l_a^b \circ l_c^d)\, l_k^l + \partial_{l_i^j}(l_a^b \circ l_k^l)\, l_c^d + l_a^b\, \partial_{l_i^j}(l_c^d \circ l_k^l) \right) + \hbar^2 \partial_{l_i^j}(l_a^b \circ l_c^d \circ l_k^l).$$

Note that the differential operators $D_k = TrL^k, k = 0, 1, 2, ...$ where $L = \|\partial_{l_i^j}\|$ maps the center $Z(U(gl(m)_\hbar))$ of the algebra $U(gl(m)_\hbar)$ into itself. This center is generated by the elements $TrM^k, k = 0, 1, 2, ...$ where $M = \|l_i^j\|$. Moreover, the matrix M is subject to an analog of Cayley-Hamilton (CH) identity with coefficients belonging to the center. (They generate the center $Z(U(gl(m)_\hbar))$ as well.) This CH identity enables us to introduce the notion of eigenvalues of the matrix M. Let us describe them for the example in question. The matrix M expressed via the generators $t,\, x,\, y,\, z$ has the following form

$$M = \begin{pmatrix} t - i\,z & -i\,x - y \\ -i\,x + y & t + i\,z \end{pmatrix}$$

It is subject to the CH identity

$$M^2 - (2\,t + \hbar)\, M + \; (t^2 + x^2 + y^2 + z^2 + \hbar\, t)\, I = 0.$$

The quantities μ_1 and μ_2 are called eigenvalues of the matrix M if they satisfy the relations

$$\mu_1 + \mu_2 = 2\,t + \hbar, \qquad \mu_1\, \mu_2 = t^2 + x^2 + y^2 + z^2 + \hbar\, t.$$

In a similar way eigenvalues of the matrix M can be defined in a general case.

These eigenvalues are a useful tool for parameterizing all central elements from $Z(U(u(2)_\hbar))$. In particular, the elements TrM^k from the example above can be presented as

$$TrM^k = \mu_1^k \frac{\mu_1 - \mu_2 - \hbar}{\mu_1 - \mu_2} + \mu_2^k \frac{\mu_2 - \mu_1 - \hbar}{\mu_2 - \mu_1}.$$

It is interesting to express the operators D_k restricted to the center $Z(U(u(2)_\hbar))$ via these eigenvalues. For instance, the Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ (which is a combination of such operators) restricted to the center $Z(U(u(2)_\hbar))$ reads

$$\Delta(f(t,\mu)) = \frac{1}{\hbar^2}(2f(t+\hbar,\mu) - f(t+\hbar,\mu-2\hbar) - f(t+\hbar,\mu+2\hbar)) +$$

$$\frac{2}{\mu\hbar}(f(t+\hbar,\mu-2\hbar) - f(t+\hbar,\mu+2\hbar))$$

where $f(t,\mu)$ is a polynomial (or even a rational function) in variables $t = (TrM)/2$ and $\mu = \mu_1 - \mu_2$.

A similar construction must lead in a higher dimensional case ($m > 2$) to a family of difference operators in involution and thus to give rise to some integrable systems based on difference operators.

Now, exhibit a way of quantizing some $SO(3)$-invariant differential operators in the frameworks of our approach. Let us consider a space equipped with a Schwarzschild type metric

$$\varphi(r)\, dt^2 - \varphi(r)^{-1}\, dr^2 - r^2 d\Omega^2.$$

Here $d\Omega^2$ is the area form of the unit sphere and $\varphi(r)$ is a rational function. It is just Schwarzschild metric provided $\varphi(r) = 1 - \frac{r_g}{r}$. The corresponding Laplace-Beltrami (LB) operator describing dynamics of a scalar massless particle is

$$\Box_{LB} = \varphi(r)^{-1}\partial_t^2 - \varphi(r)\partial_r^2 - \frac{X^2 + Y^2 + Z^2}{r^2} - \frac{1}{r^2}\partial_r(\varphi(r)\, r^2)\partial_r,$$

where $X = y\,\partial_z - z\,\partial_y,\ Y = z\,\partial_x - x\,\partial_z,\ Z = x\,\partial_y - y\,\partial_x$. By using the relation

$$\frac{X^2 + Y^2 + Z^2}{r^2} = \Delta - \partial_r^2 - \frac{2}{r}\partial_r,$$

we can represent this operator as follows

$$\Box_{LB} = \varphi(r)^{-1}\partial_t^2 - (\varphi(r) - 1)\partial_r^2 - \Delta + (\frac{2}{r}(1 - \varphi(r)) - \partial_r\varphi(r))\partial_r.$$

Introduce the operator $Q = x\partial_x + y\partial_y + z\partial_z$ and observe that in the commutative algebra $\mathrm{Sym}(u(2))$ the following formulae are valid

$$\partial_r = \frac{Q}{r}, \ \partial_r^2 = \frac{Q^2 - Q}{r^2}.$$

Thus, the operator $\Box_{LB}$ can be presented as

$$\Box_{LB} = \varphi(r)^{-1}\partial_t^2 - (\varphi(r) - 1)(\frac{Q^2 - Q}{r^2}) - \Delta + (\frac{2}{r}(1 - \varphi(r)) - \partial_r\varphi(r))\frac{Q}{r}. \quad (1)$$

Similarly to the operator Δ that Q is well defined on the algebra $U(u(2)_\hbar)$ and its action can be computed. Its restriction to the center $Z(U(u(2)_\hbar))$ is also a difference operator in t and μ. Even more, this operator can be also extended on rational functions in these variables.

Also, introduce the so-called *quantum radius* $r_\hbar = \frac{\mu}{2i}$. Now, the quantization procedure consists in replacing r in the above expression for $\Box_{LB}$ by $r_\hbar$ and by treating Δ and Q which come in this formula as operators on the algebra $U(u(2)_\hbar)$. The action of the final quantum operator can be explicitly computed and expressed in terms of t and $r_\hbar$. It will be interesting to find the spectrum of this operator.

Emphasize that our quantization procedure is $SU(2)$-covariant. Besides, there is a deformation of the algebra $U(gl(2))$ which is $U_q(sl(2))$-covariant. Describe this deformation for a general case $m \geq 2$. To this end we consider the image (denoted R and called braiding) of the universal quantum R-matrix in the space $V \otimes V$ where V is the fundamental $U_q(sl(m))$-module. Let $M = \|m_i^j\|$ be a matrix with entries m_i^j. The unital algebra generated by these entries subject to the system

$$R(M \otimes I)R(M \otimes I) - (M \otimes I)R(M \otimes I)R = \hbar(R(M \otimes I) - (M \otimes I)R)$$

is called $\hbar$-Reflection Equation Algebra and denoted $A(q, \hbar)$. It is a two-parameter deformation of that $\mathrm{Sym}(gl(m))$ (the parameters of deformation are $\hbar$ and q). The point is that the matrix M subject to this system also satisfies a version of the CH identity. Besides, on this algebra analogs of partial derivatives $\partial_{m_i^j}$ can be also introduced via proper permutation relations.

Furthermore, the algebra $A(q, \hbar)$ has the center $Z(A(q, \hbar))$ similar to that $Z(U(gl(m)_\hbar))$. Namely, it is a deformation of that $Z(U(gl(m)_\hbar))$ and it is generated by the elements $Tr_R M^k$ where Tr_R is the quantum trace. This trace can be directly extracted from the braiding R. In the example above

$$Tr_R \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{-3}a + q^{-1}d.$$

All these properties enable us to generalize a large part of the above constructions to this algebra.

We want to complete this note with a historical remark exhibiting a role of the $\hbar$-Reflection Equation Algebra in constructing our differential calculus on the algebras $U(gl(m)_\hbar)$ (or $U(u(m)_\hbar)$). In a number of papers [W,IP,FP] a version of differential calculus was constructed on a pseudogroup (in fact an RTT algebra). In [GPS1] we suggested a generalization of this construction by replacing this algebra by other Quantum Matrix Algebras (QMA). In a particular case, the role of such QMA is played by a $\hbar$-Reflection Equation Algebra (may be with $h = 0$). Thus, in [GPS2] we introduced a braided version of Weyl algebra which in the limit $q \to 1$ given rise to differential calculus presented above.

The most amazing consequence of this calculus consists in the following. Though it is defined on a Noncommutative algebra but the final operators coming in the above dynamical models are difference ones. Thus, these operators can be defined on a lattice and consequently the space-time can be considered as discrete. It will be interesting to study physical consequences of this conclusion.

References

[FP] Faddeev L., Pyatov P. *The Differential calculus on quantum linear groups*, Amer. Math. Soc. Transl. Ser. 2, 175.

[GPS1] Gurevich D., Pyatov P., Saponov P. *Braided differential operators on quantum algebras*, J. of Geometry and Physics 61 (2011), 1485–1501.

[GPS2] Gurevich D., Pyatov P., Saponov P. *Braided Weyl algebras and differential calculus on $U(u(2))$*, J. of Geometry and Physics 62 (2012), 1175–1188.

[IP] Isaev A., Pyatov P. *Covariant differential calculus complexes on quantum linear groups*, J.Phys. A 28 (1995) 8, 2227–2246.

[W] Woronowicz S. *Differential Calculus on Compact Matrix Pseudogroups (Quantum groups)*, CMP 122 (1989), 125–170.

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Bounds for tropical, determinantal and Gondran-Minoux ranks

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1 Introduction

Definition 1 The set of real numbers with the additional element $-\infty$ is called the *max-algebra* (it is sometimes also called *max-plus algebra* or *tropical algebra*), denoted $\overline{\mathbb{R}}_{\max}$, if the operations of addition, $\oplus$, and multiplication, $\otimes$, on this set are defined in the following way: $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$, correspondingly. Here *neutral element* with respect to addition is $-\infty$ and *neutral element* with respect to multiplication is 0. $\overline{\mathbb{R}}_{\max}$ with these operations is an algebraic structure, called a *semiring*.

For the purpose of this paper it is more convenient to use the exponential model of the max-algebra which is defined below and directly obtained by applying the exponent function to the ordinary model introduced above.

Definition 2 The *max-algebra* $\mathbb{R}$ is the set of non-negative real numbers with the operations of addition and multiplication defined by $a \oplus b = \max\{a, b\}$, $a \otimes b = a \cdot b$, correspondingly. Here *neutral element* with respect to addition is $0 \in \mathbb{R}$ and *neutral element* with respect to multiplication is $1 \in \mathbb{R}$.

Definition 3 The *binary Boolean semiring* $\mathbb{B}$ is the set $\{0, 1\}$ with the operations of addition and multiplication defined by $a \oplus b = \max\{a, b\}$, $a \otimes b = \min\{a, b\}$, correspondingly.

We consider matrices over $\mathbb{B}$ and $\mathbb{R}$. The set of $m \times n$ matrices with entries from a set S is denoted by $\mathcal{M}_{m\times n}(S)$ and $\mathcal{M}_n(S) = \mathcal{M}_{n\times n}(S)$. By O we denote a *zero* matrix, i.e., all elements of O are equal to 0. By $A[r_1, \ldots, r_k]$ we denote the matrix formed by the rows of the matrix A with the indexes $r_1, \ldots, r_k$. By $A[r_1, \ldots, r_k | c_1, \ldots, c_m]$ we denote the submatrix of A located on the intersection of the rows with the indexes $r_1, \ldots, r_k$ and the columns with the indexes $c_1, \ldots, c_m$. By a_{ij} we denote the entry in the ith row and jth column of A.

There are many different approaches to define the notion of rank for matrices over max-algebras and binary Boolean semirings, see for example [1] for the detailed and self-contained information on this subject. In this paper we consider the notions of tropical rank (see [1, 3, 7]), determinantal rank (see [1]) and Gondran-Minoux rank (see [1, 4, 5]). Let us start with the exact definitions of these notions.

Definition 4 The tropical *permanent* of an $n \times n$ matrix $A = (a_{ij})$ with elements from $\mathbb{B}$ or $\mathbb{R}$ is the following function

$$perm(A) = \max_{\sigma \in S_n} \{a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\}, \tag{1}$$

where S_n denotes the symmetric group on $\{1, \ldots, n\}$.

Definition 5 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$. A matrix $A \in \mathcal{M}_n(\mathcal{S})$ is called *tropically singular* if $n = 1$ and $perm(A) = 0$ or if the maximum in the expression (1) is achieved at least on two different permutations in S_n. A matrix that is not tropically singular is called *tropically nonsingular*.

Remark 1 In particular, if $n \geq 2$ and $perm(A) = 0$, then for any $\sigma \in S_n$ it holds that $a_{1,\sigma(1)} \ldots a_{n,\sigma(n)} = 0$, hence A is tropically singular.

Definition 6 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$. The *tropical rank* $trop(A)$ of a matrix $A \in \mathcal{M}_n(\mathcal{S})$ is the maximal size of a tropically nonsingular square submatrix of A. By the definition $trop(O) = 0$.

Definition 7 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$ and let $A = (a_{ij}) \in M_n(\mathcal{S})$. Elements

$$\|A\|^+ = \max_{\tau \in \mathcal{A}_n} \{a_{1,\tau(1)} \ldots a_{n,\tau(n)}\}, \; \|A\|^- = \max_{\varphi \in S_n \setminus \mathcal{A}_n} \{a_{1,\varphi(1)} \ldots a_{n,\varphi(n)}\} \in \mathcal{S}$$

are called respectively the *positive determinant* and the *negative determinant* of A. Here $\mathcal{A}_n \subset S_n$ denotes the subgroup of even permutations.

Definition 8 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$. A matrix $A \in \mathcal{M}_n(\mathcal{S})$ is called *d-nonsingular*, if $\|A\|^+ \neq \|A\|^-$. In the case $\|A\|^+ = \|A\|^-$, the matrix A is called *d-singular*.

Definition 9 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$, $A \in \mathcal{M}_n(\mathcal{S})$. The *determinantal rank* $d(A)$ is the maximal size of d-nonsingular square submatrix of A. By the definition $d(O) = 0$.

Definition 10 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$. A set $a^1, \ldots, a^n$ of vectors from $\mathcal{S}^m$, $a^i = (a_1^i, \ldots, a_m^i)^t$, is called *Gondran-Minoux dependent* (or shortly, *GM-dependent*) if there are subsets $I, J \subset \{1, \ldots, n\}$, $I \cap J = \varnothing$, $I \cup J \neq \varnothing$ and elements $\lambda_1, \ldots, \lambda_n \in \mathcal{S}$, $\lambda_t \neq 0$, $t \in I \cup J$, such that for all $k \in \{1, \ldots, m\}$ it holds that

$$\max_{i \in I}\{\lambda_i a_k^i\} = \max_{j \in J}\{\lambda_j a_k^j\}. \tag{2}$$

In the case, a set of vectors is not GM-dependent, then it is called *GM-independent*.

Definition 11 The maximal number of vectors in all GM-independent subsets of the set $a^1, \ldots, a^n$ is called *Gondran-Minoux rank (GM-rank)* of the set $a^1, \ldots, a^n$ and is denoted by $GM(a_1, \ldots, a_n)$.

Definition 12 Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$, $A \in M_n(\mathcal{S})$. GM-rank of the set of rows of A is called *row GM-rank* of A and is denoted by $GMr(A)$. GM-rank of the set of columns of A is called *column GM-rank* of A and is denoted by $GMc(A)$.

Remark 2 Note that row GM-rank and column GM-rank can be different, for details see the work of the second author [13] or Example 1 in this paper. Below by GM-rank we mean row GM-rank.

The notion of GM-independence is related to the notion of L-matrices, which can be found in [9, 12].

Definition 13 Let $A \in \mathcal{M}_n(\mathbb{R})$ be a real matrix, $c_1, \ldots, c_n$ be its columns. It is said that A is *not an L-matrix* if there are subsets $I, J \subset \{1, \ldots, n\}$, $I \cap J = \varnothing$, $I \cup J \neq \varnothing$, such that every nonzero row of the matrix A' formed by the columns $\{c_i, -c_j\}_{i \in I, j \in J}$ contains both positive and negative elements. Otherwise A is called an *L-matrix*.

Remark 3 From Definitions 10 and 13 we have that a 0-1 matrix A is an L-matrix iff the columns of A are GM-independent over $\mathbb{B}$.

The properties of GM-independence and d-nonsingularity are equivalent:

Fact 1 *[4, Chapter 5, Corollary 3.4.3] Let $\mathcal{S} = \mathbb{B}$ or $\mathbb{R}$. The following conditions are equivalent for $A \in M_n(\mathcal{S})$:*
1) The rows (columns) of A are GM-dependent;
2) $\|A\|^+ = \|A\|^-$.

Remark 4 From Remark 3 and Fact 1 it follows that a square 0-1 matrix A is an L-matrix iff A is d-nonsingular.

In this paper we introduce the notion of the tropical matrix pattern, which provides a powerful tool to investigate tropical matrices. The above approach is then illustrated by the application to the study of the properties of the Gondran-Minoux rank function. Our main result states that up to a multiplication of matrix rows by non-zero constants the Gondran-Minoux independence of the matrix rows and that of the rows of its tropical pattern are equivalent.

As a consequence of our main result we obtain that the tropical rank, $trop(A)$, and the determinantal rank, $d(A)$, of tropical matrices satisfy the following inequalities: $trop(A) \geqslant \sqrt{GMr(A)}$, $d(A) \geqslant \sqrt{GMr(A)}$, $trop(A) \geqslant \frac{d(A)+2}{3}$. As an important corollary of this result we obtain that if one of these functions is bounded then the other two are also bounded unlike the situation with the factor and Kapranov ranks.

The purpose of this note is to state our result. The detailed proofs will be available soon in [6].

2 Tropical pattern technique and its applications

We first introduce the notion of the tropical pattern of a matrix for $A \in \mathcal{M}_{n\times m}(\mathbb{R})$. It is shown that up to the multiplication of rows of a matrix by positive numbers, GM-dependence of rows of any matrix is equivalent to GM-dependence of rows of its tropical pattern. This enables us to generalize several results proved for matrices over $\mathbb{B}$ to the matrices over max-algebras.

Next we present a number of applications of the pattern technique. Some general properties of GM-rank, determinantal rank and tropical rank of matrices over $\mathbb{R}$ are investigated. We provide minimal (with respect to the size of matrices) examples where these functions are different. Also we prove the following inequalities $trop(A) \geqslant \sqrt{GMr(A)}$, $trop(A) \geqslant \frac{d(A)+2}{3}$, $d(A) \geqslant \sqrt{GMr(A)}$ for matrices with coefficients from $\mathbb{B}$ and $\mathbb{R}$.

2.1 Tropical pattern and GM-independence

Main Definition 1 *The tropical pattern of a matrix $A = (a_{ij}) \in \mathcal{M}_{n\times m}(\mathbb{R})$ is the matrix $B = (b_{ij}) \in \mathcal{M}_{n\times m}(\mathbb{B})$ defined by*

$$b_{uv} = \begin{cases} 1 & \text{if } a_{uv} = \max_{i=1}^{n}\{a_{iv}\} > 0, \\ 0 & \text{if either } a_{uv} = 0 \text{ or } a_{uv} < \max_{i=1}^{n}\{a_{iv}\}. \end{cases}$$

The tropical pattern of the matrix A is denoted by $\mathcal{P}(A)$.

Lemma 1 *Let* $A = (a_{ij}) \in \mathcal{M}_{n\times m}(\mathbb{R})$, $W = \mathcal{P}(A) \in \mathcal{M}_{n\times m}(\mathbb{B})$ *be the tropical pattern of* A. *Assume that there exist sets* $I, J \subset \{1,\dots,n\}$, $I \cap J = \emptyset$, $I \cup J = \{1,\dots,n\}$, *and nonnegative* $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ *not all equal to* 0 *such that*

$$\max_{i\in I}\{\lambda_i a_{ik}\} = \max_{j\in J}\{\lambda_j a_{jk}\} \text{ for all } k.$$

Set $\mu_t = 1$ *if* $\lambda_t = \max_{u=1}^{n}\{\lambda_u\}$ *and* $\mu_t = 0$ *if* $\lambda_t < \max_{u=1}^{n}\{\lambda_u\}$. *Then* $\max_{i\in I}\{\mu_i w_{ik}\} = \max_{j\in J}\{\mu_j w_{jk}\}$ *for all* k.

Lemma 1 states that the pattern of a matrix with GM-dependent rows has also GM-dependent rows. One of the main results of our paper states that, conversely, multiplying the rows of any GM-independent matrix A by positive numbers, we can obtain a matrix A' whose pattern has GM-independent rows as well:

Theorem 1 *Let* $A \in \mathcal{M}_{n\times m}(\mathbb{R})$. *The rows of* A *are GM-independent iff there exists a matrix* $A' \in \mathcal{M}_{n\times m}(\mathbb{R})$ *which is obtained from* A *by the multiplication of rows of* A *by positive elements from* $\mathbb{R}$ *and such that the rows of* $\mathcal{P}(A') \in \mathcal{M}_{n\times m}(\mathbb{B})$ *are GM-independent.*

2.2 Matrices for which Gondran-Minoux, determinantal and tropical ranks are different

In this subsection we provide some known relations between the rank functions under consideration and minimal examples of matrices that distinguish these functions.

Lemma 2 *Let* $\mathcal{S} = \mathbb{B}$ *or* $\mathbb{R}$, $A \in \mathcal{M}_{n\times n}(\mathcal{S})$. *Then* $trop(A) \leqslant d(A) \leqslant GMr(A)$.

Inequalities from Lemma 2 can be sharp. We provide the minimal possible (with respect to the size of matrices) examples which show that these rank functions are indeed different.

Example 1 Let

$$A = \begin{pmatrix} 1&0&1&1&0&0\\ 1&0&0&0&1&1\\ 0&0&1&1&1&1\\ 0&1&0&1&0&1\\ 0&1&1&0&1&0 \end{pmatrix} \in \mathcal{M}_{5\times 6}(\mathbb{B}) \text{ (or } \mathcal{M}_{5\times 6}(\mathbb{R})).$$

Then $GMr(A) = 5$, $GMc(A) = d(A) = 4$, $trop(A) = 3$. The matrix A contains the minimal number of rows and columns among all matrices $M \in \mathcal{M}_{n\times m}(\mathcal{S})$

such that $GMr(M) > d(M)$ and among all matrices $N \in \mathcal{M}_{n\times m}(\mathcal{S})$ such that $GMr(N) > GMc(N)$.

Example 2 Let

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{B}) \text{ (or } \mathcal{M}_{3\times 3}(\mathbb{R})).$$

Then $GMr(B) = d(B) = 3$, $trop(B) = 2$. The matrix B contains the minimal possible number of columns among all matrices $M \in \mathcal{M}_{n\times m}(\mathcal{S})$ such that $GMr(M) > trop(M)$ and among all matrices $N \in \mathcal{M}_{n\times m}(\mathcal{S})$ such that $d(N) > trop(N)$.

2.3 Inequalities for matrix rank functions over $\mathbb{B}$

In this section we state the following inequalities: $trop(A) \geqslant \sqrt{GMr(A)}$, $trop(A) \geqslant \frac{d(A)+2}{3}$, $d(A) \geqslant \sqrt{GMr(A)}$ for any $A \in \mathcal{M}_{n\times m}(\mathbb{B})$.

Theorem 2 *Let the rows of $A \in \mathcal{M}_{n\times m}(\mathbb{B})$ be GM-independent. Then $trop(A) \geqslant \sqrt{n}$.*

Theorem 3 *Let $A \in \mathcal{M}_{n\times m}(\mathbb{B})$. Then $trop(A) \geqslant \sqrt{GMr(A)}$.*

The next inequality which connects GM- and determinantal ranks for matrices over $\mathbb{B}$, is a direct consequence of Theorem 3.

Corollary 1 *Let $A \in \mathcal{M}_{n\times m}(\mathbb{B})$. Then $d(A) \geqslant \sqrt{GMr(A)}$.*

Now we are going to formulate the inequality relating determinantal and tropical ranks for matrices over $\mathbb{B}$.

Lemma 3 *Let $A \in \mathcal{M}_{n\times n}(\mathbb{B})$, and elements $p_1, \ldots, p_t \in \{1, \ldots, n\}$ be different. Assume that for any $q_1, \ldots, q_t \in \{1, \ldots, n\}$, we have that*

$$\|A[p_1, \ldots, p_t|q_1, \ldots, q_t]\|^+ = \|A[p_1, \ldots, p_t|q_1, \ldots, q_t]\|^-.$$

Then $\|A\|^+ = \|A\|^-$.

Theorem 4 *Let $A \in \mathcal{M}_{n\times n}(\mathbb{B})$ be such that $\|A\|^+ \neq \|A\|^-$. Then $trop(A) \geqslant \frac{n+2}{3}$.*

Now we are ready to state the inequality relating determinantal and tropical ranks of matrices over $\mathbb{B}$.

Theorem 5 *Let $A \in \mathcal{M}_{n\times m}(\mathbb{B})$. Then $trop(A) \geqslant \frac{d(A)+2}{3}$.*

2.4 Matrices over $\mathbb{R}$

In this subsection, using the characterization of the GM-rank of matrices via the GM-ranks of tropical patterns, we generalize the main results from the previous subsection to $\mathbb{R}$.

Theorem 6 *Let* $A \in \mathcal{M}_{n\times m}(\mathbb{R})$, $W = \mathcal{P}(A) \in \mathcal{M}_{n\times m}(\mathbb{B})$. *Then* $trop(A) \geqslant trop(W)$.

Theorem 7 *Let* $A \in \mathcal{M}_{n\times m}(\mathbb{R})$. *Then* $trop(A) \geqslant \sqrt{GMr(A)}$.

Theorem 8 *Let* $A \in \mathcal{M}_{n\times m}(\mathbb{R})$. *Then* $trop(A) \geqslant \frac{d(A)+2}{3}$.

Corollary 2 *Let* $A \in \mathcal{M}_{n\times m}(\mathbb{R})$. *Then* $d(A) \geqslant \sqrt{GMr(A)}$.

The following consequence from the obtained results is very important. In particular it underlines the difference of the functions under consideration with the factor and Kapranov ranks, cf. [1–3, 8].

Corollary 3 *Let one of the functions* $trop(A)$, $d(A)$, $GMr(A)$ *is bounded. Then the other two are also bounded.*

References

1. M. Akian, S. Gaubert, A. Guterman. Linear independence over tropical semirings and beyond // Contemporary Mathematics, AMS, **495**(2009) 1–38.
2. L. B. Beasley, T. J. Laffey. Real rank versus nonnegative rank // Linear Algebra and its Applications, **431(12)**(2009), 2330-2335.
3. M. Develin, F. Santos, B. Sturmfels. On the rank of a tropical matrix, in Discrete and Computational Geometry (E. Goodman, J. Pach and E. Welzl, eds.), MSRI Publications, Cambridge Univ., Press, 2005.
4. M. Gondran, M. Minoux. Graphs, Dioids and Semirings: New Models and Algorithms, Springer Science+Business Media, LLC, 2008.
5. M. Gondran, M. Minoux. L'independance lineaire dans les dioides, E.D.F. — Bulletin de la Direction des Etudies et Recherches, Serie C — Mathematiques, Informatique **1**(1978), 67–90.
6. A. Guterman, Ya. Shitov. Tropical patterns of matrices and the Gondran–Minoux rank function // Linear Algebra and its Applications, accepted.
7. K. H. Kim, N. F. Roush. Kapranov rank vs. tropical rank // Proc. Amer. Math. Soc., **134(9)**(2006), 2487–2494.
8. K. H. Kim, N. F. Roush. Factorization of polynomials in one variable over the tropical semiring // arxiv:math.CO/0501167.
9. V. Klee, R. Ladner, and R. Manber. Sign-solvability revisited // Linear Algebra Appl., **59**(1984), 131–158.
10. P. L. Poplin, R. E. Hartwig. Determinantal identities over commutative semirings // Linear Algebra Appl., **387**(2004), 99–132.
11. Ya. Shitov. Inequalities for Gondran-Minoux rank and idempotent semirings // Linear Algebra Appl., doi:10.1016/j.laa.2010.09.032 (2010).
12. C. Thomassen. Sign-nonsingular matrices and even cycles in directed graphs // Linear Algebra Appl. **75**(1986), 27–41.
13. Ya. Shitov. Minimal example of a matrix which distinguishes GM- and d- ranks over max-algebras // Fundamental and Applied Mathematics, **14(4)**(2008), 231–268 [in Russian].

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Singularities for tropical limit/dequantization

B. Kh. Kirshtein

To keep in mind some engineering applications, it is important to be able to estimate the proximity of solutions of equations or systems of equations to their singular values.

A natural way to do this is a transfer to the so-called tropical limit/dequantization and comparison of properties of real and complex solutions of the original equations with properties of tropical solutions of equations after the transfer to the tropical limit.

For instance, if we consider real and complex tropical roots in the sense of hyperfields theory [1], then appearance of singularities in the process of the transition to the tropical limit can be found if some of complex roots become real tropical after the transition to the tropical limit. In this way we can detect singularities working with tropical solutions before their real appearance in real or in complex settings.

We discuss two examples of this type.

In the first one we consider a numerical method which allows us to compare the number of all real solutions of polynomials in one variable with real coefficients and the number of all tropical real solutions of these polynomials after the transfer to the tropical limit.

In the second example we consider some systems of polynomial equations arising in electrical power industry. In this case we give a simple test for appearance of singularities after the transfer to the tropical limit in terms of some spanning trees of graphs of electrical networks. We can use this test as a model for predictions of the instability of electrical power networks.

This work is supported by the grant RFBR 12-01-00886-a and the joint RFBR-CNRS grant 11-01–93106-a

References

1. O.Viro, *Hyperfields in tropical geometry I. Hyperfields and dequntization*, E-print arXiv:math.AG/1006.3034v2.

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On Maslov's quantization of thermodynamics

V. N. Kolokoltsov

The program of the tunnel quantization of thermodynamics was put forward by V. P. Maslov some 20 years ago, as a possible link of Idempotent (Tropical) mathematics and statistical physics. Many papers by Maslov and his collaborators have been published since then. Very beautiful underlying ideas are scattered among technical results. Our objective is to present a personal view on these ideas, or better to say some bunch of these ideas related to two major points: quantization of critical exponents and fractional dimensions. We aim to give a clear glimps of the picture for nonexperts starting from the basics, explaining the main idealogical link number theory–economics–physics and indicating the perspectives.

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On the geometry of quantum codes. Introduction

T. E. Krenkel

A finite generalized quadrangle of order (s,t), usually denoted $GQ(s,t)$, is an incidence structure $S=(P,B,\mathrm{I})$, where P and B are disjoint (non-empty) sets of objects, called respectively points and lines, and where I is a symmetric point-line incidence relation satisfying the following axioms:

(GQ1) each point is incident with $1+t$ lines $(t \geq 1)$ and two distinct points are incident with at most one line;
(GQ2) each line is incident with $1+s$ points $(s \geq 1)$ and two distinct lines are incident with at most one point;
(GQ3) if x is a point and L is a line not incident with x, then there exists a unique pair $(y, M) \in P \times B$ for which $x\mathrm{I}M\mathrm{I}y\mathrm{I}L$.

From these axioms it readily follows that $|P| = (s+1)(st+1)$ and $|B| = (t+1)(st+1)$. If $s=t$, S is said to have order s.

Given two points x and y of S one writes $x \sim y$ and says that x and y are collinear if there exists a line L of S incident with both. For any $x \in P$ denote $x^{\perp} = \{y \in P | y \sim x\}$ and note that $x \in x^{\perp}$; obviously, $|x| = 1+s+st$.

A generalized quadrangle of order 2, denoted $W(3,2)$, consists of 15 points and 15 lines and is shown as the Payne graph in Fig.1, which consists of 15 vertices and 30 edges. The *equipped Payne graph* is shown in Fig.2 with coordinates of its vertices and is called the *Pauli graph* $\mathcal{G}_2$.

Define a symplectic bilinear form

$$B(x,y) = x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2$$

on the point set P of space $PG(3,2)$. We will call two points x and y conjugate $(x \approx y)$, if for these points the symplectic form $B(x,y)$ equals null. Every point

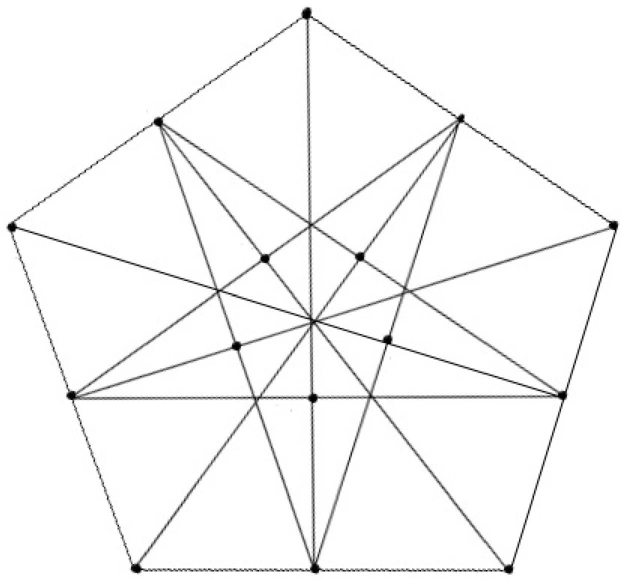

Fig. 1

is selfconjugate and a pair of points lying on the line is also conjugate. The set of conjugate points in the quadrangle $W(3,2)$ defines a bundle of isotropic lines $x^{\perp}$. The point x is called the base of the bundle of isotropic lines. The bundle consists of three lines with the common base $x \in x^{\perp}$ and $|x^{\perp}| = 7$.

Key facts used in the description of the Pauli graph $\mathcal{G}_2$ are as follows:

1) to every vertex of the Pauli graph $\mathcal{G}_2$ the Kronecker product of the Pauli matrices $\sigma_0, \sigma_x, \sigma_y$ and σ_z is ascribed;

2) the conjugation relation of points in GQ $W(3,2)$ is substituted for the commutation relation for the Kronecker products of the Pauli matrices (generalized Pauli matrices).

The Pauli matrices $\sigma_0 = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ describes the transformation of a state of a particle with spin $1/2$ in two-dimensional complex Hilbert space $\mathbb{C}^2$ (spinor space) and satisfy the commutation relation

$$\sigma_x \sigma_y = i\sigma_z, \sigma_y \sigma_z = i\sigma_x, \sigma_z \sigma_x = i\sigma_y. \tag{1}$$

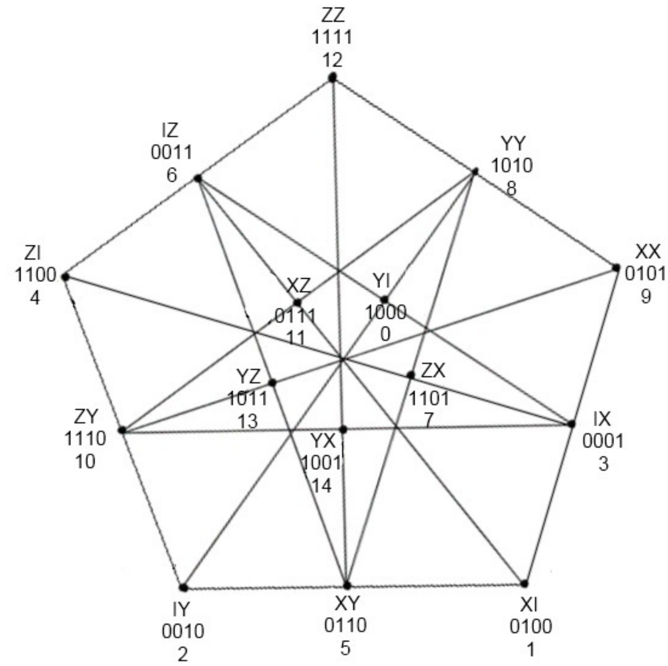

Fig. 2

Let us design the set $\mathbb{P}$ of the Kronecker products of $\sigma_\alpha \otimes \sigma_\beta, \quad \alpha, \beta = 0, x, y, z$. These generalized Pauli matrices (operators) operate in complex Hilbert space $\mathbb{C}^4$ corresponding to the four level quantum system of two particles with spin 1/2 (space of twoqubits).

Vertices of the Pauli graph $\mathcal{G}_2$ are numerated by the points of projective space as in Fig.2. Using the correspondence $\{00 \mapsto \mathbb{I}, 01 \mapsto X, 10 \mapsto Y, 11 \mapsto Z\}$, we ascribe corresponding generalized Pauli matrix to each vertex of the Pauli graph $\mathcal{G}_2$. The generalized Pauli matrices commuting among themselves, as seen in Fig.3, belong to the same bundle (with the base ZX) of isotropic lines. The number of points in the bundle equals 7.

Describe as $\mathbb{P}^\times = \mathbb{P} \setminus \{\mathbb{I} \otimes \mathbb{I}\}$ matrices that constitute finite Pauli group $\mathcal{P}_2$ of order 2^6 and nilpotency class 2. This group, as was described above, is defined as follows: $\mathcal{P}_2 = \{\mathbb{I}, X, Y, Z\}^{\otimes 2} \times \{\pm 1, \pm i\}$. It is isomorphic to the small permutation group [64,266], the group of number 266 in the sequence of small groups with cardinality 64. It also may be seen as a central product: $\mathcal{P}_2 \cong E_{32}^{\pm} * \mathbb{Z}_4$ since

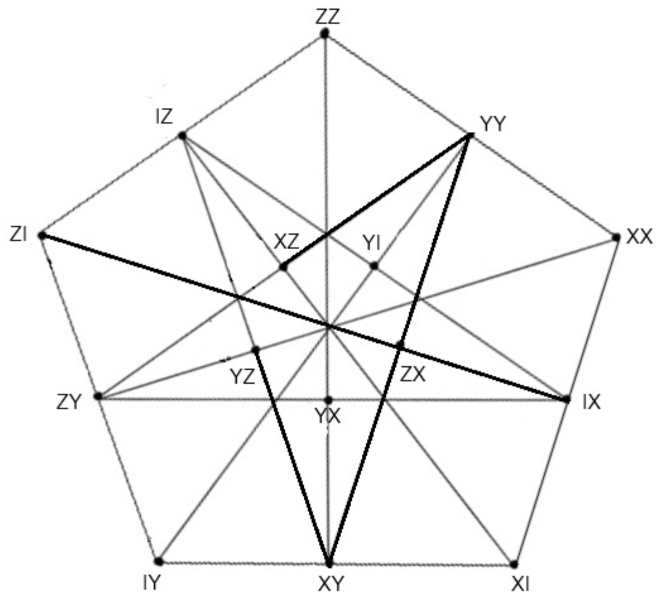

Fig. 3

[64,266] contain the extraspecial groups $E^{\pm}_{32}$ and the cyclic group $\mathbb{Z}_4$ as normal subgroups.

The commutation relation for matrices from $\mathbb{P}^{\times}$(described as a usual right group-theoretical commutator $[,]$), by definition, is equivalent to conjugation relation $\approx$ in symplectic GQ $W(3,2)$, i.e. the commuting matrices are in the same bundle of isotropic lines. The exponent of the Pauli group $\mathcal{P}_2$ is equal to 4.

Proposition 1 *(i) The derived group (commutant)* $\mathcal{P}'_2 = [\mathcal{P}_2, \mathcal{P}_2]$ *equals to the center* $\mathbf{Z}(\mathcal{P}_\mathbf{2}) = \{\pm\mathbb{I}\otimes\mathbb{I}, \pm i\mathbb{I}\otimes\mathbb{I}\}$ *of* $\mathcal{P}_2$*;*

(ii) We have the following exact sequence:

$$1 \mapsto \mathbf{Z}(\mathcal{P}_2) \mapsto \mathcal{P}_2 \mapsto V(4,2) \mapsto 1.$$

Now let us describe the quantum code $\mathcal{C}$ [[15,7]]. Codeword c consists of 15 twoqubits (bispinors) where the first seven ones are message twoqubits and last eight are error correction twoqubits. Therefore codewords are the elements of complex Hilbert space $\mathbb{C}^{60}$. The quantum codes are always linear being subspace of $V = \mathbb{C}^{2n}$.

Codeword c is written in the 15 twoqubit register as

$|**>|**>|**>|**>|**>|**>|**>|00>|00>|00>|00>|00>|00>|00>|00>$. A message bispinor (twoqubit) $|**>$ in general is $|\psi>=\alpha_{00}|00>+\alpha_{01}|01>+\alpha_{10}|10>+\alpha_{11}|11>$ where $|\alpha_{00}|^2+|\alpha_{01}|^2+|\alpha_{10}|^2+|\alpha_{11}|^2=1$. Thus the cardinality of codevectors is equal to 2^{28}.

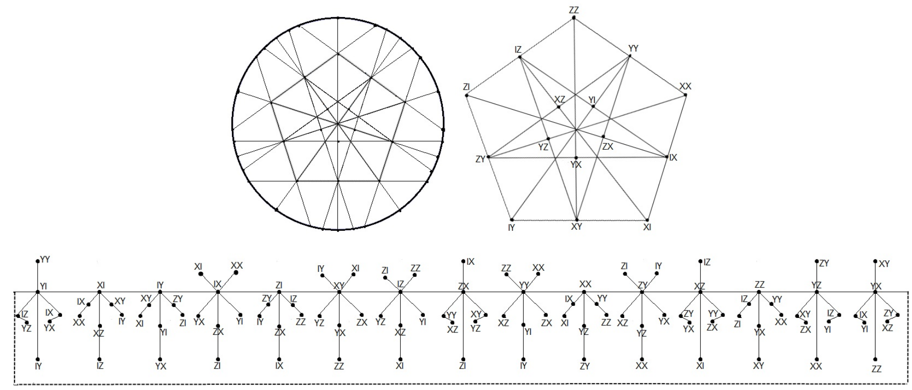

Fig. 4

The generating matrix G for quantum code [[15,7]] is now written as $G=(IX,XI,XX,ZX,ZI,YX,YI,XY,IY,ZZ,YZ,ZY,XZ,IZ,YY)$. Transformed codeword C is generated as $C=G^tc$, i.e. is the result of transformation of the content of bispinor register by the *bispinor network* represented by Pauli graph $\mathcal{G}_2$.

On Fig.4 Levi-Feynman diagram (chord diagram) and a *full isotropy system (total space)* corresponding to Pauli graph $\mathcal{G}_2$ are represented.

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Solution to an extremal problem in tropical mathematics

Nikolai Krivulin

1 Introduction

Methods and computational procedures for solving multidimensional extremal problems are among the topical lines of investigation in the linear tropical (idempotent) algebra [1–7]. We consider the problems of minimizing linear or nonlinear functionals defined on finite-dimensional semimodules over idempotent semifields, with possible additional constraints imposed on the feasible solution set in the form of linear equations and inequalities. The problems under study in the area include idempotent analogues of linear programming problems [7, 8] as well as their extensions with nonlinear objective functions [9–14]. In addition, there exist solutions to some problems where both the objective function and constraints appear to be nonlinear [15, 16].

Many extremal problems under consideration are stated and solved in terms of specific semifields (see, e.g., [7–9] which concentrate on the classical semifield $\mathbb{R}_{\max,+}$). Other results [10–14] offer solutions in more general setting which take $\mathbb{R}_{\max,+}$ as a particular case. In some cases, solutions to extremal problems are given in the form of an iterative computational algorithm that produces a solution if any, or indicate that there is no solution otherwise [7–9]. Some other problems allow one to obtain a closed-form solution as in [10–14]. It is worth noting that the existing approaches normally produce one or more particular solutions provided that they exist, rather than give a comprehensive complete solution to the problems.

In this paper, we consider a multidimensional extremal problem that is a generalization of problems examined in [10,11,14]. Based on the implementation of methods and techniques developed in [11,12,18], we obtain all the solutions of the problem in an explicit and closed form that is appropriate for both farther analysis and development of computation procedures.

2 Preliminary Definitions and Results

We start with a brief overview of basic concepts, notations, and results from [11, 12, 18] that underlie the main findings presented in the paper. Additional related details and thorough consideration can be found in [1–7].

2.1 Idempotent Semifield

Let $\mathbb{X}$ be a set equipped with two operations, addition $\oplus$ and multiplication $\otimes$, with their respective neutral elements, zero $\mathbf{0}$ and identity $\mathbf{1}$. We assume $(\mathbb{X}, \mathbf{0}, \mathbf{1}, \oplus, \otimes)$ to be a commutative semiring where the addition is idempotent and the multiplication is invertible. Since the set of nonzero elements $\mathbb{X}_+ = \mathbb{X} \setminus \{\mathbf{0}\}$ forms a group under multiplication, such semiring is usually referred to as an idempotent semifield.

The power notation with integer exponent is routinely defined in the semifield to represent iterated multiplication. Moreover, the integer power is assumed to be extendable to rational exponents, and so the semifield is taken to be radicable.

From here on, as is customary in conventional algebra, we drop the multiplication sign $\otimes$. The power notation is used in the sense of idempotent algebra.

The idempotent addition naturally induces a partial order in the semifield. We suppose that the partial order can be completed into a linear order and thus consider the semifield as totally ordered. In what follows, the relation signs are thought of as referring to this linear order.

Examples of totally ordered idempotent radicable semifields include

$$\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +), \quad \mathbb{R}_{\min,+} = (\mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, +),$$
$$\mathbb{R}_{\max,\times} = (\mathbb{R}_+ \cup \{0\}, 0, 1, \max, \times), \quad \mathbb{R}_{\min,\times} = (\mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times),$$

where $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$.

2.2 Idempotent Semimodule

Consider the Cartesian power $\mathbb{X}^n$ with its elements represented as column vectors. A vector with all zero entries is the zero vector denoted by $\mathbf{0}$. Vector addition $\oplus$ and multiplication by scalars $\otimes$ are routinely defined componentwise on the basis of the scalar addition and multiplication on $\mathbb{X}$.

The set $\mathbb{X}^n$ with the above operations is an idempotent semimodule over $\mathbb{X}$.

A vector is called regular if it has no zero entries. The set of all regular vectors of order n over $\mathbb{X}_+$ is denoted by $\mathbb{X}_+^n$.

For any nonzero column vector $\boldsymbol{x} = (x_i) \in \mathbb{X}^n$, we introduce a row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbf{0}$, and $x_i^- = \mathbf{0}$ otherwise, $i = 1, \dots, n$.

2.3 Matrix Algebra

For conforming matrices with elements in $\mathbb{X}$, matrix addition and multiplication together with multiplication by scalars are performed according to the standard rules applied to the scalar operations defined in $\mathbb{X}$.

A matrix with all zero entries is the zero matrix which is denoted by $\mathbf{0}$.

Consider the set of square matrices $\mathbb{X}^{n\times n}$. Any matrix that has all off-diagonal entries equal to $\mathbf{0}$ is called diagonal. The diagonal matrix with all diagonal entries equal to $\mathbf{1}$ is the identity matrix denoted by I. With respect to matrix addition and multiplication, the set $\mathbb{X}^{n\times n}$ is an idempotent semiring with identity.

For every matrix $A = (a_{ij})$, its trace is given by

$$\operatorname{tr} A = \bigoplus_{i=1}^{n} a_{ii}.$$

A matrix is reducible if it can be put in a block-triangular (normal) form by simultaneous permutations of rows and columns. Otherwise the matrix is irreducible.

The normal form of a matrix $A \in \mathbb{X}^{n\times n}$ is given by

$$A = \begin{pmatrix} A_{11} & \mathbf{0} & \dots & \mathbf{0} \\ A_{21} & A_{22} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix}, \tag{2}$$

where A_{ii} is an irreducible or zero matrix of order n_i, A_{ij} is an arbitrary matrix of size $n_i \times n_j$ for all $j < i$ with $i = 1, \dots, s$, and $n_1 + \cdots + n_s = n$.

2.4 The Spectrum of Matrices

Every matrix $A \in \mathbb{X}^{n\times n}$ defines on $\mathbb{X}^n$ a linear operator that possesses specific spectral properties.

If the matrix A is irreducible, then it has only one eigenvalue given by

$$\lambda = \bigoplus_{m=1}^{n} \operatorname{tr}^{1/m}(A^m). \tag{3}$$

The corresponding eigenvectors of A have no zero entries.

Suppose the matrix A is reducible and takes the normal form (2). The eigenvalues of A are found among the eigenvalues λ_i of diagonal blocks A_{ii}, $i = 1,\ldots,s$. The sum $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_s$ always provides an eigenvalue for A. The eigenvalue is given by (3), and usually referred to as the spectral radius of A.

2.5 Linear Inequalities

Given a matrix $A \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbb{X}^n$, the problem is to find the regular solutions $\boldsymbol{x} \in \mathbb{X}_+^n$ of a linear inequality

$$A\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}. \tag{4}$$

To solve the inequality we use an approach based on the application of a function $\operatorname{Tr}(A)$ that takes each square matrix A to a scalar according to the definition

$$\operatorname{Tr}(A) = \bigoplus_{m=1}^{n} \operatorname{tr} A^m.$$

The solution also involves evaluation of the matrix

$$A^* = I \oplus A \oplus \cdots \oplus A^{n-1}.$$

Consider a matrix A that has the normal form (2). We introduce matrices

$$D = \begin{pmatrix} A_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_{ss} \end{pmatrix}, \qquad T = \begin{pmatrix} \mathbf{0} & \ldots & \ldots & \mathbf{0} \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{s1} & \ldots & A_{s,s-1} & \mathbf{0} \end{pmatrix}$$

that respectively represent the diagonal part and the lower triangular part in the additive decomposition of A in the form

$$A = D \oplus T. \tag{5}$$

If the matrix A is irreducible and so $s = 1$ in (2), we put $D = A$ and $T = \mathbf{0}$.

Lemma 1 *Let $\boldsymbol{x}$ be the regular solution of inequality* (4) *with a matrix A in the form* (5). *Then the following statements hold:*

1. *if* $\mathrm{Tr}(A) \leq \mathbf{1}$, *then* $\boldsymbol{x} = (D^* T)^* D^* \boldsymbol{u}$ *for any* $\boldsymbol{u} \in \mathbb{X}_+^n$ *such that* $\boldsymbol{u} \geq \boldsymbol{b}$*;*
2. *if* $\mathrm{Tr}(A) > \mathbf{1}$, *then there are no regular solutions.*

3 Multidimensional Extremal Problems

Now we consider examples of multidimensional optimization problems formulated in the idempotent algebra setting. Such problems consist in minimizing linear or nonlinear functionals defined on finite-dimensional semimodules over idempotent semifields, and may have additional constraints imposed on the feasible solution set in the form of linear equations and inequalities.

We start with an idempotent analogue of linear programming problems investigated in [7, 8]. The problem is formulated in terms of the semimodule $\mathbb{R}_{\max,+}$ and consists in finding the vectors $\boldsymbol{x}$ to provide

$$\begin{gathered} \min \ \boldsymbol{p}^T \boldsymbol{x} \\ \text{subject to} \\ A\boldsymbol{x} \oplus \boldsymbol{b} \leq C\boldsymbol{x} \oplus \boldsymbol{d}, \end{gathered}$$

where $\boldsymbol{p}$, $\boldsymbol{b}$, and $\boldsymbol{d}$ are given vectors, and A and B are given matrices.

To solve the problem, an algorithm based on an alternating method is proposed that finds a solution with an iterative computation scheme provided that solutions exist, or indicates that there is no solution otherwise.

In [9], the above approach is extended to attack problems with nonlinear objective functions that take the form

$$\begin{gathered} \min \ (\boldsymbol{p}^T \boldsymbol{x} \oplus r)(\boldsymbol{q}^T \boldsymbol{x} \oplus s)^{-1} \\ \text{subject to} \\ A\boldsymbol{x} \oplus \boldsymbol{b} \leq C\boldsymbol{x} \oplus \boldsymbol{b}. \end{gathered}$$

There are extremal problems that can be solved explicitly to obtain closed-form solutions. Specifically, it is shown in [13] how to obtain in closed form regular solutions for the problem

$$\begin{gathered} \min \ (\boldsymbol{x}^- \boldsymbol{p} \oplus \boldsymbol{q}^- \boldsymbol{x}) \\ \text{subject to} \\ A\boldsymbol{x} \leq \boldsymbol{x}. \end{gathered}$$

Consider a problem to find vectors $\boldsymbol{x}$ that provides

$$\min_{\boldsymbol{x}\in\mathbb{X}_{+}^{n}} \boldsymbol{x}^{-}A\boldsymbol{x}.$$

Explicit solutions to the problem are proposed in [10–12, 17] on the basis of results in the spectral theory of matrices in idempotent algebra. Below we examine a more general problem and give all solutions to the problem.

4 Solution to an Optimization Problem

Given a nonzero matrix $A \in \mathbb{X}^{n\times n}$ and a vector $\boldsymbol{b} \in \mathbb{X}^{n}$, the optimization problem of interest is to evaluate

$$\min_{\boldsymbol{x}\in\mathbb{X}_{+}^{n}} (\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b}), \tag{6}$$

and find all vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that provide the minimum.

Particular cases of the problem arise in various application including estimation of the growth rate of state vector in stochastic discrete event dynamic systems [12, 17] and solution of single facility minimax location problems with Chebyshev and rectilinear distances [10, 14].

The solution to the problem is given by the following result.

Theorem 1 *Let $A \in \mathbb{X}^{n\times n}$ be a nonzero matrix in the form* (2) *with spectral radius λ, $\boldsymbol{b} \in \mathbb{X}^{n}$ be a vector, $A_{\lambda} = \lambda^{-1}A$, and $\boldsymbol{b}_{\lambda} = \lambda^{-1}\boldsymbol{b}$. Suppose $A_{\lambda} = D_{\lambda} \oplus T_{\lambda}$, where D_{λ} is the diagonal part and T_{λ} is the lower triangular part of A_{λ}.*

Then it holds that

$$\min_{\boldsymbol{x}\in\mathbb{X}_{+}^{n}} (\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b}) = \lambda,$$

where the minimum is attained if and only if

$$\boldsymbol{x} = (D_{\lambda}^{*}T_{\lambda})^{*}D_{\lambda}^{*}\boldsymbol{u}$$

for any regular vector $\boldsymbol{u} \geq \boldsymbol{b}_{\lambda}$.

Proof Let us show that λ is a lower bound for the objective function in (6). First suppose that the matrix A is irreducible and has a unique eigenvalue λ.

To verify that λ is a lower bound, we assume x_{0} to be an eigenvector of A. Since it holds that

$$\boldsymbol{x}_{0}^{-}\boldsymbol{x}_{0} = \mathbf{1}, \qquad \boldsymbol{x}\boldsymbol{x}_{0}^{-} \geq (\boldsymbol{x}^{-}\boldsymbol{x}_{0})^{-1}I,$$

for all $\boldsymbol{x} \in \mathbb{X}_{+}^{n}$ we have

$$\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b} \geq \boldsymbol{x}^{-}A\boldsymbol{x}\boldsymbol{x}_{0}^{-}\boldsymbol{x}_{0} \geq \boldsymbol{x}^{-}A\boldsymbol{x}_{0}(\boldsymbol{x}^{-}\boldsymbol{x}_{0})^{-1} = \lambda\boldsymbol{x}^{-}\boldsymbol{x}_{0}(\boldsymbol{x}^{-}\boldsymbol{x}_{0})^{-1} = \lambda.$$

Now consider an arbitrary matrix A in the form (2). Take any vector $\boldsymbol{x} \in \mathbb{X}_{+}^{n}$ and split it into subvectors $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{s}$ according to the row partition of A. By applying the previous result for irreducible matrices, we get

$$\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b} \geq \boldsymbol{x}^{-}A\boldsymbol{x} \geq \bigoplus_{i=1}^{s}\bigoplus_{j=1}^{s}\boldsymbol{x}_{i}^{-}A_{ij}\boldsymbol{x}_{j} \geq \bigoplus_{i=1}^{s}\boldsymbol{x}_{i}^{-}A_{ii}\boldsymbol{x}_{i} \geq \bigoplus_{i=1}^{s}\lambda_{i} = \lambda.$$

Let us find all regular solutions of the equation

$$\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b} = \lambda.$$

Under the condition $\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b} \geq \lambda$ for all $\boldsymbol{x} \in \mathbb{X}_{+}^{n}$, the solutions coincide with those of the inequality

$$\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{b} \leq \lambda.$$

It is not difficult to see that the last inequality is equivalent to the inequality

$$A_{\lambda}\boldsymbol{x} \oplus \boldsymbol{b}_{\lambda} \leq \boldsymbol{x}.$$

Indeed, multiplication of both sides of this inequality by $\lambda\boldsymbol{x}^{-}$ gives the first inequality. On the other hand, by multiplying the first inequality by $\lambda^{-1}\boldsymbol{x}$ and using that $\boldsymbol{x}\boldsymbol{x}^{-} \geq I$, we have

$$A_{\lambda}\boldsymbol{x} \oplus \boldsymbol{b}_{\lambda} \leq \lambda^{-1}\boldsymbol{x}\boldsymbol{x}^{-}A\boldsymbol{x} \oplus \lambda^{-1}\boldsymbol{x}\boldsymbol{x}^{-}\boldsymbol{b} \leq \boldsymbol{x},$$

and thus arrive at the second inequality.

By applying Lemma 1 to this inequality, we get the solution

$$\boldsymbol{x} = (D_{\lambda}^{*}T_{\lambda})^{*}D_{\lambda}^{*}\boldsymbol{u},$$

where $\boldsymbol{u}$ is any regular vector such that $\boldsymbol{u} \geq \boldsymbol{b}_{\lambda}$.

References

1. F. L. Baccelli, G. Cohen, G. J. Olsder, J.-P. Quadrat, *Synchronization and linearity: An algebra for discrete event systems*, Wiley, Chichester, 1993.
2. R. A. Cuninghame-Green, *Minimax algebra and applications*, Advances in Imaging and Electron Physics, Vol. 90 (1994), Academic Press, San Diego, pp. 1–121.
3. V. N. Kolokoltsov, V. P. Maslov, *Idempotent analysis and its applications*, Kluwer, Dordrecht, 1997.
4. J. S. Golan, *Semirings and affine equations over them: Theory and applications*, Springer, New York, 2003.
5. B. Heidergott, G. J. Olsder, J. van der Woude, *Max-Plus at work: Modeling and analysis of synchronized systems*, Princeton Univ. Press, Princeton, 2005.
6. G. L. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: A brief introduction*, Journal of Mathematical Sciences **140** (2007), no. 3, 426–444. E-print arXiv:math.GM/0507014.
7. P. Butkovič, *Max-linear systems: Theory and algorithms*, Springer, London, 2010.
8. P. Butkovič, A. Aminu, *Introduction to max-linear programming*, IMA Journal of Management Mathematics **20** (2008), no. 3, 233mïS-249.
9. S. Gaubert, R. D. Katz, S. Sergeev, *Tropical linear programming and parametric mean payoff games*, to be published in Journal of Symbolic Computation, 2012. E-print arXiv:1101.3431.
10. N. K. Krivulin, *An algebraic approach to multidimensional minimax location problems with Chebyshev distance*, WSEAS Transaction on Mathematics **10** (2011), no. 6, 191–200.
11. N. K. Krivulin, *Eigenvalues and eigenvectors of matrices in idempotent algebra*, Vestnik St. Petersburg University: Mathematics **39** (2006), no. 2, 72–83.
12. N. K. Krivulin, *Methods of idempotent algebra in problems of complex systems modeling and analysis*, St. Petersburg Univ. Press, St. Petersburg, 2009. (in Russian)
13. N. K. Krivulin, *A new algebraic solution to multidimensional minimax location problems with Chebyshev distance*, to be published in WSEAS Transaction on Mathematics, 2012.
14. N. K. Krivulin, *An extremal property of the eigenvalue for irreducible matrices in idempotent algebra and an algebraic solution to a Rawls location problem*, Vestnik St. Petersburg University: Mathematics **44** (2011), no. 4, 272–281.
15. K. Zimmermann, *Disjunctive optimization, max-separable problems and extremal algebras*, Theoretical Computer Science **293** (2003), no. 1, 45–54.
16. A. Tharwat, K. Zimmermann, *One class of separable optimization problems: Solution method, application*, Optimization **59** (2008), no. 5, 619–625.
17. N. K. Krivulin, *Evaluation of bounds on the mean rate of growth of the state vector of a linear dynamical stochastic system in idempotent algebra*, Vestnik St. Petersburg University: Mathematics **38** (2005), no. 2, 42–51.
18. N. K. Krivulin, *Solution of generalized linear vector equations in idempotent algebra*, Vestnik St. Petersburg University: Mathematics **39** (2006), no. 1, 16–26.

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Versions of the Engel theorem for semigroups

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1. Nilpotent and quasinilpotent semigroups

Let S be an abstract semigroup with a subsemigroup S_0. We shall say that S is an *n-extension* of its subsemigroup S_0 if every product of n elements of S is an element of S_0; in this case we shall say that S is *quasinilpotent* if S_0 is commutative. The semigroup S is called *nilpotent* if it is quasinilpotent for S_0 consisting of a single zero element 0, so $S^n = 0$, see [5]. Of course, every commutative semigroup is quasinilpotent.

2. Versions of the Engel theorem for semigroups in traditional mathematics

Theorem 1 *Let S be a nilpotent or quasinilpotent semigroup of finite dimensional complex matrices. Then there exists a common nonzero eigenvector for all elements of S.*

Theorem 2 *Let S be a nilpotent or quasinilpotent semigroup of finite dimensional matrices with nonnegative entries. Then there exists a common nonzero nonnegative eigenvector for all elements of S.*

This result is a generalization of the the well known Perron-Frobenius theorem.

3. Nilpotent and quasinilpotent semigroups of operators in Archimedean tropical linear spaces

We use the terminology from [2, 3, 6] and discuss semigroups of operators in b-complete linear idempotent spaces over the tropical semifield $\mathbb{R}_{\max}$, i.e. in tropical linear b-complete spaces. Let V and W be spaces of this type. We shall say that a linear operator $A : V \to W$ is *continuous* if $A(\sup X) = \sup A(X)$ and $A(\inf X) = \inf A(X)$ for every bounded linearly ordered subset X in V (with respect to the corresponding induced order). Note that every operator of this type is b-linear.

A linear functional ϕ on V is called *bounded*, if the set $\{x \in V | \phi(x) \preceq \mathbf{1}\}$ is bounded.

The space V is called *Archimedean* [6], if there exists a bounded continuous linear functional f on V such that $f(x)$ is nonzero for every nonzero $x \in V$. Examples of Archimedean spaces can be found in [3, 6].

Theorem 3 *Let S be an arbitrary set of commuting continuous linear operators on an Archimedean space V. Then there exists a common nonzero eigenvector for all elements of S.*

Theorem 4 *Let S be a nilpotent or quasinilpotent semigroup of continuous linear operators on an Archimedean space. Then there exists a common nonzero eigenvector for all elements of S.*

The following result is a trivial corollary of this theorem.

Theorem 5 *Let S be a nilpotent or quasinilpotent semigroup of finite dimensional matrices with tropical entries. Then there exists a common nonzero eigenvector for all elements of S.*

Note that our Theorems 2 and 5 extend a result of Katz, Schneider and Sergeev [1] on existence of a joint eigenvector of several commuting matrices. A different theorem on existence of a joint eigenvector for all matrices in a "nice" semigroup consisting of strongly regular matrices over the tropical semifield was recently obtained by Merlet [4].

The authors thank S.N. Sergeev for his comments and references.

References

1. *Katz R.D., Schneider H., Sergeev S.* On commuting matrices in max algebra and in nonnegative matrix algebra // Linear Algebra and its Applications, vol. 436, issue 2, 2012, 276-292.

2. *Litvinov G.L., Maslov V.P., Shpiz G.B.* Idempotent Functional Analysis. An algebraic approach //Mathematical Notes, vol. 69, No.5, 2001, p. 696–729. E-print math.FA/0009128 (http://arXiv.org).
3. *Litvinov G.L., Maslov V.P., Shpiz G.B.* Idempotent (asymptotic) mathematics and the representation theory. - In: V.A. Malyshev and A.M. Vershik (eds.), Asymptotic Combinatorics with Applications to Mathematical Physics. Kluwer Academic Publishers, Dordrecht et al, 2002, p. 267-278. E-print math.RT/0206025 (http://arXiv.org).
4. *Merlet G.* Semigroup of matrices acting on the max-plus projective space, Linear Algebra and its Applications, vol. 433, issue 8, 2010, 1923-1935.
5. *Nilpotent semi-group.* – In: Encyclopedia of Mathematics. E-print http://www.encyclopediaofmath.org
6. *Shpiz G.B.* A theorem on eigenvectors in idempotent spaces // Doklady Mathematics, vol. 62, No.2, 2000, p. 169–171.

This work is supported by the grant RFBR 12-01-00886-a and the joint RFBR-CNRS grant 11-01—93106-a.

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Tropical computations in Mathpar

Gennadi Malaschonok

1 Introduction

Objects studied in physics, chemistry, biology require more precise and efficient means to describe and create adequate mathematical models. Instead of simple numerical models popular in the past, we now have analytical models that require advanced analytical tools for their construction.

Computer algebra systems like MAPLE, Mathematica, CoCoA, Reduce, etc., designed for single-core machines do not allow to create compound analytical models. New software tools designed for SPMD Cluster Computing with thousands of nuclei are required today.

The history of numerical linear algebra packages (ScaLAPAC, LAPAC and others) shows that multi-processor systems require the creation of new software systems for which the data structures and algorithms have been initially focused on parallel computing systems. So we need a fundamentally new parallel programming systems for numerically-analytical calculations, in which data structures and algorithms will be designed for supercomputers.

There are several fundamental symbolic matrix algorithms developed during the last 20 years, with the same computational complexity as in the algorithms of matrix multiplication. These algorithms are basic for solving linear algebra problems in the commutative domains and in the domains of principal ideals. Most of these algorithms are implemented as parallel MPI-programs.

Mathpar is a computer algebra system aimed at performing computations in arbitrary algebras with high efficiency.

The Mathpar language is called ATeX. This is an *active* version of the TeX language, allowing to perform operations and to write procedures and functions like in TeX. Mathpar is a web service at http://mathpar.com. A tutorial of Mathpar and many help pages of this on-line mathematical service can be found on this website. In addition to traditional algebra you can perform calculations in idempotent algebras. This is the subject of the present report.

2 Environment for mathematical objects

The definition of any mathematical object, a number or a function, a matrix or a symbol, involves the definition of some environment, that is, the space which contains this object. To select the environment you have to set the *algebraic structure*. This algebraic structure is defined by numeric sets, algebraic operations in these sets and variable names.

First of all, any user has to set an environment in Mathpar. By default, the space of six integer variables $\mathbb{Z}[x, y, z]\mathbb{Z}[u, v, w]$ is defined. This is a ring of polynomials with coefficients in the ring of integer numbers. The variables are divided into two groups of three variables: variable x is the "youngest", and variable w is the "eldest". The variables are arranged from left to right.

User can change the environment, setting a new algebraic structure. For example the space $\mathbb{R}64[x, y, z]$ may be suitable to solve many problems of computational mathematics. The installation command should be as follows: "SPACE = R64 [x, y, z]; ".

Moving a mathematical object from the previous environment to the current environment is performed explicitly, using the $toNewRing()$ function. In some cases, such a transformation to the current environment is automatic.

All other names not listed as variables can be chosen by the user arbitrarily, for any mathematical object.

For example

$$a = x + 1, \quad f = \backslash sin(x + y) - a.$$

There is a rule to distinguish between commutative and noncommutative algebra. If the object name begins with a *capital letter*, then the object is an element of a *noncommutative* algebra. If the object name begins with a *lowercase letter*, then the object is an element of a *commutative* algebra.

3 Numerical sets with standard operations

Current version of the system supports the following numerical sets with standard operations.

Z — the set of integers $\mathbb{Z}$,

Zp — a finite field $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number,

Zp32 — a finite field $\mathbb{Z}/p\mathbb{Z}$ where p is less 2^{31},

Z64 — the ring of integer numbers z such that $-2^{63} \leqslant z < 2^{63}$,

Q — the set of rational numbers,

R — the set of floating point numbers to store approximate real numbers with arbitrary mantissa,

R64 — standard floating-point 64-bit numbers (52 digits for mantissa, 11 bits for the order and 1 bit for the sign),

R128 — standard floating-point 64-bit numbers, equipped with optional 64-bit for the order,

C — complexification of R,

C64 — complexification of R64,

C128 — complexification of R128,

CZ — complexification of of Z,

CZp — complexification of Zp,

CZp32 — complexification of Zp32,

CZ64 — complexification of Z64,

CQ — complexification of Q.

Examples of simple commutative polynomial rings:
SPACE = Z [x, y, z]; SPACE = R64 [u, v]; SPACE = C [x].

4 Several numerical sets

The ring $\mathbb{Z}[x, y, z]\mathbb{Z}[u, v, w]$, which has two subsets of variables, is the polynomial ring with variables u, v, w with coefficients in the polynomial ring $\mathbb{Z}[x, y, z]$.

For example, the characteristic polynomial of a matrix over the ring $\mathbb{Z}[x, y, z]$ can be obtained as a polynomial of variable u, whose coefficients are polynomials in the ring $\mathbb{Z}[x, y, z]$.

You can set algebraic space which defines several numerical sets. For example, the space $C[z]R[x,\ y]Z[n,\ m]$ allows to have five names of variables, which belong to the sets $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively. The first set is the main one.

$C[z]R[x, y]Z[n, m]$ can be viewed as a polynomial ring of five variables over $\mathbb{C}$, with additional properties. If the polynomial does not contain variables z, x,

y, then it is a polynomial with coefficients in the set $\mathbb{Z}$. If the polynomial does not contain z, then it is a polynomial with coefficients in $\mathbb{R}$. **Examples:** SPACE=Z[x, y]Z[u]; SPACE=R64[u, v]Z[a, b]; SPACE=C[x]R[y, z];

5 Group algebras

Group algebra is defined as KG, where K is a commutative ring of scalars and G — is a group of noncommutative operators with finite number of generators. Names of these generators should begin with capital letters.

For example, the following group algebras can be defined:

$SPACE = Z[x, y]G[U, V]$; (generators U, V),

$SPACE = R64[u, v]G[A, B]$; (generators A, B),

$SPACE = C[]G[X, Y, Z, T]$; (generators X, Y, Z, T).

Each element of such algebra can be considered as a sum of terms with functional coefficients.

$R64[t, y]G[X,\ Y,\ Z]$ is the free group algebra over a function field of two variables t, y over the field $\mathbb{R}64$ with three non-commuting generators X, Y, Z. For example, $A = (t^2 + 1)X + \sin(t)Y + 3X^2y^3 + (t^2 + 1)XY^3X^2Y^{-2}X^2$ is an element of such algebra.

6 Constants

It is possible to set or replace the following constants.

ACCURACY — the amount of exact decimal positions in the fractional part of real numbers of type R resulting from multiplication or division operation.

FLOATPOS —the amount of decimal positions of real number of type R or R64, which you can see in the printed form.

ZERO_R — the machine zero for R and C numbers.

ZERO_R64 — the machine zero for R64, R128, C64 and C128 numbers.

MOD32 — the module for a finite field of type Zp32, its value is not greater than 2^{31}.

MOD — the module for a finite field of type Zp.

To set the machine zero to $1/10^9$ (i.e., $1E - 9$), you can use the commands $ZERO\_R = 9$ or $ZERO\_R64 = 9$.

Example.

```
SPACE=Zp32[x, y];   MOD32=7;
f1=37x+42y+55;   f2=2f1;
\print(f1, f2 );
```

The results: $f1 = 2x - 1$; $f2 = 4x + 5$.

7 Idempotent algebra and tropical mathematics

Idempotent/tropical algebras can be used as well. In this case the signs of "addition" and "multiplication" for the infix operations can be used for operations in tropical algebra: min, max, addition, multiplication.

Each numerical set $\mathbb{R}$, $\mathbb{R}64$ or $\mathbb{Z}$ has two additional elements ∞ and $-\infty$, and two elements that play the role of zero and unit. We denote these sets $\hat{\mathbb{R}}$, $\hat{\mathbb{R}}64$, $\hat{\mathbb{Z}}$, respectively. The name of idempotent algebra consists of three components: (1) the numerical set, (2) the operation corresponding to *plus* and (3) the operation corresponding to *times*.

For example, algebras $R64MaxPlus$, $R64MinPlus$, $R64MaxMin$, $R64MinMax$, $R64MaxMult$, $R64MinMult$ are defined over the numerical set $\hat{\mathbb{R}}64$.

RMaxPlus, RMinPlus, RMaxMin, R64MinMax, RMaxMult, RMinMult are defined over the numerical set $\hat{\mathbb{R}}$.

ZMaxPlus, ZMinPlus, ZMaxMin, ZMinMax, ZMaxMult, ZMinMult are defined over the numerical set $\hat{\mathbb{Z}}$.

Let us give an example of tropical arithmetics in $ZMaxPlus$.

Example.

```
SPACE=ZMaxPlus[x, y];
a=2; b=9+x; c=a+b; d=a*b+y; \print(c, d)
```

Results: $c = x + 9$; $d = y + 2 * x + 11$.

For each algebra we defined elements **0** and **1**, $-\infty$ and ∞. For each element a we defined the operation of closure (Kleene star): $a^{\times} = 1 + a + a^2 + a^3 + ...$ This operation is an analogue of the classical $(1 - a)^{-1}$.

8 The calculations on a supercomputer

In order to solve computational problems that require large computation time or large amounts of memory, the system has special functions that provide the user with resources of supercomputer. These functions allow you to perform calculations not on a single processor and on a dedicated set of cores of supercomputer. The number of cores is ordered by the user.

You have the following functions (*parfunctions*) that can be performed on supercomputer:

1) *gbasisPar* — computation of Gröbner basis;
2) *adjointPar* — computation of adjoint matrix;
3) *adjointDetPar* — computation of adjoint matrix and determinant of a matrix;
4) *echelonFormPar* — computation of matrix echelon form;
5) *inversePar* — computation of inverse matrix;
6) *detPar* — computation of determinant of a matrix;
7) *kernelPar* — computation of kernel of a linear operator;
8) *charPolPar* — computation of characteristic polynomial;
9) *multiplyPar* — calculation of matrix product;
10) *multiplyPar* — computation of product of polynomials.

Before applying any of these functions, the user has to specify some parameters that define the parallel environment:

$TOTALPROCNUMBER$ — total number of processors (cores) provided for the computations,

$NODEPROCNUMBER$ — number of cores on a single node,

$CLUSTERTIME$ — maximum time (in minutes) which you allow for a program to execute (after which it is forced to end).

To set the number of cores on a single node the user has to know which cluster is used and how many cores are available on the node. By default, the $TOTALPROCNUMBER$ and $NODEPROCNUMBER$ are set so that all available cores are used by the node, and $CLUSTERTIME = 1$.

The user can change the number of cores on one node. This is an important feature, since the memory on a node is used by all cores on this node. Consequently, the user can regulate the size of RAM that is available at one core. For now, only users of the cluster of Tambov State University can perform parallel computing.

References

1. G. I.Malaschonok, *Project of Parallel Computer Algebra*, Tambov University Reports. Series: Natural and Technical Sciences. **15**. Issue 6. (2010), 1724–1729.
2. G. I.Malaschonok, Computer mathematics for computational network Tambov University Reports. Series: Natural and Technical Sciences. **15** Issue. 1. (2010), 322–327.
3. G. I.Malaschonok, *On the project of parallel computer algebra*, Tambov University Reports. Series: Natural and Technical Sciences. **14** Issue. 4. (2009), 744–748.

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Powers of matrices with an idempotent operation and an application to dynamics of spatial agglomerations

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Abstract The paper develops an approach to analyze dynamics of economic network structures with a considerable role of externalities, such as spatial agglomerations. The approach is based on a use of powers of matrices with an idempotent operation.

1 An idempotent algebraic model of economic development with positive externalities

The scheme of dynamic programming was studied by many authors beginning from Bellman [1] and probably most explicitly by Romanovsky [10] who showed that a sufficiently long T-step optimal path in its initial part is characterized by a so called (first) system of potentials which is a version of the Bellman's value function, and by use of this function the first part of the optimal path (up to its entrance into the turnpike) can be constructed stepwise. However these authors have missed some other natural and principally important conclusions. Similarly to the initial part, the final part of the path can be constructed "backward" (in the inverse time) stepwise by use of a function which can be referred to as the second system of potentials or the second value function. The latter corresponds the left eigenvector of the matrix of utilities in an algebra with an idempotent operation, while the first value function corresponds the right eigenvector. Each T-step optimal path under sufficiently large horizon T has a three-part structure which corresponds to the structure of the T-th power (with the idempotent operation) of the matrix of utilities (see [9]).

One of dynamic objects which can be modeled by use of the powers of a matrix with idempotent operation is an economic system with mutual externalities.

Externality is "any indirect effect that either a production or a consumption activity has on a utility function, a consumption set, or a production set. By "indirect" we mean both that the effect is created by an economic agent other than the one who is affected and that the effect is not transmitted through prices" ([5], p. 6). The role of externalities in spatial structures, such as countries, regions and cities, has been stressed by many authors following Marshall [7] who considered industry-specific district externalities as ensuring specialization and economies of scale. Jacobs [3] argued that knowledge can spill over between different complementary industries in the same location. Such sort of externalities is often referred as 'Jacobian'. Lucas [6] called them 'externalities of creative professions'. These externalities are not limited only by knowledge but can relate to any forms of mutual influence and dependence.

In [8] individual possibilities of the agents to develop can be limited by an insufficiency of the sizes of externalities created by other agents.

Let $i = 1, \cdots, n$ be economic agents (firms, authorities, groups of labor of different types, etc.). Each agent i in period t is characterized by a single positive number x_i^t referred as her *value*. It can be, for example, profit, income, welfare, present value, knowledge, etc. Development of the i-th agent is being described as changes in her value in discrete time. We make two assumptions:

1. Development of the i-th agent is limited by her own potential possibilities described by a fixed growth factor $a_{ii} > 0$:

$$x_i^{t+1} \le a_{ii} x_i^t,\ t = 0, 1, \cdots;\ i = 1, \cdots, n;$$

2. Development of the i-th agent is limited by limitations on positive externalities created by some other agents:

$$x_i^{t+1} \le a_{ij} x_j^t,\ j = 1, \cdots, n,\ j \ne i,\ t = 0, 1, \cdots;\ i = 1, \cdots, n;$$

where a_{ij} are coefficients describing limitations for the i-th agent's development caused by bounds of the positive externalities created by the j-th agent. Here $a_{ij} = +\infty$ if agent j creates no externality used by agent i or if the externality created by j never becomes binding for i. An externality is called *nonbiding* for an agent if it currently does not limit her development, i.e. the development is limited either by the agent's own growth factor or by another externality. An externality is *binding* if the development is currently limited by the externality.

If each agent maximizes her value stepwise under the present constraints this results in the following equations characterizing the equilibrium path given initial values $x_1^0, \cdots, x_n^0$:

$$x_i^{t+1} = \min_{j=1,\cdots,n} a_{ij} x_j^t, \ t = 0, 1, \cdots; \ i = 1, \cdots, n$$

which in terms of idempotent mathematics can be easily written as

$$x^{t+1} = A \otimes x^t, \ t = 0, 1, \cdots$$

where $A = (a_{ij})$ is an $n \times n$-matrix, x^t, x^{t+1} are positive n-vectors, and the product $A \otimes x^t$ is a product of a matrix and a vector with an idempotent operation $\oplus = \min$ and usual multiplication.

The behavior of the system $x^t = A^t \otimes x^0, \ t = 1, 2, \cdots$, where $A^t = A \otimes \cdots \otimes A$, is totally defined by properties of the matrix A. The system can demonstrate different patterns such as convergence to an eigenvector with stable growth factor, stable decline, convergence to a limit cycle, etc. An important result is that often a small change in an element of the matrix A can lead to a radical change of a pattern of behavior of the system (so called *butterfly effect* or *catastrophic bifurcation*).

It is important to notice that an externality can be nonbinding until an agent i is "small" but when the agent "grows up" she can face the limitation of an insufficient development of another agent or a group of agents.

An agent facing a binding externality could be interested in making payments improving her development. These payments can have numerous forms such as a redistribution of incomes through a mechanism of taxes and transfers, direct supply of products and resources (e.g., in form of barter or credit), supply of privileges, providing information, or, simply, bribes. However, as is seen from the model, a choice of a suitable direction of payment is a serious problem for the agent, which can have political consequences.

A model variable changeable coefficients can be considered. In a simple case it can be assumed that agent i is able to diminish her own development coefficient, a_{ii}, and, instead, to increase a development coefficient a_{ij} of any other agent or a coefficient of externality constraint a_{kl} for any pair of agents, k,l.

More realistic is a situation of *tranferable values* where agents are able to exchange not their development coefficients but values. For a model with matrix

$$A = \begin{pmatrix} 2 & 1 & +\infty \\ +\infty & 2 & 1 \\ +\infty & +\infty & 0.6 \end{pmatrix}$$

under the absence of redistribution the path leads to a proportional decline with a growth rate 0.6. Let the agents be able to partially exchange their values, and let the direction and the value of transfers be defined by the 1st agent (e.g. she has a majority in a parliament).

The 1st agent feels an externality from the side of the 2nd one, that is why she is interested in payments to the 2nd agent. In such case the 1st agent cares about maximization of her next current value; for her, as can be checked, it will be optimal to support the value of the 2nd agent on a level twice more than her own level. Such strategy leads to a temporary increase in the value of the 1st agent and then to a decline slower than under the absence of payments . The values of the 3rd agent are not touched by this policy at all.

However, there exists a possibility of a long-run growth of the whole system if the transfers are directed from the 1st agent not to the 2nd but to the 3rd one.

Notice that in this example an "optimal" (from the long-run point of view) governmental redistribution policy leads to a long-run growth of all agents, however, it does not give a short-run gain to the 1st agent, which defines a choice of a policy. If a subjective discounting coefficient of the 1st agent is not high enough, she will prefer a short-run gain, which will lead to a long-run decline.

The case, when the 1st agent executes a transfer to the 2nd agent whose insufficient externality is filled, can be also interpreted as a bribe, and a case, when a transfer is received by the 3rd agent with whom the 1st agent is not linked directly, as a result of an action of a fiscal system. Thus the model shows that the corruption provides an agent an evident gain in a short run but it does not prevent a decline in a long run. The governmental redistribution policy, in its turn, leads to a growth of all agents but provides no short-run gain to the 1st agent. Under a small subjective discount factor, the 1st agent will prefer a short-run gain which can be achieved through tax-avoiding and bribes.

In a general case of n agents, the model can be used to provide a choice of agents to be taxed and to define optimal values of taxes and transfers.

2 A model of spatial development

The approach can be applied to explaining such processes in spatial economic systems as migration of production factors, agglomeration and de-agglomeration (dispersion). Johansson and Quigley [4] in their discussion of agglomerations and networks mark that in the agglomeration economies externalities usually have a character of public good, while in the periphery market network relations prevail with a distribution of private institutional capital among the participants. ("When co-location is infeasible, networks may substitute for agglomeration"). However, these authors only mention a unity of the mechanisms of agglomeration and networking and do not explain the reasons of the difference in behavior of economic agents in the core and in the periphery. We try to answer this question here.

The specific character of the spatial economies lies in much in the presence of two ample sets of economic agents: (i) *stationary* agents condensed and accomplishing their activities permanently in definite points or areas in space, and (ii) *free* agents who are able to move their activities in the space more easily. Economic processes of the locational movement of productive factors and consumers, such as migration, offshoring, agglomeration and de-agglomeration can be represented as moves of the free agents generated by dynamics of economic structures connected first of all with the effectiveness of the stationary agents. From dynamics perspective these structures are related first of all with numerous mutual externalities created by agents in their locations. The concentrations of agents of different types is the most important characteristic of any geographic location.

As a simplest example of such dynamic model let us consider the case of two stationary agents: 1 in a core, 2 in a periphery and one free agent: 3. The individual potential growth factor of a stationary agent in the core is higher than in the periphery: $a_{11} > a_{22}$. Let $a_{11} = 1.2,\ a_{22} = 1.5,\ a_{33} = 1.5,\ a_{31} = 0.4,\ a_{32} = 2$. Other four elements of the matrix A are equal to $a_{ij} = +\infty$. The dynamics of the free agent is described by the equation:

$$x_3^{t+1} = \min\{a_{33}x_3^t, \max\{a_{31}x_1^t, a_{32}x_2^t\}\}.$$

Let the free agent be initially in the core, and let the initial state of the model be $(x_1^0 = 2, x_2^0 = 0.7, x_3^0 = 0.5)$. The free agent faces initially no external limitation, but on step 1 she faces a binding external limitation in the core and moves to the periphery. On step 2 she acts in the periphery and does not face binding external limitations there. Further on steps 3-6 she faces a binding limitation of an insufficient externality but still prefers to stay in the periphery. However, on step 7 the free agent chooses to return to the core.

The next stage of the research is introducing *transferable values*: now the agents are allowed to transfer a part of their values to other agents. These transfers, in particular, can be in form of an aid, an investment, a tax, a bribe, etc. In case of transferable values, growth rates of all agents and locations of free agents are endogenous variables.

To continue the example, if the free agent is interested in maximization of her own growth (but not in the maximal current value as previously) and if she can share her value with a stationary agent, she stays forever in the periphery and the periphery starts developing faster than the core.

It can be shown that the growth factor of the system, consisting of a stationary agent $i = 1$ or $i = 22$ (the cases when the stationary agent is either in the core or in the periphery) and a free agent 3, is equal to

$$a_{i3} = \frac{a_{33}(a_{ii} + a_{3i})}{a_{33} + a_{3i}}.$$

It follows that the condition that the periphery is more preferable for the free agent than the core is

$$a_{11} + a_{31}\left(1 - \frac{a_{22}}{a_{33}}\right) < a_{22} + a_{32}\left(1 - \frac{a_{11}}{a_{33}}\right).$$

The resulting *index of the i-th region*,

$$a_{ii} + a_{3i}\left(1 - \frac{a_{jj}}{a_{33}}\right), \ j \neq i,$$

includes terms related to the region's "stationary" growth factor, a_{ii} , the externality limitation in the region, a_{3i} , as well as (with opposite sign) the relative growth factor in the alternative location, $\frac{a_{jj}}{a_{33}}$.

Interesting are situations when behavior of free agents such as groups of qualified labor depend on behavior of free firms. First free firms move to the periphery and after some time the free qualified labor also tends to move to the periphery. An important question is identifying conditions under which all of them return to the core and those conditions under which they stay forever in the periphery. Both forward linkages in terms of Krugman [2] (the incentive of workers to be close to the producers of consumer goods) and backward linkages (the incentive of producers to concentrate where the market is larger) can be easily taken into account in the model.

Trade between cities, regions or countries also can be included into the present model. To do this, a part of the coefficients a_{ij} can be interpreted as limitations related to potential trade flows between the agents, and if these flows are actualized it means presence of trade between the areas.

The model provides also an insight to policy analysis. For example, trade liberalization and FDI can lead to an appearance of new external links, in particular, in the periphery.

References

[1] R. Bellman Dynamic programming. Princeton: Princeton University Press, 1957.

[2] M. Fujita, P. Krugman. The new economic geography: past, present, and the future. Papers Reg. Sci. bf83 (2004), 139–164.

[3] J. Jacobs The economy of cities. New York: Random House, 1969.

[4] B. Johansson and J.M. Quigley Agglomeration and networks in spatial economies. Papers Reg. Sci. **83** (2004), 165–176.

[5] J.-J. Laffont. Fundamentals of public economics. Cambridge, MA: MIT Press, 1988.

[6] R.E. Lucas. On the mechanics of economic development. Journal of Monetary Economics **22** (1988), 3–42.

[7] A. Marshall Principles of economics. London: Macmillan, 1890.

[8] V. Matveenko Development with positive externalities: The case of the Russian economy. Journal of Policy Modeling **17** (1995), 207-221.

[9] V. Matveenko Optimal paths in oriented graphs and eigenvectors in max-$\otimes$ systems. Discrete Mathematics and Applications **19** (2009), 389-409.

[10] J.V. Romanovsky Optimization of stationary policy of discrete deterministic process. Kibernetika **2** (1967), 71-83 (In Russian).

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Games of network disruption and idempotent algorithm complexity

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Abstract We consider a game on the space of network disruptions. An application in command and control is used as a guide for the development of the model. The outcome of any set of physical actions depends on the information available to the controller. We suppose that information flows along a network of humans and machines. The opposing player may act to intermittently block information flow along the network. This may be combined with physical actions such that our controller might be making decisions based on information that is not as current as would be possible without the induced delays. We find that the model of information delay dynamics is best captured as a controlled min-plus linear system. We also find that the minimax value function may be represented as a min-plus convex functional over the space of delay vectors. This is a max-min linear space. Backward dynamic programming propagation of the value function leads to (max-min) sum and product compositions. This yields a particularly nice solution algorithm. Computational and solution-representation complexity are examined.

1 Introduction

We consider a game on the space of network disruptions. An application in command and control is used as a guide for the development of the model. However, it should be clear that the problem class addressed by the theory is much larger. There is a physical domain via which the ultimate rewards are obtained. The outcome of any set of physical actions depends on the information available to the controller. We suppose that information flows along a network of humans and machines, where information processing may occur may occur at some of the nodes. The opposing player may act to intermittently block information flow along the network. This may be combined with physical actions such that our controller might be making decisions based on information that is not as current as would be possible without the disruption.

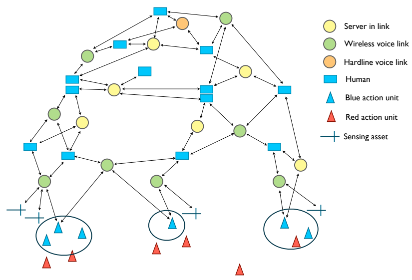
We will find that the model of information delay is best captured as a min-plus linear system. Specifically, the delay of information from node g at node γ is the minimum over the delays along available information-routes from node g to node γ. The opposing control may act to increase delays while our controller may seek to reduce such delays, leading to a game with controlled min-plus linear dynamics.

We will find that the value function may take the form of a min-plus convex functional over the space of delay vectors. Note that in this component of the analysis, we work in min-plus vector spaces. Recall that the space of standard-sense convex functions is a max-plus linear space (c.f., [12]), where finite-complexity convex functions are those given as pointwise maxima of a finite set of standard-sense affine functions. That is, they may be represented as max-plus linear combinations of standard-sense linear functions. In the work here, the standard-sense algebra is replaced by the min-plus algebra and the max-plus algebra is replaced by the max-min algebra. Consequently, min-plus convex functions are represented as max-min linear combinations of min-plus linear functions.

We will want to consider larger scale networks. Thus, the complexity of the solution representation is a critical question. We are led to questions of the complexity of max-min sums and products of finite-complexity min-plus functions. We find that this complexity grows more slowly that one would intuitively expect. We also consider max-min linear projections as a means of complexity-growth attenuation.

2 Motivational Application

We will refer to the figure given below. Suppose there exists a network consisting of sensing nodes, action nodes, communication nodes and processing/decision nodes. An example is given in the figure. The network may be considered to be belonging to the Blue player, although mathematically this is somewhat irrelevant. The opponent is designated as the Red player. Information will flow from the Blue sensing nodes along the network. Processing and/or decisions may be made by the relevant nodes, and the results will flow to the Blue action nodes. (Of course, there would be a network associated with Red as well as with Blue, but that is outside the scope of our analysis here.) There are two levels to the game. At the physical level, the Blue action nodes (represented as groups of blue triangles in the figure), might interact with the corresponding Red action nodes. (In the figure, the Red action nodes are placed in proximity to Blue nodes in order to intuitively indicate some physical proximity, although this is unrelated to the network graph structure.) The outcome of any such interaction will depend not only on the physical state, but also on the information available to Blue at the time of such interaction. We will associate a payoff with the outcome. The dependence of outcomes on information in a purely game model is discussed in [9], and we will review that here. However, in this study, we will be concerned with a game of network disruption. Any network disruptions will induce delays in the information available to the Blue action nodes, which will degrade the value of the information. Although the game of network disruption will be the focus, some amount of discussion of the physical level game is needed in order to motivate the payoff. The authors will attempt to ensure that the reader does not become confused between these two sub-games.



3 Problem Definition

Prior to development of the problem model, we need to define the relevant mathematical objects. We first introduce the relevant idempotent algebras. The min-plus algebra (i.e., semifield) is given by

$$a \oplus b \doteq \min\{a, b\}, \qquad a \otimes b \doteq a + b,$$

operating on $I\!R^+ \doteq I\!R \cup \{+\infty\}$. The max-plus algebra (i.e., semifield) is given by

$$a \oplus^{\vee} b \doteq \max\{a, b\}, \qquad a \otimes^{\vee} b \doteq a + b,$$

operating on $I\!R^- \doteq I\!R \cup \{-\infty\}$. In the max-min algebra (i.e., semiring), the operations are defined as

$$a \vee b \doteq \max\{a, b\}, \qquad a \wedge b \doteq \min\{a, b\},$$

operating on $\overline{I\!R} \doteq I\!R \cup \{-\infty\} \cup \{+\infty\}$, where we note that $-\infty \vee b = b$ for all $b \in \overline{I\!R}$ and $-\infty \wedge b = -\infty$ for all $b \in \overline{I\!R}$ (c.f., [6]).

3.1 Payoff origin

We briefly describe the physical level game which will yield the payoff for the network disruption game that is our main focus. This will be a zero-sum game. Consider time t_k with $k \in]0, K[\doteq \{0, 1, 2 \dots K\}$, where we note that throughout we will use the notation $]a, b[$ to denote $\{a, a+1, a+2 \dots b\}$ for integers $a \leq b$. Suppose that at this time, there is an interaction between the Blue and Red action nodes. The possible actions for Blue are denoted by $v \in \mathcal{V} =]1, V[$. The true Red action node configuration is $x \in \mathcal{X} =]1, X[$. Given true asset configuration, x, Blue would obtain an action payoff $c(x, v)$. Let $C(v)$ be the vector of length X with components $c(x, v)$.

It is natural to use the max-plus probability structure (c.f., [1, 4, 9, 14] and the references therein) for deterministic games. (Note that, as with the networks, we model only the effect of Blue partial information.) Suppose Blue's knowledge of the true configuration is described by max-plus probability distribution, $q \in S^{\oplus^{\vee} X}$, where

$$S^{\oplus^{\vee} X} \doteq \Big\{ q \in [-\infty, 0]^X \,\Big|\, \bigoplus_{x \in \mathcal{X}}^{\vee} q_x = 0 \Big\},$$

where $[-\infty, 0]$ denotes $(-\infty, 0] \cup \{-\infty\}$ and the X superscript indicates outer product X times. (Also, the $\bigoplus^{\vee}$ symbol indicates max-plus summation.) It is useful to recall that $[-\infty, 0]$ is analogous to $[0, 1]$ in the standard algebra, where 0 takes the place of 1 (the multiplicative identity), and $-\infty$ takes the place of 0 (the additive identity). We may interpret each component, q_x, as the additive inverse of the (relative) cost to Red to cause Blue to believe that the configuration is x. This will be become more clear below. The expected payoff for action $v \in \mathcal{V}$ given max-plus distribution q at time t_K, is as follows. Letting max-plus random variable ξ be distributed according to q, and $\mathbf{E}_q^{\oplus^\vee}$ denote max-plus expectation according to this q, the expected payoff is

$$\hat{J}(q,v) = \mathbf{E}_q^{\oplus^\vee}[c(v,\xi)] = \bigoplus_{x\in\mathcal{X}}^{\vee} c(v,x) \otimes^\vee q_x = C(v) \odot^\vee q,$$

where $\odot^\vee$ denotes the max-plus dot product. Given that Blue wants to minimize (make more negative) the physical level payoff, the value of information q is

$$\phi(q) \doteq \min_{v\in\mathcal{V}} \hat{J}(q,v) = \bigwedge_{v\in\mathcal{V}} [C(v) \odot^\vee q]. \tag{1}$$

We see that if information is represented by a max-plus probability distribution over a finite set (and one has a finite set of controls), then *the value of information takes the form of a min-max sum of max-plus linear functionals over a max-plus probability simplex.* We also emphasize again that the above game merely provides the payoff for the network disruption game to follow.

3.2 Network game dynamics

It may again be helpful to refer to our main figure (see above). We suppose that the Blue network will be defined as a graph, $(\mathcal{G},\mathcal{E})$, where $\mathcal{G}$ denotes the set of nodes, and $\mathcal{E}$ denotes the set of edges. The set of nodes will be decomposed as $\mathcal{G} = \mathcal{G}_s \cup \mathcal{G}_a \cup \mathcal{G}_c \cup \mathcal{G}_d$ where $\mathcal{G}_s$ denotes the set of sensing nodes (air vehicle icons in the example figure), $\mathcal{G}_a$ denotes the set of action nodes (blue triangular icons in the example figure), $\mathcal{G}_c$ denotes the set of communication nodes (colored disc icons in the example figure) and $\mathcal{G}_d$ denotes the set of decision/analysis nodes (blue rectangular icons in the example figure). Let $\mathcal{G} =]1,n[$, and $\mathcal{E} = \{(g_i^1, g_i^2) \,|\, g_i^1, g_i^2 \in \mathcal{G}\, \forall i \in]1, n_e[\}$, where the elements of $\mathcal{E}$ are unordered pairs. Let $G_s = \#\mathcal{G}_s$, $G_a = \#\mathcal{G}_a$, $G_c = \#\mathcal{G}_c$ and $G_d = \#\mathcal{G}_d$. We suppose that for each action node, say $\alpha \in \mathcal{G}_a$, there exists a set of relevant sensing nodes, $\hat{\mathcal{G}}_s(\alpha) \subseteq \mathcal{G}_s$ such that information from these sensing elements affects the min-plus probability distribution describing information relevant to action node α. Further, given $\alpha \in \mathcal{G}_a$ and $\sigma \in \hat{\mathcal{G}}_s(\alpha)$, there exists an ordered set of decision/analysis nodes that information from σ must pass through prior to use by α. Let this path be denoted as

$$I^{\sigma,\alpha} = (\sigma, g_2, g_3 \ldots g_{\bar{n}(\sigma,\alpha)-1}, \alpha)$$

with $g_k \in \mathcal{G}_d$ for all $k \in]2, \bar{n}(\sigma,\alpha)-1[$. We also let $\mathcal{P}_N^{\sigma,\alpha}$ denote the set of ordered sequences of maximum length N given by

$$\mathcal{P}_N^{\sigma,\alpha}\Big\{\{\gamma_i\}_{i=1}^{\hat{n}}\,\Big|\,\hat{n}\in]1,N[,\,(\gamma_i,\gamma_{i+1})\in\mathcal{E}\,\forall i\in]1,\hat{n}-1[,$$
$$\text{and s.t. } \exists \text{ subsequence } \{\gamma_{i_j}\}_{j=1}^{\bar{n}(\sigma,\alpha)} \text{ s.t. } i_1=1,$$
$$i_{\bar{n}(\sigma,\alpha)}=\hat{n},\,\gamma_{i_j}=g_j\in I^{\sigma,\alpha}\,\forall j\in]1,\bar{n}(\sigma,\alpha)[\,\Big\}.$$

Note that the elements of $\mathcal{P}_N^{\sigma,\alpha}$ are feasible paths through the graph passing through the required nodes of $I^{\sigma,\alpha}$ in the required order.

We now begin the discussion of the network disruption game. We use a fixed time-step model where, without loss of generality, $t_{k+1}-t_k=1$ for all k. Let the delay in transfer of information along edge $(\gamma,g)\in\mathcal{E}$ be denoted by $\delta^e_{\gamma,g}\in[0,\infty)$, and the processing delay at node $\gamma\in\mathcal{G}$ be $\delta^p_\gamma\in[0,\infty)$. We let the delay at time t_k in arrival of information originating from node $\sigma\in\mathcal{G}_s$ at node $g\in\mathcal{G}$ be denoted by $d^\sigma_{g,k}$. We also let the vector of length n of such delays be d^σ_k. The delays in information arrival originating at all nodes in $\mathcal{G}_s$ will be $D_k=\{d^\sigma_k\,|\,\sigma\in\mathcal{G}_s\}$. Notationally, it will be helpful to arrange D_k as a vector of length nG_s. Of course, there may be multiple routes from node σ to node g. For $g\in\mathcal{G}$, let $\mathcal{N}_g\doteq\{\gamma\in\mathcal{G}\,|\,(\gamma,g)\in\mathcal{E}\}$. It is easy to see that the dynamics of $d^\sigma_{g,k}$ are given by

$$d^\sigma_{g,k+1}=\bigwedge_{\gamma\in\mathcal{N}_g}\bar{\delta}_{\gamma,g}+d^\sigma_{g,k}=\bigoplus_{\gamma\in\mathcal{N}_g}\bar{\delta}_{\gamma,g}\otimes d^\sigma_{g,k}, \tag{2}$$

where $\bar{\delta}_{\gamma,g}=\delta^e_{\gamma,g}+\delta^p_\gamma$ and the $\bigoplus$ symbol denotes min-plus summation. Let T be the $n\times n$ matrix with elements

$$T_{g,\gamma}=\begin{cases}\bar{\delta}_{\gamma,g} & \text{if } \gamma\in\mathcal{N}_g,\\ +\infty & \text{otherwise.}\end{cases}$$

Then,

$$d^\sigma_{g,k+1}=\bigoplus_{\gamma\in\mathcal{G}}T_{g,\gamma}\otimes d^\sigma_{\gamma,k}\quad\forall\,\sigma\in\mathcal{G}_s,\,g\in\mathcal{G},$$

and arranging this in vector form,

$$d^\sigma_{k+1}=T\otimes d^\sigma_k\quad\forall\,\sigma\in\mathcal{G}_s, \tag{3}$$

where here $\otimes$ indicates min-plus matrix-vector multiplication. Also, letting $\overline{T}$ be the block-diagonal $nG_s\times nG_s$ matrix consisting of T blocks, with $+\infty$ (the min-plus zero) elsewhere, we may write

$$D_{k+1}=\overline{T}\otimes D_k. \tag{4}$$

Now suppose that Red may act to increase delays, and Blue may act to counter this. For example, in the case of radio transmissions, short delays may be induced by jamming, and with inclusion of intervening routers, delays may also be induced by other means. Finally, we arrive at

$$d^{\sigma}_{k+1} = T(u^b_k, u^r_k) \otimes d^{\sigma}_k \quad \forall \sigma \in \mathcal{G}_s, \tag{5}$$

or, equivalently,

$$D_{k+1} = \overline{T}(u^b_k, u^r_k) \otimes D_k, \tag{6}$$

where the Blue and Red controls at time t_k are $u^b_k \in \mathcal{U}^b$ and $u^r_k \in \mathcal{U}^r$, respectively, and where we assume $U^b = \#\mathcal{U}^b < \infty$ and $U^r = \#\mathcal{U}^r < \infty$. Note that (5), or equivalently (6), will define the dynamics in the network disruption game. *Importantly, we have a system with controlled (min-plus) linear dynamics.*

3.3 Network game payoff and value

We now begin definition of the payoff for this game. The payoff will be based on the information value given in (1), where we now need to determine the effect of delay on this quantity. The max-plus probability distribution, q, in (1) propagates in a surprisingly similar fashion to standard-sense probability distributions [9, 10].

Recall from Section 3.1 that we model Blue information states via some $q \in S^{\oplus^{\vee} X}$. Let us now add some more structure to this general form. We suppose the state can be decomposed according to domains partitioned by the action nodes. That is, we suppose the physical state, $x \in \mathcal{X}$, can be decomposed as $x = (x^1, x^2, \dots x^{G_a})$, with $x^a \in \mathcal{X}^a$ for all $a \in \mathcal{G}_a$, and $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2 \times \cdots \times \mathcal{X}^{G_a}$. With this decomposition, we may also let $q^a \in S^{\oplus^{\vee} X^a}$ where $X^a = \#\mathcal{X}^a$ for all $a \in \mathcal{G}_a$. For any $x \in \mathcal{X}$, $x = (x^1, x^2, \dots x^{G_a})$, we have $q_x = \bigotimes^{\vee}_{a \in \mathcal{G}_a} q^a_{x^a}$, and we note that by the usual summation process, one still has $q \in S^{\oplus^{\vee} X}$. This decomposition will be helpful in determining the cost of delay.

As discussed in [9, 10], we may suppose that in the absence of observations, information state, q_k, propagates as a max-plus Markov chain. That is,

$$q_{k+1} = \mathbb{P}^T \otimes^{\vee} q_k,$$

where $\mathbb{P}$ is a max-plus probability transition matrix. In particular, $\mathbb{P}_{i,j} \in [-\infty, 0]$ for all $i, j \in]1, X[$ and $\bigoplus^{\vee X}_{j=1} \mathbb{P}_{i,j} = 0$.

Observation processing yields an update formula that is analogous to (standard-sense) Bayes rule (c.f., [9, 10]). We briefly recall how this occurs. Suppose q^a_k is the max-plus probability distribution of state component x^a at time

t_k, prior to observation. Suppose that Blue obtains observation $y \in \mathcal{Y}$ (which we recall may be at least partially controlled by Red). Here, in order to reduce notation, we do not include the possible dependence of $\mathcal{Y}$ on domain. The resulting cost for any true state $x^a \in \mathcal{X}^a$ would be

$$\hat{q}^a_{x^a,k} = p^{\oplus^\vee}(y|x^a) + q^a_{x^a,k} = p^{\oplus^\vee}(y|x^a) \otimes^\vee q^a_{x^a,k}.$$

In solving the optimization problem, we are concerned only with the relative costs, and so we may normalize so that the max-plus sum over $x \in \mathcal{X}$ is zero. Let $\hat{q}^a_k$ (and q^a_k) denote normalized costs, where we want $\bigoplus^\vee_{x^a \in \mathcal{X}^a} \hat{q}_{x^a,k} = \bigoplus^\vee_{x^a \in \mathcal{X}^a} q_{x^a,k} = 0$. The normalized cost is

$$\begin{aligned} \hat{q}^a_{x^a,k} &= p^{\oplus^\vee}(y|x^a) \otimes^\vee q^a_{x^a,k} - \left\{ \bigoplus^\vee_{\zeta \in \mathcal{X}} \left[p^{\oplus^\vee}(y|\zeta^a) \otimes^\vee q^a_{\zeta^a,k} \right] \right\} \\ &= p^{\oplus^\vee}(y|x^a) \otimes^\vee q^a_{x^a,k} \oslash^\vee \left\{ \bigoplus^\vee_{\zeta^a \in \mathcal{X}^a} \left[p^{\oplus^\vee}(y|\zeta^a) \otimes^\vee q^a_{\zeta^a,k} \right] \right\}, \end{aligned} \tag{7}$$

where $\oslash^\vee$ indicates max-plus division (standard-sense subtraction). We may interpret each component of the resulting max-plus probability at time t_k, $q_{x,k}$, as the additive inverse of the minimal relative cost to Red for modification of the observation process to yield observed sequence $\{y_0, y_1, \dots y_k\}$ given true state x. We may denote the max-plus Bayes rule update as

$$\hat{q}^a_k = \mathcal{B}^y[q^a_k]. \tag{8}$$

Fix an action domain, indexed by $a \in \mathcal{G}^a$. For $y \in \mathcal{Y}$, let C^y be the $X^a \times X^a$ diagonal matrix with diagonal elements $p^{\oplus^\vee}(y|x^a)$. Also let R^y be the X^a-length vector with elements $p^{\oplus^\vee}(y|x^a)$. Written in vector form, update (7) takes the form

$$\hat{q}^a_k = \mathcal{B}^y[q^a_k] = [C^y \otimes^\vee q^a_k] \oslash^\vee [(R^y) \odot^\vee q^a_k]. \tag{9}$$

Now, suppose that there are multiple observation sources, indexed by $\sigma \in \hat{\mathcal{G}}_s(a)$. We would then have multiple observation processes for each domain, and we could denote the corresponding updates as

$$\hat{q}^a_k = \mathcal{B}^y_\sigma[q^a_k] = [C^y_\sigma \otimes^\vee q^a_k] \oslash^\vee [(R^y_\sigma) \odot^\vee q^a_k], \tag{10}$$

where the σ subscripts indicate the appropriate conditional probabilities corresponding to sensing node $\sigma \in \hat{\mathcal{G}}_s(a)$. Lastly, by padding these matrices and vectors appropriately, we may extend the updates to updates for the entire distribution, $q \in S^{\oplus^\vee X}$. That is, we may write the full update by sensing node $\sigma \in \mathcal{G}_s$ as

$$\hat{q}_k = \bar{\mathcal{B}}^y_\sigma[q_k] = [\bar{C}^y_\sigma \otimes^\vee q_k] \oslash^\vee [(\bar{R}^y_\sigma) \odot^\vee q_k], \tag{11}$$

For simplicity, let us suppose that each sensing node produces a single observation during each time-step. Then, the nominal dynamics of $q.$ take the form

$$q_{k+1} = I\!\!P^T \otimes^\vee \left(\prod_{\sigma \in \mathcal{G}_s} \bar{\mathcal{B}}_\sigma^y \right) [q_k], \tag{12}$$

where the $\prod$ notation indicates operator composition.

Now, recall again from Section 3.1 that we suppose Blue has some possible set of actions, possibly in response to Red actions, indexed by $v \in \mathcal{V}$. We assume Blue would like to minimize the max-plus expected cost given distribution $q \in S^{\oplus^\vee X}$. That is, the resulting payoff, assuming optimal Blue actions takes the form (as in Section 3.1)

$$\bigwedge_{v \in \mathcal{V}} \mathbf{E}_q^{\oplus^\vee}[c(v,\xi)] = \bigwedge_{v \in \mathcal{V}} C(v) \odot^\vee q.$$

We now examine how this depends on network delays. The total expected payoff at time k of action at time $k+1$ given current distribution q_k then becomes

$$\psi^0(q_k) = \mathbf{E}^{\oplus^\vee} \left\{ \bigwedge_{v \in \mathcal{V}} C(v) \odot^\vee q_{k+1} \right\}, \tag{13}$$

where the expectation must be taken over the incoming observations. For notational simplicity, assume for the moment that there is only one sensor, σ. Substituting (11) and (12) into (13) yields

$$\begin{aligned}
\psi^0(q_k) &= \bigoplus_{y \in \mathcal{Y}}^\vee \left\{ \left[\bigwedge_{v \in \mathcal{V}} C(v) \odot^\vee I\!\!P^T \otimes^\vee [\bar{C}_\sigma^y \otimes^\vee q_k] \right.\right. \\
&\quad \left.\left. \oslash^\vee [(\bar{R}_\sigma^y) \odot^\vee q_k] \right] \otimes^\vee \bigoplus_{\zeta \in \mathcal{X}}^\vee p^{\oplus^\vee}(y|\zeta) \otimes^\vee q_{\zeta,k} \right\} \\
&= \bigoplus_{y \in \mathcal{Y}}^\vee \left\{ \left[\bigwedge_{v \in \mathcal{V}} C(v) \odot^\vee I\!\!P^T \otimes^\vee [\bar{C}_\sigma^y \otimes^\vee q_k] \oslash^\vee [(\bar{R}_\sigma^y) \odot^\vee q_k] \right] \otimes^\vee (\bar{R}_\sigma^y) \odot^\vee q_k \right\} \\
&= \bigoplus_{y \in \mathcal{Y}}^\vee \left\{ \bigwedge_{v \in \mathcal{V}} \left(\bar{C}_\sigma^y \otimes^\vee I\!\!P \otimes^\vee C(v) \right) \odot^\vee q_k \right\}.
\end{aligned}$$

Now, returning to the case of multiple sensor nodes, and continuing to assume the same observation set for each, we obtain payoff form

$$\psi^0(q_k) = \bigoplus_{\mathbf{y} \in \mathcal{Y}^{G_s}}^\vee \left\{ \bigwedge_{v \in \mathcal{V}} \left[\left(\bigotimes_{\sigma \in \mathcal{G}_s}^\vee \bar{C}_\sigma^{y_\sigma} \right) \otimes^\vee I\!\!P \otimes^\vee C(v) \right] \odot^\vee q_k \right\}. \tag{14}$$

Next, we model the effects of delays through this payoff model. For the moment, we suppose these delays are all discretized to the same time-step as the dynamics, where k indexes time-step t_k. If information from node $\sigma \in \hat{\mathcal{G}}_s(\alpha)$

is delayed at $\alpha \in \mathcal{G}_a$ by some time, d_α^σ, then the observation updates in (14) for time-steps more recent than d_α^σ time back, will not take place. Suppose that at time k, the maximum delay is $d_k^* = \max_{\alpha\in\mathcal{G}_a} \max_{\sigma\in\hat{\mathcal{G}}_s(\alpha)} d_{\alpha,k}^\sigma$. Let $j^* = j^*(k) \doteq \lfloor(k - d_k^*)$.

For simplicity, we assume $\bigcup_{\alpha\in\mathcal{G}_a} \hat{\mathcal{G}}_s(\alpha) = \mathcal{G}_s$ for each $\hat{\mathcal{G}}_s(\alpha_1) \cap \hat{\mathcal{G}}_s(\alpha_2) = \emptyset$ if $\alpha_1 \neq \alpha_2$. That is, there is a unique action node, $\alpha \in \mathcal{G}_a$, corresponding to each sensing node, $\sigma \in \mathcal{G}_s$, and we denote this as $\hat{\alpha}(\sigma)$. For all $\sigma \in \mathcal{G}_s$, let $\hat{d}_k^\sigma \doteq d_{\hat{\alpha}(\sigma),k}^\sigma$.

Suppose that at time k, the least integer upper bound on delays is $d^* = d^*(k) \doteq \lceil\max\{\hat{d}_k^\sigma \,|\, \sigma \in \mathcal{G}_s\}$. Also let $j^* = j^*(k) \doteq k - d^*$. For each $j \in]j^*(k), k[$, let

$$\bar{\mathcal{G}}_s(j,k) \doteq \{\sigma \in \mathcal{G}_s \,|\, \hat{d}_k^\sigma < k - j\}$$

and $\bar{G}_s(j,k) = \#\bar{\mathcal{G}}_s(j,k)$.

For $\mathbf{y} \in \mathcal{Y}^{\bar{G}_s(j,k)}$ indexed by $\sigma \in \bar{\mathcal{G}}_s(j,k)$, let $\widetilde{C}^{\mathbf{y}} \doteq \bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}_\sigma^{y_\sigma}$. Next, let $\hat{\mathcal{Y}}_k \doteq \mathcal{Y}^{\bar{G}_s(j^*,k)} \times \mathcal{Y}^{\bar{G}_s(j^*+1,k)} \times \cdots \times \mathcal{Y}^{\bar{G}_s(k,k)}$. For $\mathbf{y} \in \hat{\mathcal{Y}}_k$, let

$$\mathcal{O}^{\mathbf{y}} \doteq \bigotimes^{\vee}_{j\in]j^*,k[} \left\{ \left[\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}_\sigma^{y_{\sigma,j}} \right] \otimes^{\vee} I\!P \right\}. \tag{15}$$

For the purposes of measuring the cost of delay, we suppose that the information state system is in some state, $\bar{q}$, at the initial time. The cost of delays D_k is given by

$$\psi = \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\mathcal{O}^{\mathbf{y}} \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}.$$

More generally, for delay vectors, $\delta \in \mathcal{W}^{G_s} \doteq \big(\{0, 1, \ldots \infty\}\big)^{G_s}$, the cost of such delays is

$$\psi(\delta) = \psi(\delta; \bar{q}) = \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\mathcal{O}^{\mathbf{y}} \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}. \tag{16}$$

3.4 Min-plus convex functions

Before proceeding further with this analysis, we review some theory in idempotent convexity. In general, we say $\phi : \Omega \subseteq (I\!R^-)^n \to I\!R^-$ is max-plus hypo-convex if the hypograph of ϕ is max-plus convex. See [5], and more generally, [2, 15], for further information on this discussion regarding max-plus hypo-convex functions. Let $\mathcal{C}^{\oplus^\vee}(\Omega) = \mathcal{C}^{\oplus^\vee}(\Omega; I\!R^-)$ denote this space of max-plus hypo-convex functions. This is also known as the space of sub-topical functions [15]. Note that $\mathcal{C}^{\oplus^\vee}(\Omega)$ is a min-max linear space (i.e., a min-max moduloid or semi-module). Alternatively, $\mathcal{C}^{\oplus^\vee}(\Omega)$ is the space of functions which are monotonically increasing (with respect to the partial order on $(I\!R^-)^n$) and globally Lipschitz with constant one with respect to the L_∞ norm on the domain.

We may similarly define the space of min-plus convex functions (i.e., min-plus epi-convex function) as the space of $\phi : \Omega \subseteq (I\!R^+)^n \to I\!R^+$ such that the epigraph of ϕ is min-plus convex, and we denote this space as $\mathcal{C}^\oplus(\Omega) = \mathcal{C}^\oplus(\Omega; I\!R^+)$. This is a max-min linear space. Note that the reversal between hypograph and epigraph is related to the reversal of orientation in the range induced by the change from minimum to maximum (i.e., from a min-max linear space to a max-min linear space) in the linear space (see [5] for some discussion). Alternatively, this is the space of functions which are monotonically decreasing (with respect to the partial order on $(I\!R^+)^n$) and globally Lipschitz with constant one with respect to the L_∞ norm on the domain.

For $L \in (0,\infty)$, we let $\mathcal{C}_L^\oplus(\Omega) = \mathcal{C}_L^\oplus(\Omega; I\!R^+)$ be the space of functions which are monotonically decreasing (with respect to the partial order on $(I\!R^+)^n$) and globally Lipschitz with constant L with respect to the L_∞ norm on the domain. Note that if $\phi \in \mathcal{C}_L^\oplus(\Omega_\Delta)$, where we let $\Omega_\Delta \doteq \big([0,\infty)\big)^{G_s}$, then $\hat\phi \in \mathcal{C}^\oplus(\Omega_\Delta)$ where $\hat\phi(\delta) \doteq \phi(\delta/L)$. We will refer to $\mathcal{C}_L^\oplus$ as the space of scaled min-plus convex functions. In practice, it can be computationally preferable to scale each component of the domain space by a different factor, but it is better to skip this complicating detail here.

We recall that the space of convex functions is a max-plus linear space, and that the set of linear functionals with rational coefficients forms a countable max-plus basis for the space, c.f. [11] among many others. (For a more general discussion of max-plus analysis, see for example, [7, 8].) Similarly, the set of max-min linear functionals with rational coefficients forms a countable max-min basis for $\mathcal{C}^\oplus$ (c.f., [5] for a discussion in the completely analogous case of $\mathcal{C}^{\oplus^\vee}$). That is, if $\{b^i \,|\, i \in \mathbf{N}\}$ is dense in $I\!R^n$, then given $\phi \in \mathcal{C}^\oplus(\Omega_\Delta)$, there exists $\{a^i \,|\, i \in \mathbf{N}\} \subset I\!R$ such that $\phi(\delta) = \bigvee_{i\in\mathbf{N}}[a^i \wedge b^i \odot \delta]$ for all $\delta \in \Omega_\Delta$, where for simplicity of exposition, we suppose the range is restricted to $I\!R$. Of course, this implies that for $\psi \in \mathcal{C}_L^\oplus(\Omega_\Delta)$ (and $\{b^i \,|\, i \in \mathbf{N}\}$ is dense in $I\!R^n$), there exist $\{a^i \,|\, i \in \mathbf{N}\} \subset I\!R$ such that $\phi(\delta) = \bigvee_{i\in\mathbf{N}}[a^i \wedge b^i \odot L\delta]$ for all $\delta \in \Omega_\Delta$.

Now we return to the specific case of the delay cost, ψ given by (16).

Lemma 1 *There exists $L \in (0,\infty)$ such that $\psi \in \mathcal{C}_L^\oplus(\mathcal{W}^{G_s})$.*

Proof It is sufficient to show that ψ is monotonically decreasing and globally Lipschitz with constant L with respect to the L_∞ norm. We begin with the former assertion.

Let $\delta \in \mathcal{W}^{G_s}$. Let $\bar{\sigma} \in \mathcal{G}_s$. Let $e \in \mathcal{W}^{G_s}$ be given by $e_\sigma = 1$ is $\sigma = \bar{\sigma}$ and $e_\sigma = 0$ otherwise. Let $\hat{\delta} = \delta + e$. For simplicity of proof, we only consider the case where $d_{\bar{\sigma}} < d^*$. By (15) and (16), and letting $j_1 = k - (d_{\bar{\sigma}} + 1)$,

$$\psi(\hat{\delta}) = \bigoplus^{\vee}_{(\mathbf{y},y^0)\in\hat{\mathcal{Y}}_k\times\mathcal{Y}} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\left\{ \bigotimes^{\vee}_{j\in]j^*,j_1-1[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \right\} \right.\right.$$
$$\otimes^{\vee} \left\{ \left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j_1,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} \bar{C}^{y^0}_{\bar{\sigma}} \otimes^{\vee} I\!P \right\}$$
$$\left.\left. \otimes^{\vee} \left\{ \bigotimes^{\vee}_{j\in]j_1+1,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \right\} \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$
$$= \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \bigoplus^{\vee}_{y^0\in\mathcal{Y}} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\bar{C}^{y^0}_{\bar{\sigma}} \otimes^{\vee} \bigotimes^{\vee}_{j\in]j^*,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\} \tag{17}$$
$$\leq \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\left(\bigoplus^{\vee}_{y^0\in\mathcal{Y}} \bar{C}^{y^0}_{\bar{\sigma}} \right) \otimes^{\vee} \left[\right.\right.\right. \tag{18}$$
$$\left.\left.\left. \bigotimes^{\vee}_{j\in]j^*,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \otimes^{\vee} C(v) \right] \right] \odot^{\vee} \bar{q} \right\}.$$

Now recall that $[\bar{C}^y_\sigma]_{x,x} = p^{\oplus^\vee}(y|x)$ and $[\bar{C}^y_\sigma]_{x,z} = -\infty$ is $z \neq x$. Consequently, $\bigoplus^{\vee}_{y\in\mathcal{Y}}[\bar{C}^y_\sigma]_{x,x} = 0$ for all (x,σ), and we see $\bigoplus^{\vee}_{y\in\mathcal{Y}}[\bar{C}^y_\sigma] = I^{\oplus^\vee}$, where $I^{\oplus^\vee}$ denotes the max-plus identity matrix. Using this insight in (18), we see that

$$\psi(\hat{\delta}) \leq \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \left\{ \bigwedge_{v\in\mathcal{V}} \left[\left\{ \bigotimes^{\vee}_{j\in]j^*,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \right\} \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$
$$= \psi(\delta),$$

which implies the asserted monotonicity.

Now let $L \doteq \max_{(y,\sigma,x)}[-\bar{C}^y_\sigma]_{x,x} < \infty$. This implies $\bigoplus^{\vee}_{y\in\mathcal{Y}} \bar{C}^y_{\bar{\sigma}} \geq -L \otimes^{\vee} I^{\oplus^\vee}$. Applying this in (17), we see that

$$\psi(\hat{\delta}) \geq \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \bigoplus^{\vee}_{y^0\in\mathcal{Y}} \left\{ \bigwedge_{v\in\mathcal{V}} \left[(-L) \otimes^{\vee} \bigotimes^{\vee}_{j\in]j^*,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$
$$= \bigoplus^{\vee}_{\mathbf{y}\in\hat{\mathcal{Y}}_k} \bigoplus^{\vee}_{y^0\in\mathcal{Y}} \left\{ \bigoplus_{v\in\mathcal{V}} \left[(-L) \otimes \bigotimes^{\vee}_{j\in]j^*,k[} \left(\left(\bigotimes^{\vee}_{\sigma\in\bar{\mathcal{G}}_s(j,k)} \bar{C}^{y_{\sigma,j}}_{\sigma} \right) \otimes^{\vee} I\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\},$$

which upon pulling the min-plus multiplication by a product out of the min-plus sum, and reverting to the previous symbology,

$$= \bigoplus_{\mathbf{y}\in\hat{\mathcal{Y}}_k}^{\vee} \bigoplus_{y^0\in\mathcal{Y}}^{\vee} \left\{ (-L)\otimes^{\vee} \bigoplus_{v\in\mathcal{V}}^{\vee} \left[\bigotimes_{j\in]j^*,k[}^{\vee} \left(\left(\bigotimes_{\sigma\in\bar{\mathcal{G}}_s(j,k)}^{\vee} \bar{C}_{\sigma}^{y_{\sigma,j}} \right) \otimes^{\vee} I\!\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$

$$= \bigoplus_{\mathbf{y}\in\hat{\mathcal{Y}}_k}^{\vee} \left\{ (-L)\otimes^{\vee} \bigoplus_{v\in\mathcal{V}}^{\vee} \left[\bigotimes_{j\in]j^*,k[}^{\vee} \left(\left(\bigotimes_{\sigma\in\bar{\mathcal{G}}_s(j,k)}^{\vee} \bar{C}_{\sigma}^{y_{\sigma,j}} \right) \otimes^{\vee} I\!\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$

$$= (-L) + \bigoplus_{\mathbf{y}\in\hat{\mathcal{Y}}_k}^{\vee} \left\{ \bigoplus_{v\in\mathcal{V}}^{\vee} \left[\bigotimes_{j\in]j^*,k[}^{\vee} \left(\left(\bigotimes_{\sigma\in\bar{\mathcal{G}}_s(j,k)}^{\vee} \bar{C}_{\sigma}^{y_{\sigma,j}} \right) \otimes^{\vee} I\!\!P \right) \otimes^{\vee} C(v) \right] \odot^{\vee} \bar{q} \right\}$$

$$= (-L) + \psi(\delta).$$

We see that $\psi(\hat{\delta}) - \psi(\delta) \geq (-L)\cdot 1$, where we note $\|\hat{\delta} - \delta\|_\infty = 1$. Since ψ is monotonically decreasing and $\bar{\sigma} \in \mathcal{G}_s$ was arbitrary, we see that ψ satisfies the required Lipschitz condition.

It's worth noting that, as the delays are bounded by d^*, the Lipschitz condition obtained above on $\mathcal{W}^{G_s}$ would be immediate if we restricted the domain to $\big(]0, d^*[\big)^{G_s}$. In any case, we will want to extend ψ to the continuum, and that this is possible is given in the next result which we present without proof.

Lemma 2 *Suppose* $\psi \in \mathcal{C}_L^{\oplus}(\mathcal{W}^{G_s}; I\!\!R)$. *Then, there exists* $\tilde{\psi} \in \mathcal{C}_L^{\oplus}(\Omega_\Delta; I\!\!R)$ *such that* $\tilde{\psi}(\delta) = \psi(\delta)$ *for all* $\delta \in \big(]0, d^*[\big)^{G_s}$.

Henceforth, we abuse notation by letting ψ denote also this extension to domain Ω_Δ. Also noting that we are only interested in the cost relative to no delay, we let

$$\hat{\psi}(\delta) \doteq \psi(\delta) - \psi(0) = \psi(\delta; \bar{q}) - \psi(0; \bar{q}) \quad \forall\, \delta \in \Omega_\Delta. \tag{19}$$

For ease of presentation, we suppose the above-indicated scaling has been performed, so that

$$\hat{\psi} \in \mathcal{C}^{\oplus}(\Omega_\Delta; I\!\!R).$$

Consequently, using the max-min basis representation available for functions in the space of min-plus convex functions, we may write

$$\hat{\psi} = \bigvee_{i\in\mathbf{N}} \big[a^i \oplus b^i \odot \delta\big].$$

Lastly, due to the finite cardinality of $\mathcal{W}^{G_s}$, we claim that without loss of generality, we can take

$$\hat{\psi} = \bigvee_{z\in\mathcal{Z}} \big[a^z \oplus b^z \odot \delta\big], \tag{20}$$

where $\#\mathcal{Z} \leq \#[\mathcal{W}^{G_s}]$.

The function, $\hat{\psi}$ will be the running cost in our delay game. We suppose a finite time-horizon problem formulation, with terminal time K (i.e., t_K). We also take a maximum over the time-domain. With this, the payoff for our game with initial time, k_0, and initial state, D_{k_0}, is given by

$$\bar{J}(k_0, D_{k_0}, u^b_\cdot, u^r_\cdot) = \bigvee_{k\in]k_o,K[} \hat{\psi}(D_k), \tag{21}$$

for $u^b_\cdot \in (\mathcal{U}^b)^{K-k_0}$ and $u^r_\cdot \in (\mathcal{U}^r)^{K-k_0}$ where the superscript $K - k_0$ indicates outer product, and where we abuse notation by using the full D_k vector as the argument for $\hat{\psi}$, which only depends on the $d^\sigma_{\hat{\alpha}(\sigma),l}$ components. Of course, it is implicit that the dynamics for $D_\cdot$ are given by (4).

As we desire a risk-averse/worst-case analysis, we consider the upper value of the game. Let

$$\mathcal{R}_{k_0} \doteq \{\rho :\, (\mathcal{U}^b)^{K-k_0} \to (\mathcal{U}^r)^{K-k_0} \,|\, \text{nonanticipative}\}.$$

The upper value will be

$$\overline{W}(k_0, D_{k_0}) = \bigvee_{\rho\in\mathcal{R}_{k_0}} \bigwedge_{u^b_\cdot\in(\mathcal{U}^b)^{K-k_0}} \bar{J}(k_0, D_{k_0}, u^b_\cdot, \rho[u^b_\cdot]). \tag{22}$$

4 Min-Plus Convex Form Propagation

We now have a payoff, $\bar{J}$, given as a maximum over time, i.e., as a max-plus sum over time, rather than as a standard-sense sum over time. The more-complex continuous-time case has been considered in, for example [4] among numerous sources. Here, the finite max-plus summation leads to a simpler analysis. We briefly discuss this analysis, yielding the dynamic program, as it impacts the rather interesting propagation form.

Consider an initial time $k \in]k_0, K[$. The value is given by

$$\overline{W}(k, \delta) = \bigvee_{\rho\in\mathcal{R}_k} \bigwedge_{u^b_\cdot\in(\mathcal{U}^b)^{K-k}} \bigvee_{j\in]k,K[} \hat{\psi}(D_j),$$

where $D_\cdot$ satisfies (4) with controls $u^b_\cdot$ and $\rho(u^b_\cdot)$ and initial state $D_k = \delta$. With a little work, one finds

$$\begin{aligned}
\overline{W}(k,\delta) = & \bigwedge_{u^b_k\in\mathcal{U}^b} \bigvee_{u^r_k\in\mathcal{U}^r} \Big\{ \hat{\psi}(D_k) \vee \\
& \Big[\bigvee_{\rho\in\mathcal{R}_{k+1}} \bigwedge_{u^b_\cdot\in(\mathcal{U}^b)^{K-(k+1)}} \bigvee_{j\in]k+1,K[} \hat{\psi}(D_j) \Big] \Big\} \\
= & \bigwedge_{u^b\in\mathcal{U}^b} \bigvee_{u^r\in\mathcal{U}^r} \Big\{ \hat{\psi}(D_k) \vee \overline{W}(k+1, D_{k+1}) \Big\} \\
= & \hat{\psi}(D_k) \vee \Big[\bigwedge_{u^b\in\mathcal{U}^b} \bigvee_{u^r\in\mathcal{U}^r} \overline{W}(k+1, D_{k+1}) \Big],
\end{aligned} \tag{23}$$

where $D_{k+1} = \overline{T}(u^b, u^r)\delta$.

Now suppose

$$\overline{W}(k+1,\delta) = \bigvee_{\omega\in\Omega_{k+1}} \Big[\bar{a}^\omega_{k+1} \oplus \bar{b}^\omega_{k+1} \odot \delta \Big], \tag{24}$$

which is certainly true at time t_K, where $\overline{W}(K,\delta) = \hat{\psi}(\delta)$. Then by (23) and (20)

$$\begin{aligned}
\overline{W}(k,\delta) = & \Big[\bigvee_{\tilde{z}\in\tilde{\mathcal{Z}}} \big(a^{\tilde{z}}_2 \oplus b^{\tilde{z}}_2 \odot \delta \big) \Big] \vee \Big[\bigwedge_{u^b\in\mathcal{U}^b} \bigvee_{u^r\in\mathcal{U}^r} \bigvee_{\omega\in\Omega_{k+1}} \Big(\bar{a}^\omega_{k+1} \oplus \bar{b}^\omega_{k+1} \odot D_{k+1} \Big) \Big] \\
= & \Big[\bigvee_{\tilde{z}\in\tilde{\mathcal{Z}}} \big(a^{\tilde{z}}_2 \oplus b^{\tilde{z}}_2 \odot \delta \big) \Big] \\
& \vee \Big[\bigwedge_{u^b\in\mathcal{U}^b} \bigvee_{u^r\in\mathcal{U}^r} \bigvee_{\omega\in\Omega_{k+1}} \Big(\bar{a}^\omega_{k+1} \oplus (\overline{T}^T(u^b,u^r) \otimes \bar{b}^\omega_{k+1}) \odot \delta \Big) \Big],
\end{aligned}$$

and letting $\hat{b}^{\omega,u^b,u^r}_k \doteq \overline{T}^T(u^b,u^r) \otimes \bar{b}^\omega_{k+1}$,

$$= \Big[\bigvee_{\tilde{z}\in\tilde{\mathcal{Z}}} \big(a^{\tilde{z}}_2 \oplus b^{\tilde{z}}_2 \odot \delta \big) \Big] \vee \Big[\bigwedge_{u^b\in\mathcal{U}^b} \bigvee_{(\omega,u^r)\in\Omega_{k+1}\times\mathcal{U}^r} \Big(\bar{a}^\omega_{k+1} \oplus \hat{b}^{\omega,u^b,u^r}_k \odot \delta \Big) \Big]. \tag{25}$$

The last term in square brackets in (25) is a max-min product of min-plus finite-complexity convex functions. We discuss this in more detail in Section 5, but first, note that by applying the max-min distributive property to (25), one obtains

$$\overline{W}(k,\delta) = \left[\bigvee_{\tilde{z}\in\tilde{\mathcal{Z}}} \left(a_2^{\tilde{z}} \oplus b_2^{\tilde{z}} \odot \delta \right) \right] \vee \left[\bigvee_{\{(\omega,u^r)_{u^b}\}\in(\Omega_{k+1}\times\mathcal{U}^r)^{U^b}} \bigoplus_{u^b\in\mathcal{U}^b} \left(\bar{a}_{k+1}^{\omega} \oplus \hat{b}_k^{\omega,u^b,u^r} \odot \delta \right) \right],$$

and letting $\tilde{a}_k^{\{\omega_{u^b}\}} \doteq \bigoplus_{u^b\in\mathcal{U}^b} \bar{a}_{k+1}^{\omega_{u^b}}$ and $\tilde{b}_k^{\{(\omega,u^r)_{u^b}\}} \doteq \bigoplus_{u^b\in\mathcal{U}^b} \hat{b}_k^{(\omega,u^r)_{u^b},u^b}$, this becomes

$$= \left[\bigvee_{\tilde{z}\in\tilde{\mathcal{Z}}} \left(a_2^{\tilde{z}} \oplus b_2^{\tilde{z}} \odot \delta \right) \right] \vee \left[\bigvee_{\{(\omega,u^r)_{u^b}\}\in(\Omega_{k+1}\times\mathcal{U}^r)^{U^b}} \left(\tilde{a}_k^{\{\omega_{u^b}\}} \oplus \tilde{b}_k^{\{(\omega,u^r)_{u^b}\}} \odot \delta \right) \right]$$
$$= \bigvee_{\omega\in\Omega_k} \left[\bar{a}_k^{\omega} \oplus \bar{b}_k^{\omega} \odot \delta \right] = \bigvee_{\omega\in\Omega_k} \left[\bar{a}_k^{\omega} \wedge \bar{b}_k^{\omega} \odot \delta \right], \tag{26}$$

where in the last form, we emphasize that this is a max-min linear combination of min-plus linear functionals (i.e., a finite-complexity min-plus convex functional). We have

Theorem 1 *For all* $k \in]0,K[$, $\overline{W}$ *takes the form (26). Further, one obtains* $\{\bar{a}_k^{\omega} \,|\, \omega \in \Omega_k\}$ *and* $\{\bar{b}_k^{\omega} \,|\, \omega \in \Omega_k\}$ *from* $\{\bar{a}_{k+1}^{\omega} \,|\, \omega \in \Omega_{k+1}\}$ *and* $\{\bar{b}_{k+1}^{\omega} \,|\, \omega \in \Omega_{k+1}\}$ *by idempotent linear algebraic operations.*

In short, value function propagation reduces to idempotent linear operations.

5 Complexity Analysis

We see that the solution of our network disruption game reduces to propagation of coefficients by idempotent linear operations, where the value function is given as a max-min linear combination of min-plus linear functionals at each time-step. However, the apparent cardinality of the set of coefficients grows extremely rapidly as one back-propagates. In this section, we discuss the complexity of such forms and optimal max-min linear projections for complexity-growth attenuation.

5.1 Complexity of max-min sums and products of min-plus linear functions

From an examination of the value function propagation derived in the previous section, one sees that there appear to be a tremendous number of new min-plus affine functions added to the value-function representation at each time-step. In practice, the overwhelming majority of these play no role. Typically they are either duplicates of existing functions, or they are inactive, by which we mean that they nowhere achieve the pointwise maximum in (26). We say a component affine function is strictly active if it is the unique maximizing function at some point. Consequently, they can be eliminated from the representation with no loss of accuracy. Within the limited space here, let us give some short synopsis of the theory underlying max-min sums and products of min-plus affine functions.

First, we deal with finite-complexity min-plus convex functions. The general form is

$$f(d) = \bigvee_{j \in \mathcal{J}} \left[a^j \wedge b^j \odot d\right] = \bigvee_{j \in \mathcal{J}} \left[a^j \oplus b^j \odot d\right] \doteq \bigvee_{j \in \mathcal{J}} h^j(d), \tag{27}$$

where $\mathcal{J} =]1, J[$, $d \in (I\!R^+)^n$, $a^j \in I\!R^+$ and $b^j \in (I\!R^+)^n$ for all $j \in \mathcal{J}$. Let $\mathcal{N} =]1, n[$. (Here, for simplicity, we suppose the functions are defined over all of $(I\!R^+)^n$ rather than only $[0, +\infty]^n$.) An important object with respect to each of the min-plus affine components, $h^j(\cdot)$, is its crux. The crux value of h^j is $v^j \doteq a^j$. The crux location, c^j, is given component-wise as $c_i^j = v^j - b_i^j$ for all $i \in]1, n[$. The crux is the pair, (c^j, v^j). Intuitively speaking, the crux is the unique point where the $n+1$ hyperplane sections which form the graph of h^j intersect. The importance of the crux with regard to these expansions is evident from the following simple result. Note that we suppose that duplicates have already been removed from representation (27) (i.e., there do not exist $\bar{j} \neq \hat{j}$ such that $b^{\bar{j}} = b^{\hat{j}}$ and $a^{\bar{j}} = a^{\hat{j}}$).

Lemma 1 *A min-plus affine functional is strictly active in (27) if and only if it is strictly active at its crux location.*

Proof Sufficiency is obvious, and so we only prove necessity. Suppose affine function $h^{\bar{j}}$ is strictly active. Then, there exists $\hat{d} \in I\!R^n$ such that

$$h^{\bar{j}}(\hat{d}) = a^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > a^j \oplus b^j \odot \hat{d} = h^j(\hat{d}) \quad \forall j \in \mathcal{J} \setminus \{\bar{j}\}. \tag{28}$$

Fix any $j \neq \bar{j}$.

Suppose $a^j \leq b^j \odot \hat{d}$. Then, by (28), $a^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > a^j$. This implies

$$h^{\bar{j}}(c^{\hat{j}}) = v^{\hat{j}} = a^{\hat{j}} > a^j \geq a^j \oplus b^j \odot c^{\bar{j}} = h^j(c^{\bar{j}}),$$

which is the desired result.

Now instead, suppose $a^j > b^j \odot \hat{d}$. Then, by (28), $a^{\bar{j}} \oplus b^{\bar{j}} \odot \hat{d} > b^j \odot \hat{d}$, which implies

$$b^{\bar{j}} \odot \hat{d} > b^j \odot \hat{d}. \tag{29}$$

Let $\tilde{i} \in \mathcal{N}$ be such that

$$b^j_{\tilde{i}} \otimes \hat{d}_{\tilde{i}} = b^j \odot \hat{d}. \tag{30}$$

By (29) and (30), $b^{\bar{j}}_{\tilde{i}} \otimes \hat{d}_{\tilde{i}} > b^j_{\tilde{i}} \otimes \hat{d}_{\tilde{i}}$, which implies $b^{\bar{j}}_{\tilde{i}} > b^j_{\tilde{i}}$, and consequently, $\bigoplus_{i \in \mathcal{N}} b^j_i - b^{\bar{j}}_i < 0$. This implies

$$\begin{aligned} h^{\bar{j}}(c^{\bar{j}}) = v^{\bar{j}} &> \bigoplus_{i \in \mathcal{N}} b^j_i + v^{\bar{j}} - b^{\bar{j}}_i = \bigoplus_{i \in \mathcal{N}} b^j_i + c^{\bar{j}} \\ &= b^j \odot c^{\bar{j}} \geq a^j \oplus b^j \odot c^{\bar{j}} = h^j(c^{\bar{j}}). \end{aligned}$$

We remark that the cruxes in a finte-complexity min-plus convex function can also be defined geometrically, without recourse to a representation such as (27), and we do not include the details. One finds the following.

Lemma 2 *The number of active affine functions in f is exactly the number of cruxes.*

We say that a finite-complexity min-plus convex functional given as (27) is in minimal form if it does not contain any inactive affine functionals. We also note the following useful result, and do not include the straight-forward proof.

Theorem 1 *Suppose finite-complexity min-plus convex function $f : (I\!\!R^+)^n \to I\!\!R^+$ has exactly J cruxes, and these are $\mathcal{C} \doteq \{(c^j, v^j) \,|\, j \in \mathcal{J}\}$ where $\mathcal{J} =]1, J[$. Let $\hat{f} : (I\!\!R^+)^n \to I\!\!R^+$ be given by $\hat{f}(d) = \bigvee_{j \in \mathcal{J}} a^j \otimes b^j \odot d$, where $a^j = v^j$ and $b^j_i = v^j - c^j_i$ for all $j \in \mathcal{J}$ and all $i \in \mathcal{N}$. Then, $f = \hat{f}$, and this is the unique minimal form.*

Max-min sums of finite-complexity min-plus convex functions increase the complexity at most linearly, with this growth often being tempered by crux dominance. On the other hand, in the value function propagation derivation, we had max-min *products* of finite-complexity min-plus convex functions. Very interestingly, because of the max-min distributive property, these may continue to be represented as finite-complexity min-plus convex functions. However, the complexity appears to grow very rapidly: With a product over K groups of J-complexity min-plus convex functions, the apparent complexity, purely from examination of the distributive property, is the cardinality of sequences of length K of elements of J, i.e., J^K. However, there is another bound, induced by the geometry, that can be significantly lower. There is some similarity in this discussion to the standard-sense finite-complexity discussion in [16]. We very briefly describe the results. Suppose

$$g(d) = \bigwedge_{k\in\mathcal{K}} \bigvee_{j\in\mathcal{J}} h^{k,j}(d) = \bigwedge_{k\in\mathcal{K}} \bigvee_{j\in\mathcal{J}} \Big[a^{k,j} \oplus b^{k,j} \odot d\Big]. \tag{31}$$

In the one-dimensional case, $n = 1$, a very nice complexity bound is obtained. (Because of space limitations, we do not include a proof.)

Theorem 2 *In the case of domain, $I\!R^+$, the complexity of (31) is at most $KJ-(K-1)$, where this bound is tight.*

Obviously, this can be much better than J^K. Tight bounds in higher dimensions remain an open question. However, one known bound is $(KJ)^{n-1}$, where again, space limitations do not allow for a proof.

5.2 Max-min projection and complexity attenuation

Regardless of complexity bounds, there is still a need for complexity-growth attenuation. For the general class of finite-complexity min-plus convex functions, we extend the results of [5]. There are two questions. First, what is the optimal (minimum error) complexity-reduction representation? Second, how does one compute this representation? Due to space limitations, we only briefly indicate the results.

In regard to the first question, results for finite-complexity standard-sense convex functions were given in [12]. These results were extended to finite-complexity max-plus convex functions in [5]. In the case at hand, one trivially transfers the results for the max-plus case to the min-plus case. Suppose one has $f(d) = \bigvee_{j\in\mathcal{J}} h^j(d)$ with $J = \#\mathcal{J}$, and wishes to find an approximation in the form $g(d) = \bigvee_{m\in\mathcal{M}} \hat{h}^m(d)$ with $M = \#\mathcal{M} < J$. Note that as this is a pointwise maximum, any error such that $g(\tilde{d}) > f(\tilde{d})$ for some $\tilde{d} \in I\!R^n$ cannot be corrected by the addition of more terms to g. Consequently, one seeks an approximation from below.

For given $x \in I\!R^k$ for some $k \in \mathbf{N}$, w define the downward cone as $\mathcal{D}(x) = \{\hat{x} \in I\!R^k \,|\, \hat{x} \preceq x\}$ where $\preceq$ denotes the partial order on $I\!R^k$. Next, for a set of points, $X \subset I\!R^k$, we let $\langle X\rangle$ denote the convex hull of X. Then, we may define the *min-plus cornice* of X as

$$\mathcal{C}^{\oplus}(X) = \bigcup_{x\in\langle X\rangle} \mathcal{D}(x).$$

One may show that our optimal complexity reduction problem reduces to maximization of a min-plus convex, monotonically increasing function over an outer-product of cornices, where the cornices are formed from the coefficients describing f. We find, that the optimal solution (also the optimal min-plus projection) is obtained by pruning the set of constituent min-plus affine functions describing f down to M elements (see [5]).

The second question regards the computational cost of this optimal projection/pruning. Noting the discussion of cruxes above, the reader may be able to see that the optimal projection may be obtained by evaluation of all of the constituent affine functions at each of the active cruxes, with a complexity-bound proportional to J^2. Due to space limitations, we do not include the details.

6 Concluding Remarks

This paper considers a network disruption game, where the payoff derives from the reduced effectiveness of a player's controls caused by information flow delays. We find that the problem may be formulated with a min-plus convex cost and controlled min-plus linear dynamics. We also find that the solution may be obtained purely by idempotent linear operations. Complexity growth is the most significant issue. We find that the growth is severely limited by geometric considerations. We also find that complexity-growth attenuation is relatively straight-forward as well. More generally, computations are easily instantiated (and bear some similarity to those appearing in [9] for a different problem class), although space limitations prevent inclusion of examples here.

The first author was supported in part by AFOSR.

References

1. M. Akian, "Densities of idempotent measures and large deviations", Trans. Amer. Math. Soc., 109 (1999), 79–111.
2. G. Cohen, S. Gaubert, J.-P. Quadrat and I. Singer, *Max-plus convex sets and functions,* Idempotent Mathematics and Mathematical Physics, G. L. Litvinov and V. P. Maslov (Eds.), Contemporary Mathematics, Amer. Math. Soc., 377 (2005), 105–129.
3. P. Del Moral and M. Doisy, "Maslov idempotent probability calculus", Theory Prob. Appl., 43 (1999), 562–576.
4. W.H. Fleming, H. Kaise and S.-J. Sheu, "Max-Plus Stochastic Control and Risk-Sensitivity", Applied Math. and Optim., 62 (2010), 81–144.
5. S. Gaubert and W.M. McEneaney, "Min-max spaces and complexity reduction in min-max expansions", Applied Math. and Optim., (to appear).
6. B. Heidergott, G.J. Olsder and J. van der Woude, *Max-Plus at Work: Modeling and Analysis of Synchronized Systems,* Princeton Univ. Press, 2006.
7. V.N. Kolokoltsov and V.P. Maslov, *Idempotent Analysis and Its Applications,* Kluwer, 1997.
8. G.L. Litvinov, V.P. Maslov and G.B. Shpiz, *Idempotent Functional Analysis: An Algebraic Approach,* Mathematical Notes, 69 (2001), 696–729.

9. W.M. McEneaney, “Idempotent Method for Deception Games and Complexity Attenuation”, Proc. 2011 IFAC.
10. W.M. McEneaney, “Idempotent Method for Deception Games”, Proc. 2011 ACC, 4051–4056.
11. W.M. McEneaney, *Max-Plus Methods for Nonlinear Control and Estimation*, Birkhauser, Boston, 2006.
12. W.M. McEneaney, “Complexity Reduction, Cornices and Pruning”, Proc. of the Intl. Conf. on Tropical and Idempotent Mathematics, G.L. Litvinov and S.N. Sergeev (Eds.), Contemporary Math. 495, Amer. Math. Soc. (2009), 293–303.
13. W.M. McEneaney, “Idempotent Method for Dynamic Games and Complexity Reduction in Min-Max Expansions”, Proc. IEEE CDC 2009.
14. A. Puhalskii, *Large Deviations and Idempotent Probability,* Chapman and Hall/CRC Press, 2001.
15. A.M. Rubinov and I. Singer, “Topical and Sub-Topical Functions, Downward Sets and Abstract Convexity”, Optimization, 50 (2001), 307–351.
16. E.D. Sontag, “VC dimension of neural networks”, Neural Networks and Machine Learning, Ed. C.M. Bishop, Springer (1998), 69–95.

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A minimum-weight perfect matching process for cost functions of concave type in 1D

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1. Preliminaries. Following R. McCann [5], we call a function $w\colon \mathbf{R}^2 \to \mathbf{R} \cup \{-\infty\}$ a *cost function of concave type* if for any $x_1, x_2, y_1, y_2 \in \mathbf{R}$ the inequality $w(x_1, y_1) + w(x_2, y_2) \leq w(x_1, y_2) + w(x_2, y_1)$ implies that the intervals connecting x_1 to y_1 and x_2 to y_2 are either disjoint or one of them is contained in the other.

Examples are $w(x, y) = |x - y|^\alpha$ with $0 < \alpha < 1$ or $w(x, y) = \log |x - y|$ extended to the diagonal $x = y$ by $-\infty$. In fact whenever a cost function w of concave type is spatially homogeneous and symmetric, i.e., $w(x, y) = g(|x - y|)$, the function $g\colon \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$ must be strictly increasing and strictly concave [5].

Let now $x_1 < x_2 < \cdots < x_{2n}$ be an even number of points on the real line $\mathbf{R}$. Consider the complete graph K_{2n} on these points, each of whose edges (x_i, x_j) is equipped with weight $w(x_i, x_j)$. We look for a minimum-weight perfect matching in the graph K_{2n}, i.e., a set of n nonintersecting edges such that the sum of their weights is minimal.

A bipartite version of this problem has been thoroughly treated for costs of concave type in the continuous setting in [5]. Similar discrete problems have also been considered in the algorithmics literature for the specific case of the distance $|x - y|$ [1, 4, 6] and for a general cost function of a concave type in [2, 3].

Call a matching *nested* if, for any two arcs (x_i, x_j) and $(x_{i'}, x_{j'})$ that are present in the matching, the corresponding intervals in $\mathbf{R}$ are either disjoint or one of them is contained in the other.

Lemma 1 ([1, 5]) *A minimum-weight matching is nested.*

Corollary 1 *In a minimum-weight perfect matching, points with even numbers are matched to points with odd numbers.*

2. Stabilization of optimal matchings. Suppose $X = \{x_i\}_{1 \le i \le 2n}$ with $x_1 < x_2 < \cdots < x_{2n}$ and $X' = \{x'_{i'}\}_{1 \le i' \le 2n'}$ with $x'_1 < x'_2 < \cdots < x'_{2n'}$ be two sets such that $x_{2n} < x'_1$, i.e., X' lies to the right of X. We will refer to minimum-weight perfect matchings on X and X' as *partial matchings* and to the minimum-weight perfect matching on $X \cup X'$ as *joint matching.*

Call an arc (x_i, x_j) in a nested matching *exposed* if there is no arc $(x_{i'}, x_{j'})$ with x_i, x_j contained between $x_{i'}$ and $x_{j'}$. We call all other arcs in a nested matching non-exposed or *hidden.* Intuitively, exposed arcs are those visible "from above" and hidden arcs are those covered with exposed ones.

It turns out that minimum-weight matchings enjoy a stabilization property: adding new points beside an existing point configuration does not affect the hidden arcs in the optimal matching thereon.

Theorem 1 ([3]) *Whenever an arc (x_i, x_j) is hidden in the partial matching on X, it belongs to the joint optimal matching and is hidden there too.*

Proof By contradiction, assume that some of hidden arcs in the partial matching on X do not belong to the joint matching. Then there will be at least one exposed arc (x_ℓ, x_r) in the partial matching on X such that some points x_i with $x_\ell < x_i < x_r$ are connected in the joint matching to points outside (x_ℓ, x_r).

Indeed, if points inside every exposed arc (x_ℓ, x_r) would be matched in the joint matching only among themselves, then their matching could be without loss of generality taken the same as in the partial matching on X, and therefore all hidden arcs between x_ℓ and x_r would be preserved in the joint matching.

Suppose (x_ℓ, x_r) is, e.g., the leftmost arc of the above kind. Denote all the points in the segment $[x_\ell, x_r]$ that are connected in the joint matching to points on the left of x_ℓ by $z_1 < z_2 < \cdots < z_k$; denote the opposite endpoints of the corresponding arcs by $y_1 > y_2 > \cdots > y_k$, where the inequalities follow from the fact that the joint matching is nested. Likewise denote those points from $[x_\ell, x_r]$ that are connected in the joint matching to points on the right of x_r by $z'_1 > z'_2 > \cdots > z'_{k'}$ and their counterparts in the joint matching by $y'_1 < y'_2 < \cdots < y'_{k'}$.

Although k or k' may be zero, the number $k + k'$ must be positive and even. Indeed, by Corollary 1 the segment $[x_\ell, x_r]$ contains an even number of points and all of them must be matched in a perfect matching; removing from the joint matching all arcs whose ends both lie in $[x_\ell, x_r]$, we are left with an even number of points that are matched outside this segment.

Consider now a matching on the segment $[x_\ell, x_r]$ that consists of the following arcs: those arcs of the joint matching whose both ends belong to $[x_\ell, x_r]$; the arcs (z_1, z_2), …, $(z_{2\kappa-1}, z_{2\kappa})$, where[1] $\kappa = \lfloor k/2 \rfloor$; the arcs (z'_2, z'_1), …, $(z'_{2\kappa'}, z'_{2\kappa'-1})$, where $\kappa' = \lfloor k'/2 \rfloor$; and $(z_k, z'_{k'})$ if both k and k' are odd. Denote by W' the weight of this matching. It cannot be smaller than the weight W'_0 of the restriction of the optimal partial matching on X to $[x_\ell, x_r]$. For the total weight W of the joint matching on $X \cup X'$ we thus have

$$W \geq W - W' + W'_0. \tag{1}$$

We now show that by a suitable sequence of uncrossings the right-hand side here can be further reduced to a matching whose weight is strictly less than W.

Step 1. The arcs (z_1, y_1) and (x_ℓ, x_r) are crossing, so that $w(y_1, z_1) + w(x_\ell, x_r) > w(y_1, x_\ell) + w(z_1, x_r)$. Uncrossing these arcs strictly reduces the right-hand side of (1):

$$W > W - W' + W'_0 - w(y_1, z_1) - w(x_\ell, x_r) + w(y_1, x_\ell) + w(z_1, x_r).$$

Now the arcs (y_2, z_2) and (z_1, x_r) are crossing, so $w(y_2, z_2) + w(z_1, x_r) - w(z_1, z_2) > w(y_2, x_r)$ and therefore

$$W > W - W' + W'_0 - w(y_1, z_1) - w(y_2, z_2) - w(x_\ell, x_r) + w(y_1, x_\ell) + w(z_1, z_2) + w(y_2, x_r).$$

Repeating this step $\kappa = \lfloor k/2 \rfloor$ times gives the inequality

$$\begin{aligned} W > W - W' + W'_0 - w(x_\ell, x_r) - \sum_{1 \leq i \leq 2\kappa} w(y_i, z_i) \\ + \sum_{1 \leq i \leq \kappa} w(z_{2i-1}, z_{2i}) + \sum_{1 \leq i \leq \kappa} w(y_{2i-1}, y_{2i-2}) + w(y_{2\kappa}, x_r), \end{aligned}$$

where in the rightmost sum y_0 is defined to be x_ℓ. Note that at this stage all arcs coming to points $z_1, z_2, \ldots$ from outside $[x_\ell, x_r]$ are eliminated from the matching, except possibly (y_k, z_k) if k is odd.

Step 2. It is now clear by symmetry that a similar reduction step can be performed on arcs going from $z'_1, z'_2, \ldots$ to the right.

Step 3. If k and k' are odd, we uncross the pair of arcs (y_k, x_k) and $(y_{k-1}, y'_{k'-1}$ and finally the pair $(z_k, y'_{k'-1})$ and $(z'_{k'}, y'_{k'})$.

[1] $\lfloor \xi \rfloor$ is the largest integer n such that $n \leq \xi$.

The final estimate for W has the form

$$\begin{aligned} W > W - W' + W'_0 - w(x_\ell, x_r) - \sum_{1\le i\le k} w(y_i, z_i) - \sum_{1\le i'\le k'} w(z'_{i'}, y'_{i'}) \\ + \sum_{1\le i\le \kappa} w(z_{2i-1}, z_{2i}) + \sum_{1\le i'\le \kappa'} w(z'_{2i'}, z'_{2i'-1}) + w(z_k, z'_{k'}) \cdot [k,\ k' \text{ are odd}] \\ + \sum_{1\le i\le \kappa} w(y_{2i-1}, y_{2i-2}) + \sum_{1\le i'\le \kappa'} w(y'_{2i'-2}, y'_{2i'-1}) + w(y_k, y'_{k'}) \cdot [k,\ k' \text{ are even}], \end{aligned} \tag{2}$$

where notation such as $[k,\ k'$ are odd$]$ means 1 if k, k' are odd and 0 otherwise.

The right-hand side of (2) contains four groups of terms: first,

$$W - \sum_{1\le i\le k} w(y_i, z_i) - \sum_{1\le i'\le k'} w(z'_{i'}, y'_{i'}),$$

corresponding to the joint matching without the arcs connecting points inside $[x_\ell, x_r]$ to points outside this segment; second,

$$W' - \sum_{1\le i\le \kappa} w(z_{2i-1}, z_{2i}) - \sum_{1\le i'\le \kappa'} w(z'_{2i'}, z'_{2i'-1}) - w(z_k, z'_{k'}) \cdot [k,\ k' \text{ are odd}],$$

which comes with a negative sign and corresponds to the arcs of the joint matching with both ends inside $[x_\ell, x_r]$, and cancels them from the total; third, $W'_0 - w(x_\ell, x_r)$, with positive sign, which corresponds to the hidden arcs of the partial matching on X inside the exposed arc (x_ℓ, x_r), not including the latter; and finally the terms in the last line of (2), corresponding to the arcs matching x_ℓ, x_r, and points $y_1, \dots, y_k, y'_1, \dots, y'_{k'}$, i.e., those points outside $[x_\ell, x_r]$ that were connected in the joint matching to points inside this segment.

Gathering together contributions of these four groups of terms, we observe that all negative terms cancel out and what is left corresponds to a perfect matching with a weight strictly smaller than W, in which all arcs hidden by (x_ℓ, x_r) in the partial matching on X are restored. There may still be some crossings caused by terms of the fourth group and *not* involving the hidden arcs in $[x_\ell, x_r]$; uncrossing them if necessary gives a nested perfect matching whose weight is strictly less than that of the joint matching. This contradicts the assumption that the latter is the minimum-weight matching on $X \cup X'$. Therefore all hidden arcs in the partial matching on X (and, by symmetry, those in the partial matching on X') belong to the joint matching.

3. The minimum-weight matching process. For indices i, j of opposite parity and such that $i < j$, let $W_{i,j}$ be the weight of the minimum-weight perfect matching on the $j-i+1$ points $x_i < x_{i+1} < \cdots < x_j$. It is convenient to organize weights $W_{i,j}$ with $i < j$ into a pyramidal table:

$$\begin{array}{ccccccccc} & & & & W_{1,2n} & & & & \\ & & & \multicolumn{3}{c}{\cdots\cdots\cdots\cdots} & & & \\ & & W_{1,6} & W_{2,7} & W_{3,8} & W_{4,9} & \ldots & & \\ & W_{1,4} & W_{2,5} & W_{3,6} & W_{4,7} & W_{5,8} & \ldots & W_{2n-3,2n} & \\ W_{1,2} & W_{2,3} & W_{3,4} & W_{4,5} & W_{5,6} & W_{6,7} & \ldots & W_{2n-2,2n-1} & W_{2n-1,2n} \end{array}$$

Theorem 2 ([3]) *For all indices i, j of opposite parity with $1 \le i < j \le 2n$, weights $W_{i,j}$ satisfy the recursion*

$$W_{i,j} = \min\left[w(x_i, x_j) + W_{i+1,j-1},\ W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}\right] \tag{3}$$

with "initial conditions"

$$W_{i,i-1} = 0, \quad W_{i+2,i-1} = -w(x_i, x_{i+1}). \tag{4}$$

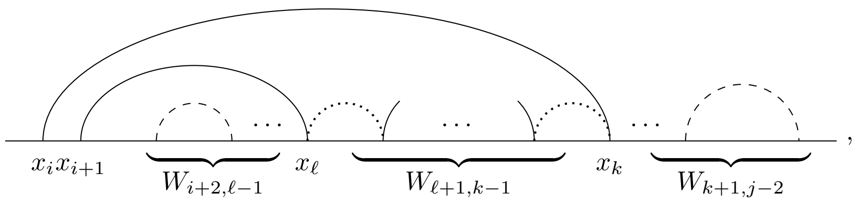
Proof For simplicity we will refer to the minimum-weight perfect matching on points $x_r < x_{r+1} < \cdots < x_s$ as the "matching $W_{r,s}$."

Consider first the matching that consists of the arc (x_i, x_j) and all arcs of the matching $W_{i+1,j-1}$, and observe that by optimality the latter its weight $w(x_i, x_j) + W_{i+1,j-1}$ is minimal among all matchings that contain (x_i, x_j).

We now examine the meaning of the expression $W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}$. Denote the point connected in the matching $W_{i,j-2}$ to x_i by x_k and the point connected to x_{i+1} by x_ℓ. By Corollary 1, the pairs of indices i, k and $i+1, \ell$ both have opposite parity. Assume first that

$$x_{i+1} < x_\ell < x_k \le x_{j-2}. \tag{5}$$

Applying Theorem 1 to the sets $X = \{x_i, x_{i+1}\}$ and $X' = \{x_{i+2}, \ldots, x_{j-2}\}$ and taking into account parity of k and ℓ, we see that x_k and x_ℓ (as well as their neighbors x_{k+1} and $x_{\ell-1}$ if they are contained in $[x_{i+2}, x_{j-2}]$) belong to exposed arcs of the matching $W_{i+2,j-2}$. Thus the matching $W_{i,j-2}$ has the following structure:



where dashed (resp., dotted) arcs correspond to those exposed arcs of the matching $W_{i+2,j-2}$ that belong (resp., do not belong) to $W_{i,j-2}$.

Since points $x_{\ell-1}$ and x_{k+1} belong to exposed arcs in the matching $W_{i+2,j-2}$, the (possibly empty) parts of this matching that correspond to points $x_{i+2} < \cdots < x_{\ell-1}$ and $x_{k+1} < \cdots < x_{j-2}$ coincide with the (possibly empty) matchings $W_{i+2,\ell-1}$ and $W_{k+1,j-2}$. For the same reason the (possibly empty) part of the matching $W_{i,j-2}$ supported on $x_{\ell+1} < \cdots < x_{k-1}$ coincides with $W_{\ell+1,k-1}$. Therefore

$$W_{i,j-2} = w(x_i, x_k) + w(x_{i+1}, x_\ell) + W_{i+2,\ell-1} + W_{\ell+1,k-1} + W_{k+1,j-2}. \tag{6}$$

Taking into account (4), observe that in the case $k = i + 1$ and $\ell = i$, which was left out in (5), this expression still gives the correct formula $W_{i,j-2} = w(x_i, x_{i+1}) + W_{i+2,j-2}$.

Now assume that in the matching $W_{i+1,j}$ the point x_j is connected to $x_{\ell'}$ and the point x_{j-1} to $x_{k'}$. A similar argument gives

$$W_{i+2,j} = W_{i+2,\ell'-1} + W_{\ell'+1,k'-1} + W_{k'+1,j-2} + w(x_{\ell'}, x_j) + w(x_{k'}, x_{j-1}); \tag{7}$$

in particular, if $\ell' = j - 1$ and $k' = j$, then $W_{i+2,j} = W_{i+2,j-2} + w(x_{j-1}, x_j)$.

Suppose that $x_k < x_{\ell'}$. Taking into account that x_k, x_{k+1}, $x_{\ell'-1}$, and $x_{\ell'}$ all belong to exposed arcs in $W_{i+2,j-2}$, we can write

$$W_{k+1,j-2} = W_{k+1,\ell'-1} + W_{\ell',j-2}, \quad W_{i+2,\ell'-1} = W_{i+2,k} + W_{k+1,\ell'-1} \tag{8}$$

and

$$W_{i+2,j-2} = W_{i+2,k} + W_{k+1,\ell'-1} + W_{\ell',j-2}. \tag{9}$$

Substituting (8) into (6) and (7) and taking into account (9), we obtain

$$\begin{aligned} W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2} &= w(x_i, x_k) + w(x_{i+1}, x_\ell) + W_{i+2,\ell-1} + W_{\ell+1,k-1} \\ &\quad + W_{k+1,\ell'-1} + w(x_{\ell'}, x_j) + W_{\ell'+1,k'-1} + w(x_{k'}, x_{j-1}) + W_{k'+1,j-2}. \end{aligned}$$

The right-hand side of this expression corresponds to a matching that coincides with $W_{i,j-2}$ on $[x_i, x_k]$, with $W_{i+2,j-2}$ on $[x_{k+1}, x_{\ell'-1}]$, and with $W_{i+1,j}$ on $[x_{\ell'}, x_j]$. By optimality, this matching cannot be improved on any of these three segments and is therefore optimal among all matchings in which x_i and x_j belong to different exposed arcs.

It follows that under the assumption that $x_k < x_{\ell'}$ the expression in the right-hand side of (3) gives the minimum weight of all matchings on $x_i < x_{i+1} < \cdots < x_j$. Moreover, the only possible candidates for the optimal matching are those constructed above: one that corresponds to $w(x_i, x_j) + W_{i+1,j-1}$ and one given by the right-hand side of the latter formula.

It remains to consider the case $x_k \geq x_{\ell'}$. Since $x_k \neq x_{\ell'}$ for parity reasons, it follows that $x_k > x_{\ell'}$; now a construction similar to the above yields a matching which corresponds to $W_{i,j-2} + W_{i+2,j} - W_{i+2,j-2}$ and in which the arcs (x_i, x_k) and $(x_{\ell'}, x_j)$ are crossed. Uncrossing them leads to a matching with strictly smaller weight, which contains the arc (x_i, x_j) and therefore cannot be better than $w(x_i, x_j) + W_{i+1,j-1}$. This means that (3) holds in this case too with $W_{i,j} = w(x_i, x_j) + W_{i+1,j-1}$.

We can now define a *minimum-weight matching process* on any locally finite configuration of points (x_i) in $\mathbf{R}$ as the recursion relations (3) and inital conditions (4) extended to all $i \in \mathbf{Z}$.

4. The formal continuum limit. Introduce the "space" and "time" variables $z = h \cdot (i+j)/2$, $t = \tau \cdot (j-i)/2$, where $h > 0$ and $\tau > 0$ are space and time meshes, denote $W_{i,j} = W(z, t)$, $w(x_{i-1}, x_{j+1}) = w(z, t+\tau)$, and rewrite (3) in the form

$$W(z, t+\tau) = \min\big[w(z, t+\tau) + W(z, t), W(z-h, t) + W(z+h, t) - W(z, t-\tau)\big].$$

Subtracting $W(z, t+\tau)$ from both sides of this relation, we arrive at

$$\begin{aligned}\min\big\{&w(z, t+\tau) - [W(z, t+\tau) - W(z, t)],\\ &W(z-h, t) - 2W(z, t) + W(z+h, t)\\ &\quad - W(z, t+\tau) + 2W(z, t) - W(z, t-\tau)\big\} = 0,\end{aligned}$$

where the term $2W(z, t)$ has been added and subtracted. This can be rewritten as

$$\begin{aligned}\min\big[&w(z, t+\tau) - \partial_t W(z, t+\tau)\,\tau + o(\tau),\\ &\partial_z^2 W(z, t)\,h^2 - \partial_t^2 W(z, t)\,\tau^2 + o(h^2 + \tau^2)\big] = 0.\end{aligned}$$

Here ∂_t (or ∂_z) denotes the operator of partial derivative with respect to t (or z).

Next we set $h = c\tau$ with a fixed constant c and divide the latter equality by $-\tau^2 < 0$, thus changing min to max, to get $\max\big\{\tau^{-1}[\partial_t W - \tau^{-1} w + o(1)], \partial_t^2 W - c^2 \partial_z^2 W + o(1)\big\} = 0$. Setting formally $\lim_{\tau \downarrow 0} \tau^{-1} w(z, t) = \omega(z, t)$, we finally arrive at the rescaled partial differential equation

$$\max\big[\partial_t W - \omega,\ \Box W\big] = 0, \tag{10}$$

where $\Box W = \partial_t^2 W - c^2 \partial_z^2 W$ is the d'Alembert wave operator. The initial conditions (4) are transformed into

$$\partial_t W(z,0) = \omega(z,0), \quad W(z,0) = 0. \tag{11}$$

Eq. (10) and initial conditions (11) constitute a formal continuous limit of (3), (4).

References

[1] A. Aggarwal, A. Bar-Noy, S. Khuller, D. Kravets, and B. Schieber, *Efficient minimum cost matching using quadrangle inequality*, J. Algorithms **19** (July 1995), no. 1, 116–143, DOI doi:10.1006/jagm.1995.1030.

[2] J. Delon, J. Salomon, and A. Sobolevski, *Local matching indicators for concave transport costs*, C. R. Acad. Sci. Paris, Série I Math. **348** (Aug. 2010), no. 15–16, 901–905, DOI doi:10.1016/j.crma.2010.07.010.

[3] ———, *Minimum-weight perfect matching for non-intrinsic distances on the line*, Feb. 2011, `arXiv:1102.1558`.

[4] R. M. Karp and S. Y. R. Li, *Two special cases of the assignment problem*, Discrete Mathematics **13** (1975), 129–142.

[5] R. J. McCann, *Exact solutions to the transportation problem on the line*, Proc. R. Soc. A: Math., Phys. and Eng. Sci. **455** (1999), 1341–1380.

[6] M. Werman, S. Peleg, R. Melter, and T. Y. Kong, *Bipartite graph matching for points on a line or a circle*, J. Algorithms **7** (1986), no. 2, 277–284.

The work was partially supported by the joint RFBR-CNRS grant 11–01–93106-a.

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Nonlinear dynamical systems over idempotent semirings for modelling of single agent motion in uncertain environment

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1 Introduction

In the present article we introduce a new approach to modelling of single agent motion in discrete dynamic unboundedly uncertain fully observable environment based on idempotent algebra. Such complex dynamical processes have been investigated, algorithmically, in the artificial intelligence [1], but they were not well understood in mathematical sense. We still do not know the underlying equations of single greedy agent motion. An agent is called greedy if he uses greedy strategy for decision making and chooses a suboptimal solution at each planning stage. The conventional control theory deals with situations without uncertainty [2] or uses deterministic unknown-but-bounded [3] and stochastic description [4] of it. We do not assume any description of uncertainty to be available, and to emphasize this we call it unbounded. It is very difficult to choose appropriate mathematical tools to formalize the process of acting under unbounded uncertainty. This situation is also closely related to game theory [5], because the motion of agent in uncertain environment may be considered as a dynamical game of two players, one of which may be nature itself [6]. There is a series of works about some interconnections between mean payoff games and tropical convexity [7], but they are not applicable to our case. This article shows how to obtain equations of motion in terms of idempotent semirings, which is the the principal novelty and advantage of our approach in contrast with other existing approaches. We also introduce idempotent fusion semirings, i.e., idempotent semirings with idempotent addition and fusion product playing the role of multiplication, and construct nonlinear dynamical systems over them for modelling single agent's motion in discrete unboundedly uncertain fully observable environment.

Idempotent mathematics is a branch of semiring theory which studies mostly semirings with idempotent addition, $a \oplus a = a$. Well-studied examples of such algebraic structures are $\mathbf{R}_{min}$ and $\mathbf{R}_{max}$ semirings, also called tropical. $\mathbf{R}_{min}$ is the set $\mathbf{R} \cup \{+\infty\}$ equipped with addition $a \oplus b = \min(a, b)$, multiplication $a \odot b = a + b$, zero $\mathbf{0} = +\infty$, and unity $\mathbf{1} = 0$. Their study is motivated by many practical applications, arising in discrete event systems, optimal scheduling, and modelling of synchronization problems in multiprocessor interactive systems [8]. Tropical semirings and dynamical systems over them are too restrictive to model processes of acting under unbounded uncertainty. Therefore, in the present article we have to develop a different approach based on a new "category" of semirings, namely, idempotent fusion semirings $\mathbf{F}_{min}$ and $\mathbf{F}_{max}$, and build a new class of nonlinear dynamical systems over them. Notice that similar algebraic structures have already appeared under different names in graph enumeration problems [9, 10], automata theory [11], and formal languages [12]. We define new selective addition operations $a \oplus b = \mathrm{lexmin}(\mathrm{argmin}_{z\in\{a,b\}}|z|)$ and $a \oplus b = \mathrm{lexmax}(\mathrm{argmax}_{z\in\{a,b\}}|z|)$ to build new one-valued idempotent semirings $\mathbf{F}_{min}$ and $\mathbf{F}_{max}$, as opposed to the previous works, where analogous semirings were multi-valued and therefore inconvenient for solving game-theoretical, control-theoretical, and optimization problems.

2 Idempotent fusion semirings

Idempotent fusion semiring is the set of all finite elementary words over the alphabet of natural numbers $\mathbf{N}^{\bigstar}|_e \cup \{\chi\}$ equipped with idempotent addition $a \oplus b = \mathrm{lexmin}(\mathrm{argmin}_{z\in\{a,b\}}|z|)$, noncommutative multiplication, also called fusion product, $a \odot b = \alpha_1 \dots \alpha_k \beta_2 \dots \beta_l$ if $\alpha_k = \beta_1$ and empty set $\emptyset$ otherwise, zero $\mathbf{0} = \emptyset$, and unit $\mathbf{1} = \chi$ where $\mathbf{N}^k$ is the kth Cartesian power of the natural numbers set $\mathbf{N}$, $\bigstar$ is the Kleene operator with respect to the union $\cup$ and the Cartesian product $\times$ operations, $|_e$ is the constraint predicate restricting $\mathbf{N}^{\bigstar}$ to its subset of all words with distinct symbols, χ is a formal symbol representing the unity of algebraic structure, $|\cdot|$ is the word's length, i.e. the number of symbols, lexmin is the lexicographic minimum of two words $a = \alpha_1 \dots \alpha_k, b = \beta_1 \dots \beta_l \in \mathbf{N}^{\bigstar}$. This algebraic structure forms a semiring if we suppose that $|\emptyset| = +\infty$ and $|\chi| = 0$. Similarly, $\mathbf{F}_{max}$ is defined with addition $a \oplus b = \mathrm{lexmax}(\mathrm{argmax}_{z\in\{a,b\}}|z|)$. It is easy to see that $\mathbf{F}_{min}$ and $\mathbf{F}_{max}$ are closely related to tropical semirings and even partially homomorphic to them. However, idempotent fusion semirings have additional features, which will play a crucial role at the stage of modelling. The Cayley tables, containing only a selected finite subset of $\mathbf{N}^{\star}$ are shown, for the operations in $\mathbf{F}_{min}$, in Tables 1, 2. Letters of a word, i.e., natural numbers, are separated by hyphen.

Table 1 Cayley table for addition in $\mathbf{F}_{min}$

$\oplus$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
$\emptyset$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ	χ
1	1	χ	1	1	1	1	1	1	1	1	1
2	2	χ	1	2	2	2	2	2	2	2	2
3	3	χ	1	2	3	3	3	3	3	3	3
1-2	1-2	χ	1	2	3	1-2	1-2	1-2	1-2	1-2	1-2
1-3	1-3	χ	1	2	3	1-2	1-3	1-3	1-3	1-3	1-3
2-1	2-1	χ	1	2	3	1-2	1-3	2-1	2-1	2-1	2-1
2-3	2-3	χ	1	2	3	1-2	1-3	2-1	2-3	2-3	2-3
3-1	3-1	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-1
3-2	3-2	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2

Table 2 Cayley table for multiplication in $\mathbf{F}_{min}$

$\odot$	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
χ	$\emptyset$	χ	1	2	3	1-2	1-3	2-1	2-3	3-1	3-2
1	$\emptyset$	1	1	$\emptyset$	$\emptyset$	1-2	1-3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset$	2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	2-1	2-3	$\emptyset$	$\emptyset$	$\emptyset$
3	$\emptyset$	3	$\emptyset$	$\emptyset$	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	3-1	3-2
1-2	$\emptyset$	1-2	$\emptyset$	1-2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1-2-3	$\emptyset$	$\emptyset$
1-3	$\emptyset$	1-3	$\emptyset$	$\emptyset$	1-3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1-3-2
2-1	$\emptyset$	2-1	2-1	$\emptyset$	$\emptyset$	$\emptyset$	2-1-3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
2-3	$\emptyset$	2-3	$\emptyset$	2-3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	2-3-1	$\emptyset$
3-1	$\emptyset$	3-1	3-1	$\emptyset$	$\emptyset$	3-1-2	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3-2	$\emptyset$	3-2	$\emptyset$	3-2	$\emptyset$	$\emptyset$	$\emptyset$	3-2-1	$\emptyset$	$\emptyset$	$\emptyset$

Such algebraic structures are widely used for solving some practical problems. Each word over natural numbers alphabet can represent a path on a graph, a sequence of some system states etc [12]. We continue this section by recalling necessary background in advanced linear algebra for our purposes, see [13, 14]. The operations of the semiring $\mathbf{F} \in \{\mathbf{F}_{min}, \mathbf{F}_{max}\}$ are extended to matrices and vectors just as in the conventional linear algebra. For any matrices $A = [a_{ij}] \in \mathbf{F}^{m\times n}, B = [b_{ij}] \in \mathbf{F}^{m\times n}, C = [c_{ij}] \in \mathbf{F}^{n\times l}$ addition and multiplication are defined as usual:

$$\{A \oplus B\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{BC\}_{ij} = \bigoplus_{k=0}^{n} b_{ik} c_{kj}. \tag{1}$$

In the case of square matrices $A \in \mathbf{F}^{n \times n}$ we define the Kleene operator as the sum of infinite power series

$$A^{\bigstar} = \bigoplus_{k=0}^{+\infty} A^k = I \oplus A^1 \oplus \ldots \oplus A^n \oplus \ldots, \tag{2}$$

where $I = A^0$ is identity matrix. For matrices $A = [a_{ij}] \in \mathbf{F}^{m \times n}, B = [b_{ij}] \in \mathbf{F}^{m \times n}$, consider also the Hadamard product [13], and for vectors $x \in \mathbf{F}^m, y \in \mathbf{F}^n$, consider the outer product [13] defined, respectively, by

$$\{A \circledast B\}_{ij} = a_{ij} b_{ij}, \quad \{xy^T\}_{ij} = x_i y_j, \tag{3}$$

where T is the matrix transposition.

Now we need to define some exotic algebraic notions. Binarization operator $\flat : \mathbf{F} \to \mathbf{F}^n$ turns scalar $\alpha_1 \ldots \alpha_k \in \mathbf{F}$ into the vector of appropriate size n with identities in α_1th, ..., α_kth positions and zeroes in all other positions. Vectorization operator $v : \mathbf{F}^{m \times n} \to \mathbf{F}^{mn}$, widely known as vec-operator [13], creates a column vector from a matrix $A = [a_1 \ldots a_n] \in \mathbf{F}^{m \times n}$ by stacking the column vectors $a_1, \ldots a_n$ below one another

$$A^v = \mathrm{vec} A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{4}$$

Negation operator $\neg : \mathbf{F} \to \mathbf{F}$ turns non-zero elements to zeros and zeros to unities, denoted as $\bar{a}$. Function $\mathrm{slice}_{i,j} : \mathbf{F} \to \mathbf{F}$ extracts the subword from the ith to the jth symbols inclusively from a word $a = \alpha_1 \ldots \alpha_k \in \mathbf{F}$ and, in the special case when $i = j$, function $\mathrm{pop}_i : \mathbf{F} \to \mathbf{F}$ extracts the ith symbol from the argument word a. These functions are defined by

$$\mathrm{slice}_{i,j}(a) = \begin{cases} \alpha_1 \ldots \alpha_j, i \le j < k, \\ \alpha_i \ldots \alpha_k, i \le k \le j, \\ \alpha_k, k < i \le j, \end{cases} \quad \mathrm{pop}_i(a) = \begin{cases} \alpha_i, i < k, \\ \alpha_k, k \le i. \end{cases} \tag{5}$$

By slice_i we denote the one-parametric function equal to $\text{slice}_{1,i}$. For a given dimension $n = 9$, word $a = 5\text{-}9\text{-}10\text{-}1\text{-}4\text{-}6\text{-}7\text{-}12\text{-}15$, and indices $i = 4, j = 7$, the following expression holds

$$\text{slice}^{\flat}_{4,7}(5\text{-}9\text{-}10\text{-}\underline{1\text{-}4\text{-}6\text{-}7}\text{-}12\text{-}15) = (1\text{-}4\text{-}6\text{-}7)^{\flat} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1}\ \mathbf{1}\ \mathbf{1} \\ \mathbf{0}\ \mathbf{0}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{1}\ \mathbf{0} \end{bmatrix}^{v}. \tag{6}$$

3 Motion equations

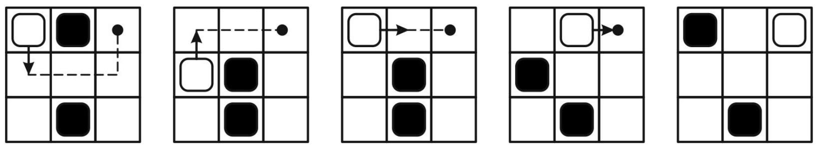
Let us list all assumptions of the model. The agent is required to achieve its own goal, while avoiding obstacles. The environment is considered fully observable, i.e., locations of obstacles are known to the agent at each moment of time. We call the agent greedy, if he chooses a suboptimal solution at each moment of time. Uncertainty is related to the fact that the agent is not being aware how the surrounding world works. Environment is called unboundedly uncertain, if we have no description of uncertainty available. Single greedy agent motion in discrete unboundedly uncertain fully observable environment can be described by the following nonlinear dynamical system over idempotent fusion semiring $\mathbf{F} = \mathbf{F}_{min}$

$$\begin{cases} \gamma[t] = x^T[t-1]\Big(\bar{u}[t]\bar{u}^T[t] \circledast A\Big)^{\bigstar} g[t], \\ x[t] = \begin{cases} \text{pop}^{\flat}_{1+c[t]}(\gamma[t]) & \text{if } \gamma[t] \neq \mathbf{0}, \\ x[t-1] & \text{if } \gamma[t] = \mathbf{0}, \end{cases} \\ y[t] = \begin{cases} \text{slice}^{\flat}_{1+c[t]}(\gamma[t]) & \text{if } \gamma[t] \neq \mathbf{0}, \\ x[t-1] & \text{if } \gamma[t] = \mathbf{0}, \end{cases} \\ x[0] = x_0,\ u[t] = \theta[t],\ t \in \mathbf{N}, \end{cases} \tag{7}$$

where $u[t] \in \mathbf{F}^n$ is the input, $x[t] \in \mathbf{F}^n$ is the state, $u[t] \in \mathbf{F}^n$ is the output, $A \in \mathbf{F}^{n\times n}$ is the matrix of basic actions, $g[t] \in \mathbf{F}^n$ is the goal state, $\circledast$ is the Hadamard product, $u[t]u^T[t] \in \mathbf{F}^{n\times n}$ is the outer product of $u[t] \in \mathbf{F}^n$ with itself, $\bigstar$ is Kleene star, $\flat$ is the binarization operator, $\gamma[t] \in \mathbf{F}$ is the momentum-optimal action, $c[t] \in \mathbf{N}$ is the velocity, slice_i and pop_i are the nonlinear functions defined above.

Thus we have obtained an algebraic representation of single agent system dynamics in the form of nonlinear dynamical system over idempotent fusion-semiring $\mathbf{F}_{min}$. These equations, which allow the dynamics of a single agent system to be represented in a convenient and unified manner well suited for analytical treatment, also provide the basis for the development of efficient simulation procedures. But there is no guarantee that the system converges in a finite or even infinite number of iterations to a stable state if the latter exists. The instability problem is well-known to specialists in artificial intelligence, see [1] for more details. These equations are recursive and have strongly nonlinear effects. This is why there is no obvious way to solve them. Let us consider discretized plane. Example in Table 3 shows the agent motion. The agent is denoted by white square. Environment states or obstacles are denoted by black squares. The goal states are denoted by black circles, the momentum-optimal action is denoted by dashed line, and arrow denotes the suboptimal part of it.

Table 3 Example of agent motion



$u[1]$	$u[2]$	$u[3]$	$u[4]$	$u[5]$
$\begin{bmatrix}0&1&0\\0&0&0\\0&1&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&0\\0&1&0\\0&1&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&0\\0&1&0\\0&1&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}^v$	$\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}^v$
$x[1]$	$x[2]$	$x[3]$	$x[4]$	$x[5]$
$\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}^v$
$y[1]$	$y[2]$	$y[3]$	$y[4]$	$y[5]$
$\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}1&1&0\\0&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}0&1&1\\0&0&0\\0&0&0\end{bmatrix}^v$	$\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}^v$

Table 3 shows the correspondence between algebraic and geometric representations with help of the vectorization operator v. After embedding of the agent into the environment, the system produces sequence of states $x[1], x[2], x[3], x[4], x[5]$ and outputs $y[1], y[2], y[3], y[4], y[5]$ based on inputs $u[1], u[2], u[3], u[4], u[5]$. It is easy to see that Bellman's optimality principle does not hold. In other words, the resulting trajectory is rather good, but not really optimal. Note that this system is not asymptotically stable in general case. Situations where the system does not converge to some stable state are very rare in practice. This is why such single-agents with greedy suboptimal decision making strategy found many applications in robotics, computer graphics, and some other areas related to artificial intelligence.

References

1. *S. Russell and P. Norvig* Artificial Intelligence : a Modern Approach. London: Prentice Hall, 2010.
2. *V.N. Kolokoltsov and V.P. Maslov* Idempotent analysis and its applications. Mathematics and its Applications, vol. 401, Dordrecht: Kluwer Academic Publishers Group, 1997.
3. *L. Jaulin, M. Keiffer, O. Didrit, and E. Walter* Applied Interval Analysis. London: Springer, 2007.
4. *R. Tempo, G. Calafiore, and F. Dabbene* Randomized Algorithms for Analysis and Control of Uncertain Systems, London: Springer-Verlag, 2005.
5. *V.N. Kolokoltsov and O.A. Malafeyev* Understanding Game Theory : Introduction to the Analysis of Many Agent Systems with Competition and Cooperation. Singapore: World Scientific, 2010.
6. *T. Bazar and G.J. Olsder* Dynamic Noncooperative Game Theory: New Models and Algorithms. Philadelphia: Society for Industrial and Applied Mathematics, 1999.
7. *X. Allamigeon, S. Gaubert, and R. Katz* Tropical polar cones, hypergraph transversals, and mean payoff games. Linear algebra and its Applications, vol. 435, no. 7, 2011, 1549-1574.
8. F. Baccelli, G. Cohen, G.J. Olsder, J.-P. Quadrat Synchronization and Linearity : an algebra for discrete event systems. New York: Wiley, 1992.
9. *B. Carre* Graphs and Networks. London: Oxford University Press, 1979.
10. *M. Gondran and M. Minoux* Graphs, Dioids and Semirings: New Models and Algorithms. Heidelberg: Springer Science + Business Media, 2008.
11. *J. Hopcroft, R. Motwani, and J. Ullman* Introduction to Automata Theory, Languages, and Computation. Addison-Wesley, 2001.
12. *J.S. Golan* Semirings and their applications. Dordrecht: Kluwer Academic Publishers, 1999.
13. *L. Hogben* Handbook of Linear Algebra. London: Chapman & Hall / CRC, 2007.
14. *G.H. Golub and C.F. Van Loan* Matrix Computations. London: John Hopkins Studies in the Mathematical Sciences, 1996.

The work was partially supported by RFBR grant 11-07-00580-a.

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Idempotent algebra methods for modelling of hierarchical multiagent systems motion

Dmitry Nikolayev

1 Introduction

We propose a new approach to modelling of hierarchical greedy multiagent systems motion based on nonlinear dynamical 2D-systems, i.e., systems with two dimensional parameter [1], over idempotent fusion semirings. Agent is called greedy if he uses greedy strategy for decision making and chooses a suboptimal solution at each planning stage. We consider multiagent systems, where all agents are greedy and have unique priorities. Since the early 1980s, motion planning has been an intensive area of study in robotics and computational geometry [2]. Most of the works in this area were focused on algorithmic motion planning, emphasizing theoretical algorithmic analysis of the problem, seeking worst-case asymptotic bounds, new heuristic approaches to the problem, and others [3]. However, the cooperative motion process haven't been sufficiently well understood in mathematical sense. We still don't know the underlying dynamical equations of hierarchically interacting greedy agents motion, because it is very difficult to find appropriate mathematical tools to formalize such kind of processes. This situation is also closely related to control theory [4, 5] and game theory [6–8]. This article shows how to obtain equations of motion in terms of idempotent semirings, which is the the principal novelty and advantage of our approach in contrast with other existing approaches.

2 Idempotent algebra

Idempotent fusion semiring is the set of all finite elementary words over the alphabet of natural numbers $\mathbf{N}^\bigstar|_e \cup \{\chi\}$ equipped with idempotent addition $a \oplus b = \text{lexmin}(\text{argmin}_{z\in\{a,b\}}|z|)$, noncommutative multiplication, also called fusion product, $a \odot b = \alpha_1 \dots \alpha_k \beta_2 \dots \beta_l$ if $\alpha_k = \beta_1$ and empty set $\emptyset$ otherwise, zero $\mathbf{0} = \emptyset$, and unit $\mathbf{1} = \chi$ where $\mathbf{N}^k$ is the kth Cartesian power of the natural numbers set $\mathbf{N}$, $\bigstar$ is the Kleene operator with respect to the union $\cup$ and the Cartesian product $\times$ operations, $|_e$ is the constraint predicate restricting $\mathbf{N}^\bigstar$ to it subset of all words with distinct symbols, χ is a formal symbol representing the unit of algebraic structure, $|\cdot|$ is a word's length, i.e. the number of symbols, lexmin is a lexicographic minimum of two words $a = \alpha_1 \dots \alpha_k, b = \beta_1 \dots \beta_l \in \mathbf{N}^\bigstar$. This algebraic structure forms a semiring if we suppose that $|\emptyset| = +\infty$ and $|\chi| = 0$. Similarly, $\mathbf{F}_{max}$ is defined with addition $a \oplus b = \text{lexmax}(\text{argmax}_{z\in\{a,b\}}|z|)$. It is easy to see that $\mathbf{F}_{min}$ and $\mathbf{F}_{max}$ are closely related to tropical semirings and even partially homomorphic to them. However, idempotent fusion semirings have additional features, which at the stage of modelling will play a crucial role.

Such algebraic structures are widely used for solving some practical problems. Each word over natural numbers alphabet can represent a path on a graph, a sequence of some system states [9–12]. We continue this section by recalling some necessary background on advanced linear algebra for our purposes, see [13, 14]. The operations of the semiring $\mathbf{F} \in \{\mathbf{F}_{\min}, \mathbf{F}_{\max}\}$ are extended to the matrices and vectors just as in the conventional linear algebra. In the case of square matrices $A \in \mathbf{F}^{n\times n}$, the Kleene operator is defined as the sum of infinite power series $A^\bigstar = A^0 \oplus A^1 \oplus \dots \oplus A^n \oplus \dots$, where $A^0 = I$ is the identity matrix. Also for matrices $A = [a_{ij}] \in \mathbf{F}^{m\times n}, B = [b_{ij}] \in \mathbf{F}^{m\times n}$ the Hadamard product and for vectors $x \in \mathbf{F}^m, y \in \mathbf{F}^n$ the outer product are defined by f$\{A \circledast B\}_{ij} = a_{ij}b_{ij}, \{xy^T\}_{ij} = x_i y_j$, where T is the matrix transposition.

Now we need to define some exotic algebraic notions. Binarization operator $\flat : \mathbf{F} \to \mathbf{F}^n$ turns scalar $\alpha_1 \dots \alpha_k \in \mathbf{F}$ into the vector of appropriate size n with identities on α_1th, ..., α_kth positions and zeroes on all other positions. Vectorization operator $v : \mathbf{F}^{m\times n} \to \mathbf{F}^{mn}$, widely known as vec-operator, creates a column vector from a matrix $A = [a_1 \dots a_n] \in \mathbf{F}^{m\times n}$ by stacking the column vectors $a_1, \dots a_n$ below one another. Negation operator $\neg : \mathbf{F} \to \mathbf{F}$ turns non-zero elements to zeros and zeros to unities, denoted as $\bar{a}$. Function $\text{slice}_{i,j} : \mathbf{F} \to \mathbf{F}$ extracts the subword from the i-th to the j-th symbols inclusively from the argument word $a = \alpha_1 \dots \alpha_k \in \mathbf{F}$, and function $\text{pop}_i : \mathbf{F} \to \mathbf{F}$ picks the i-th symbol from argument word. If function $\text{slice}_{i,j}$ is parametrized by one parameter

slice_i instead of two, we assume that this parameter is j and $i = 1$. The listed functions are defined by

$$\text{slice}_{i,j}(a) = \begin{cases} \alpha_1 \ldots \alpha_j, i \le j < k, \\ \alpha_i \ldots \alpha_k, i \le k \le j, \\ \alpha_k, k < i \le j, \end{cases} \qquad \text{pop}_i(a) = \begin{cases} \alpha_i, i < k, \\ \alpha_k, k \le i. \end{cases} \tag{1}$$

3 Motion equations for hierarchical multiagent systems

Prioritized motion planning for multiple agents was introduced by Erdmann and Lozano-Perez [15] in 1987. It works as follows. Each of the agents is assigned a unique priority. Then in order of increasing priority, the agents are picked. For each picked agent a trajectory is planned, avoiding collisions with the nonstationary obstacles as well as the previously picked agents, which are also considered as obstacles. This reduces the multiagent motion planning problem to the single agent motion planning problem in a discrete unboundedly uncertain dynamic fully observable environment [7].

In the language of semiring theory, the problem is stated as follows. Given a group of p agents and a two-dimensional discrete environment in which they can move, let us define for each agent the matrix of basic actions $A[r,t] \in \mathbf{F}^{n \times n}$ covering all possible moves in the discrete state space, the initial state $x[r,0] \in \mathbf{F}^n$, the goal state $g[r,t] \in \mathbf{F}^n$, and the speed $c[r,t] \in \mathbf{N}$, i.e. the number of adjacent discrete states that an agent can pass in one unit of time. Each agent is required to achieve its own goal state, while avoiding nonstationary obstacles arising from nature and other agents. The shapes and moves of environment are unconstrained and we assume no description of uncertainty to be available. Such environment is called unboundedly uncertain. The environment is considered fully observable, i.e., locations of obstacles are known to every agent at every moment of time. The group consists only of greedy agents, which in every moment choose a suboptimal solution. Uncertainty is related to the fact that agents are not being aware how the surrounding world works. Additionally, each agent has its input $u[r,t] \in \mathbf{F}^n$ (including information about environment), state $x[r,t] \in \mathbf{F}^n$ (including information about position), and output $y[r,t] \in \mathbf{F}^n$ (including information about position and intention).

A fundamental characteristic of multiagent systems is that individual agents communicate and interact to make their actions coordinated. This is accomplished through the exchange of messages and, to understand each other, it is crucial that agents agree on the format and semantics of these messages. The communication graph for a group of four agents with the hierarchical model of interaction and nature in the role of player 0 is shown in Figure 1.

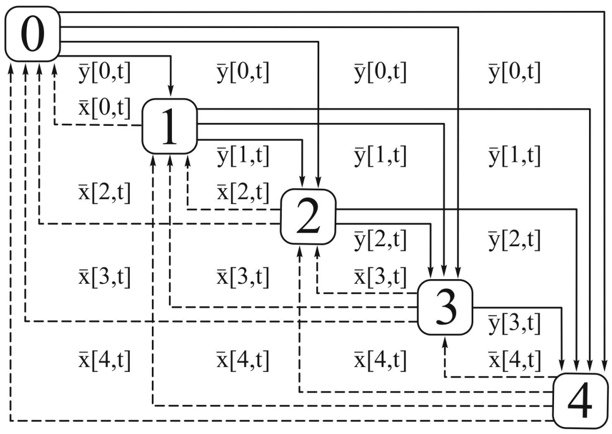

Fig. 1 Communication graph

Let us write down the communication block matrix, which looks like an adjacency matrix of a graph, but contains matrices as block elements instead of scalars. Zero blocks are denoted by O.

$$M_C[t] = \begin{bmatrix} O & y[0,t] & y[0,t] & y[0,t] & y[0,t] \\ x[1,t] & O & y[1,t] & y[1,t] & y[1,t] \\ x[2,t] & x[2,t] & O & y[2,t] & y[2,t] \\ x[3,t] & x[3,t] & x[3,t] & O & y[3,t] \\ x[4,t] & x[4,t] & x[4,t] & x[4,t] & O \end{bmatrix} \tag{2}$$

Now we are able to express the input $u[r,t]$ for each of the four agents excluding nature as sums of the communication matrix blocks in the corresponding columns:

$$\begin{aligned} u[1,t] &= y[0,t] \oplus x[2,t] \oplus x[3,t] \oplus x[4,t] \\ u[2,t] &= y[0,t] \oplus y[1,t] \oplus x[3,t] \oplus x[4,t] \\ u[3,t] &= y[0,t] \oplus y[1,t] \oplus y[2,t] \oplus x[4,t] \\ u[4,t] &= y[0,t] \oplus y[1,t] \oplus y[2,t] \oplus y[3,t] \end{aligned} \tag{3}$$

These formulas show how naturally the multiagent interaction is modelled by idempotent mathematics. Each agent just sums the incoming information with the help of idempotent addition. In our hierarchical game, agent's perception is generally defined as follows:

$$u[r,t] = \bigoplus_{s=0}^{r-1} y[s,t] \oplus \bigoplus_{s=r+1}^{p} x[s,t]. \tag{4}$$

This is the way how hierarchical multiagent motion models can be reduced to single agent motion models in the discrete dynamic unboundedly uncertain environment. The motion of hierarchical greedy multiagent systems can be described by the following nonlinear dynamical 2D-system over the idempotent fusion semiring $\mathbf{F} = \mathbf{F}_{\min}$

$$\begin{cases} u[r,t] = \bigoplus_{s=0}^{r-1} y[s,t] \oplus \bigoplus_{s=r+1}^{p} x[s,t-1], \\ \gamma[r,t] = x^T[r,t-1]\Big(\bar{u}[r,t]\bar{u}^T[r,t] \circledast A\Big)^{\bigstar} g[r,t], \\ x[r,t] = \begin{cases} \operatorname{pop}^{\flat}_{1+c[r,t]}(\gamma[r,t]) & \text{if } \gamma[r,t] \neq \mathbf{0}, \\ x[r,t-1] & \text{if } \gamma[r,t] = \mathbf{0}, \end{cases} \\ y[r,t] = \begin{cases} \operatorname{slice}^{\flat}_{1+c[r,t]}(\gamma[r,t]) & \text{if } \gamma[r,t] \neq \mathbf{0}, \\ x[r,t-1] & \text{if } \gamma[r,t] = \mathbf{0}, \end{cases} \\ x[r,0] = x_0[r],\ y[0,t] = y_0[t],\ r \in \mathbf{Z}_{1+p},\ t \in \mathbf{N}, \end{cases} \tag{5}$$

where $u[r,t]$ is the input, $x[r,t]$ is the state, $y[r,t]$ is the output, $g[r,t]$ is the goal state, $\circledast$ is the Hadamard product, $u[r,t]u^T[r,t] \in \mathbf{F}^{n\times n}$ is the outer product of $u[r,t]$ with itself, $\bigstar$ is the Kleene operator, $\flat$ is the binarization operator, slice_i and pop_i are the nonlinear functions defined earlier, $x_0[r]$ is the initial condition, $y_0[t]$ is the boundary condition. This system contains two-parametric dynamics, where the first parameter r is agent's type and the second parameter t is time. So the two dimensional manifold $\mathbf{Z}_{1+p} \times \mathbf{N}$ is an infinetely long cylinder, where $\mathbf{Z}_{1+p}$ is the ring of integers modulo $1+p$ and p is the number of agents.

Thus we have obtained an algebraic representation of multiagent systems dynamics in the form of the nonlinear dynamical 2D-system over idempotent fusion semiring $\mathbf{F}_{\min}$. These equations, which allow the dynamics of multiagent systems to be represented in a convenient and unified manner well suited to analytical treatment, also provide the basis for the development of efficient simulation procedures. But there is no guarantee that the system converges in a finite or even infinite number of iterations to a stable state if the latter exists. The instability problem is well-known to specialists in artificial intelligence, see [3] for more details. These equations are recursive and have strongly nonlinear effects. That

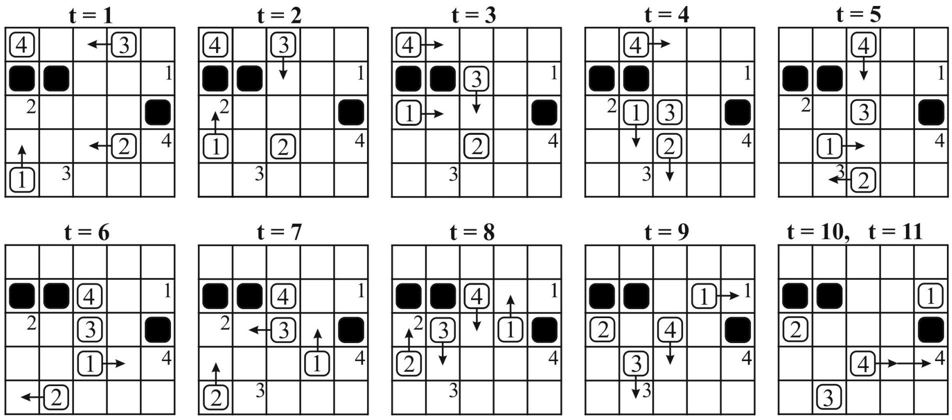

Fig. 2 Multiagent dynamics

is why there is no obvious way to solve them. Example in Figure 3 shows the motion of four agents with hierarchical model of interaction in discrete dynamic undoundedly uncertain fully observable environment. The agents are denoted by white squares with some digits inside them. Black squares symbolize discrete states occupied by nature. Goal states are denoted by digits in left top part of cells. The results of this paper have applications in robotics, computer graphics, and some other areas related to artificial intelligence.

References

1. *N.K. Bose* Multidimensional Systems Theory and Applications. Dordrecht: Springer, 2003.
2. *J.E. Goodman and J. O'Rourke* Handbook of Discrete and Computational Geometry. London: Chapman & Hall / CRC, 2004.
3. *S. Russell and P. Norvig* Artificial Intelligence : a Modern Approach. London: Prentice Hall, 2010.
4. *V.N. Kolokoltsov and V.P. Maslov* Idempotent analysis and its applications. Mathematics and its Applications, vol. 401, Dordrecht: Kluwer Academic Publishers Group, 1997.
5. F. Baccelli, G. Cohen, G.J. Olsder, J.-P. Quadrat Synchronization and Linearity : an algebra for discrete event systems. New York: Wiley, 1992.
6. *V.N. Kolokoltsov and O.A. Malafeyev* Understanding Game Theory : Introduction to the Analysis of Many Agent Systems with Competition and Cooperation. Singapore: World Scientific, 2010.
7. *T. Bazar and G.J. Olsder* Dynamic Noncooperative Game Theory: New Models and Algorithms. Philadelphia: Society for Industrial and Applied Mathematics, 1999.
8. *X. Allamigeon, S. Gaubert, and R. Katz* Tropical polar cones, hypergraph transversals, and mean payoff games. Linear algebra and its Applications, vol. 435, no. 7, 2011, 1549-1574.
9. *B. Carre* Graphs and Networks. London: Oxford University Press, 1979.
10. *M. Gondran and M. Minoux* Graphs, Dioids and Semirings: New Models and Algorithms. Heidelberg: Springer Science + Business Media, 2008.
11. *J. Hopcroft, R. Motwani, and J. Ullman* Introduction to Automata Theory, Languages, and Computation. Addison-Wesley, 2001.
12. *J.S. Golan* Semirings and their applications. Dordrecht: Kluwer Academic Publishers, 1999.

13. *L. Hogben* Handbook of Linear Algebra. London: Chapman & Hall / CRC, 2007.
14. *G.H. Golub and C.F. Van Loan* Matrix Computations. London: John Hopkins Studies in the Mathematical Sciences, 1996.
15. *M. Erdmann and T. Lozano-Perez* On Multiple Moving Objects, Algorithmica, vol. 2, 1987, pp. 477 – 521.

The work was partially supported by RFBR grant 11-07-00580-a.

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The structure of max-plus hemispaces

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1 Introduction

The work is a continuation of [5], [6] and [7], where max-plus segments, max-plus semispaces, and max-plus hyperplanes were studied. Here we describe the structure of max-plus hemispaces. We recall that $R_{\max} := R \cup \{-\infty\}$ is a semifield with the operations $\oplus = \max, \otimes = +$, and that $R^n_{\max} := R_{\max} \times ... \times R_{\max}$ (n times) is a semimodule over $R_{\max}$. For $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in R^n_{\max}$, $\alpha \in R_{\max}$ define

$$x \oplus y := (\max(x_1, y_1), ..., \max(x_n, y_n)), \quad \alpha x := (\alpha + x_1, ..., \alpha + x_n). \tag{1}$$

These are the basic operations in $R^n_{\max}$.

In analogy to the case of the real linear space, one can introduce canonically the notion of convexity in a semimodule over a semiring (see [8], [9], [3]). If $x, y \in R^n_{\max}$, the set $[x, y] := \{(\alpha x) \oplus (\beta y) \in R^n_{\max} | \, \alpha, \beta \in R_{\max}, \alpha \oplus \beta = 0\}$ where 0 is the neutral element of $\otimes = +$ in $R_{\max}$, is called the *max-plus segment joining x and y*. A subset G of $R^n_{\max}$ is said to be *max-plus convex* if along with any two points it contains the whole segment joining them, i.e., if $x, y \in G \Rightarrow [x, y] \subseteq G$.

A *(max-plus) hyperplane H* in $R^n_{\max}$ is defined as the set of points x satisfying:

$$a_1 x_1 \oplus a_2 x_2 \oplus ... \oplus a_n x_n \oplus a_{n+1} = b_1 x_1 \oplus b_2 x_2 \oplus ... \oplus b_n x_n \oplus b_{n+1}, \tag{2}$$

with $a_1, ..., a_n, a_{n+1}, b_1, ..., b_n, b_{n+1} \in R_{\max}$, where each side of (2) contains at least one term, and where $a_i \neq b_i$ for at least one index i. Formula (2) is called *the equation of the hyperplane H*. In contrast to the case of the usual linear space R^n, here one needs an *affine function* on each side; indeed, one cannot simplify the equation (2) by moving one of the terms to the other side, since the operation $\oplus$ does not admit an inverse operation. Among the papers investigating hyperplanes we mention [4], [3] and [7]. If one replaces the equality sign in (2) by $\leq$, the set of solutions of the inequality is called *max-plus halfspace.* Due to the equivalence

$$a \leq b \quad \leftrightarrow \quad a \oplus b = b, \quad a, b \in R^{+}_{\max},$$

it follows that any halfspace is also a hyperplane.

Another class of convex sets in $R^n_{\max}$ is given by semispaces. When $z \in R^n_{\max}$, a subset $S(z)$ of $R^n_{\max}$ is called a *(max-plus) semispace at* z, if it is a maximal max-plus convex subset of $R^n_{\max} \backslash \{z\}$; a subset S of $R^n_{\max}$ will be called a *(max-plus) semispace,* if there exists $z \in R^n_{\max}$ such that $S = S(z)$. We will refer to the point z as the *center* of the semispace. It is shown in [5] and [6] that in $R^n_{\max}$ there exist at most $n+1$ semispaces at each point, and exactly $n+1$ at each finite point; in particular, each max-plus convex set is contained in at least one of those semispaces.

The structure of max-plus hemispaces in $R^n_{\max}$ is the main topic of the talk. A *max-plus hemispace* is a max-plus convex set with max-plus convex complementary. The hemispaces are relevant due to Stone–Kakutani theorem: For any two disjoint convex sets $C_1, C_2 \subset R^n_{\max}$ there exists a hemispace $D \subset R^n_{\max}$ such that $C_1 \subset D, C_2 \subset R^n_{\max} \setminus D$.

Our description of hemispaces is combinatorial and geometric, and is partially motivated by [7], in which one can find a striking relationship between nondegenerate max-plus hyperplanes, that is, those for which the equation (2) contains all the variables $x_1, \ldots, x_n$, and max-plus semispaces. More precisely, the set corresponding to a nondegenerate max-plus hyperplane can be described as the union of two sets: the first one is a union of complements of max-plus semispaces and the second one is the boundary of a union of complements of max-plus semispaces. All semispaces that appear have a common center (or apex). Moreover, any union as above coincides with a max-plus hyperplane. If the hyperplane does not contain the second set above, it is also a halfspace, and this prevents the hyperplane from being a hemispace. We will show that the hemispaces are essentially halfspaces. One also needs to understand the max-plus convex decomposition in two sets of their boundary.

2 Hemispaces in $R^2_{\max}$

We describe the structure of hemispaces in $R^2_{\max}$. The following theorem is [5, Theorem 3.1].

Theorem 1 (Theorem 3.1 [5]) *Let* $z = (z_1, z_2) \in R^2_{\max}$.

1. *If z is finite, then there are three semispaces at z:*

$$S_0(z) = \{x = (x_1, x_2) \in R^2_{\max} \mid 0 < \max(x_1 - z_1, x_2 - z_2)\},$$
$$S_1(z) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_1 < \max(z_1 + x_2 - z_2, z_1)\},$$
$$S_2(z) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_2 < \max(z_2 + x_1 - z_1, z_2)\}.$$

2. *If $z = (-\infty, z_2)$, $z_2 > -\infty$, then there are two semispaces at z:*

$$S_3(z) = \{(-\infty, x_2) \in R^2_{\max} \mid x_2 < z_2\} \cup \{(x_1, -\infty) \in R^2_{\max} \mid x_1 > -\infty\} \cup R^2,$$
$$S_4(z) = \{(-\infty, x_2) \in R^2_{\max} \mid x_2 > z_2\} \cup \{(x_1, -\infty) \in R^2_{\max} \mid x_1 > -\infty\} \cup R^2.$$

3. *If $z = (z_1, -\infty)$, $z_1 > -\infty$, then there are two semispaces at z:*

$$S_5(z) = \{(x_1, -\infty) \in R^2_{\max} \mid x_1 < z_1\} \cup \{(-\infty, x_2) \in R^2_{\max} \mid x_2 > -\infty\} \cup R^2,$$
$$S_6(z) = \{(x_1, -\infty) \in R^2_{\max} \mid x_1 > z_1\} \cup \{(-\infty, x_2) \in R^2_{\max} \mid x_2 > -\infty\} \cup R^2.$$

4. *If $z = (-\infty, -\infty)$, then there is one semispace at z:*

$$S_7(z) = R^2_{\max} \setminus \{(-\infty, -\infty)\}.$$

The following theorem is [5, Theorem 5.2] for case $n = 2$.

Theorem 2 *The family of semispaces of Theorem 1 is the smallest intersectional basis for the family of all proper convex sets in $R^2_{\max}$, that is, any proper convex set $S \subset R^2_{\max}$ is equal to the intersection of the semispaces containing S. Consequently, the complement of S is equal to the union of the complements of the semispaces containing S.*

In what follows we denote by $\complement S$ the complement of the set S.

Proposition 1 *Let $z = (z_1, z_2) \in R^2_{\max}$ and let $0 \leq k \leq 7$. Denote $T_k(z) = \complement S_k(z)$.*

(0) For $k = 0$ and z finite, denote $\xi = (\xi_1, \xi_2) := (z_1, z_2)$ and

$$T_0(\xi, (1,1)) = T_0(z),$$
$$T_0(\xi, (0,1)) = T_0(z) \setminus \{x \in R^2_{\max} \mid x_1 = \xi_1\},$$
$$T_0(\xi, (1,0)) = T_0(z) \setminus \{x \in R^2_{\max} \mid x_2 = \xi_2\},$$
$$T_0(\xi, (0,0)) = T_0(z) \setminus \left(\{x \in R^2_{\max} \mid x_1 = \xi_1\} \cup \{x \in R^2_{\max} \mid x_2 = \xi_2\}\right),$$
$$T_0((\xi_1, \infty), (1)) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_1 \leq \xi_1\}$$
$$T_0((\xi_1, \infty), (0)) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_1 < \xi_1\}$$
$$T_0((\infty, \xi_2), (1)) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_2 \leq \xi_2\}$$
$$T_0((-\infty, \xi_2), (0)) = \{x = (x_1, x_2) \in R^2_{\max} \mid x_2 < \xi_2\}.$$

(1) *For* $k = 1$ *and* z *finite, denote* $\xi = (\xi_1, \xi_2) := (-z_1, z_2 - z_1)$ *and*

$$T_1(\xi, (1,1)) = T_1(z)$$
$$T_1(\xi, (0,1)) = T_1(z) \setminus \{x \in R^2_{\max} |\ 0 = x_1 + \xi_1\}$$
$$T_1(\xi, (1,0)) = T_1(z) \setminus \{x \in R^2_{\max} |\ x_2 = x_1 + \xi_2\}$$
$$T_1(\xi, (0,0)) = T_1(z) \setminus \big(\{x \in R^2_{\max} |\ 0 = x_1 + \xi_1\} \cup \{x \in R^2_{\max} |\ x_2 = x_1 + \xi_2\}\big)$$
$$T_1((\xi_1, \infty), (1)) = \{x = (x_1, x_2) \in R^2_{\max} |\ 0 \le x_1 + \xi_1\}$$
$$T_1((\xi_1, \infty), (0)) = \{x = (x_1, x_2) \in R^2_{\max} |\ 0 < x_1 + \xi_1\}$$
$$T_1((-\infty, \xi_2), (1)) = \{x = (x_1, x_2) \in R^2_{\max} |\ x_2 \le x_1 + \xi_2\}$$
$$T_1((-\infty, \xi_2), (0)) = \{x = (x_1, x_2) \in R^2_{\max} |\ x_2 < x_1 + \xi_2\}.$$

(2) *For* $k = 2$ *and* z *finite, denote* $\xi = (\xi_1, \xi_2) := (z_1 - z_2, -z_2)$ *and*

$$T_2(\xi, (1,1)) = T_2(z)$$
$$T_2(\xi, (0,1)) = T_2(z) \setminus \{x \in R^2_{\max} |\ x_1 = x_2 + \xi_1\}$$
$$T_2(\xi, (1,0)) = T_2(z) \setminus \{x \in R^2_{\max} |\ 0 = x_2 + \xi_2\}$$
$$T_2(\xi, (0,0)) = T_2(z) \setminus \big(\{x \in R^2_{\max} |\ x_1 = x_2 + \xi_1\} \cup \{x \in R^2_{\max} |\ 0 = x_2 + \xi_2\}\big)$$
$$T_2((\xi_1, -\infty), (1)) = \{x = (x_1, x_2) \in R^2_{\max} |\ x_1 \le x_2 + \xi_1\}$$
$$T_2((\xi_1, -\infty), (0)) = \{x = (x_1, x_2) \in R^2_{\max} |\ x_1 < x_2 + \xi_1\}$$
$$T_2((-\infty, \xi_2), (1)) = \{x = (x_1, x_2) \in R^2_{\max} |\ 0 \le x_2 + \xi_2\}$$
$$T_2((-\infty, \xi_2), (0)) = \{x = (x_1, x_2) \in R^2_{\max} |\ 0 < x_2 + \xi_2\}.$$

(3) *For* $k = 3$ *and* $z = (-\infty, z_2)$*,* $z_2 > -\infty$*, denote*

$$T_3(z, (1)) = T_3(z)$$
$$T_3(z, (0)) = T_3(z) \setminus \{(-\infty, z_2)\}$$
$$T_3((-\infty, -\infty)) = \{x = (-\infty, z_2) \in R^2_{\max} | z_2 > -\infty\}.$$

(4) *For* $k = 4$ *and* $z = (-\infty, z_2)$*,* $z_2 > -\infty$*, denote*

$$T_4(z, (1)) = T_4(z)$$
$$T_4(z, (0)) = T_4(z) \setminus \{(-\infty, z_2)\}$$
$$T_4((-\infty, \infty)) = \{x = (-\infty, z_2) \in R^2_{\max}\}.$$

(5) *For* $k = 5$ *and* $z = (z_1, -\infty)$*,* $z_1 > -\infty$*, denote*

$$T_5(z, (1)) = T_5(z)$$
$$T_5(z, (0)) = T_5(z) \setminus \{(z_1, -\infty)\}$$
$$T_5((-\infty, -\infty)) = \{x = (z_1, -\infty) \in R^2_{\max} | z_1 > -\infty\}.$$

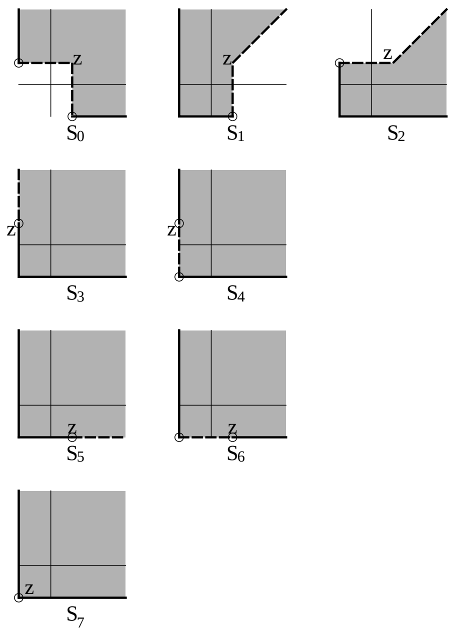

Fig. 1 Semispaces in dimension 2

(6) For $k = 6$ and $z = (z_1, -\infty)$, $z_1 > -\infty$, denote

$$T_6(z,(1)) = T_6(z)$$
$$T_6(z,(0)) = T_6(z) \setminus \{(z_1, -\infty)\}$$
$$T_6((\infty, -\infty)) = \{x = (z_1, -\infty) \in R^2_{\max}\}.$$

(7) For $k = 7$, denote $T_7(-\infty, -\infty) = \{(-\infty, -\infty)\}$.

All sets $T_k, 0 \le k \le 7$, introduced above are max-plus hemispaces.

See Figures 2 and 3 for illustrations of the sets in Theorem 2.

Definition 1 Let $S \subset R^n_{\max}$. The *max-plus convex hull* conv(S) of the set S is the smallest convex set containing S. Equivalently, conv(S) is the intersection of all convex sets containing S.

The proof of Proposition 1 is based on the observation that each hemispace can be represented as a union of complements of semispaces, which consist of just one sector in the two-dimensional case. The following observation is behind the description of "two-dimensional" hemispaces in Proposition 1 and Figure 2.

Proposition 2 *Let $I \neq \emptyset$ be a set, let $z^j = (z^j_1, z^j_2) \in R^2_{\max}, j \in I$ be finite, let $0 \le k \le 2$, and let F be the max-plus convex hull of $\cup_{j\in I} \complement S_k(z^j)$.*

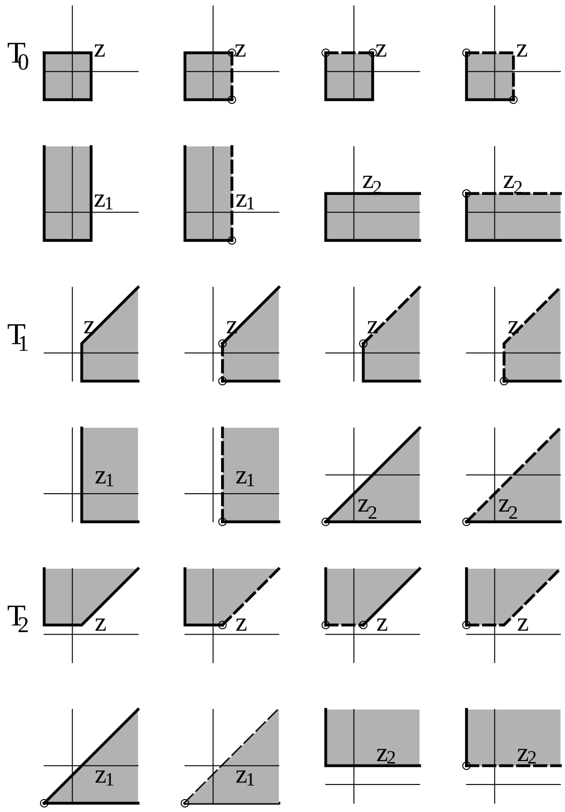

Fig. 2 Hemispaces in dimension 2, I

(0) If $k=0$, *let* $(\xi_1^k,\ \xi_2^k) := (z_1^k,\ z_2^k)$,
(1) If $k=1$, *let* $(\xi_1^k,\ \xi_2^k) := (-z_1^k,\ z_2^k - z_1^k)$,
(2) If $k=2$, *let* $(\xi_1^k,\ \xi_2^k) := (z_1^k - z_2^k,\ -z_2^k)$.

In all cases, let $(\xi_1, \xi_2) := (\sup_{j\in I} \xi_1^j,\ \sup_{j\in I} \xi_2^j)$. *Then*

(i) If $\xi_1 < \infty, \xi_2 < \infty$, *then* $F = T_k(\xi, (\ell_1, \ell_2))$. *The label* $\ell_i, 1 \le i \le 2$, *is* 1 *if the maximum in* $\sup_{j\in I} \xi_i^j$ *is reached, and* 0 *otherwise.*
(ii) If $\xi_1 = \infty, \xi_2 < \infty$, *then* $F = T_k((\infty, \xi_2), (\ell))$. *The label* ℓ *is* 1 *if the maximum in* $\sup_{j\in I} \xi_2^j$ *is reached, and* 0 *otherwise.*
(iii) If $\xi_1 < \infty, \xi_2 = \infty$, *then* $F = T_k((\xi_1, \infty), (\ell))$. *The label* ℓ *is* 1 *if the maximum in* $\sup_{j\in I} \xi_1^j$ *is reached, and* 0 *otherwise.*
(iv) If $\xi_1 = \infty, \xi_2 = \infty$, *then* $F = R^2_{\max}$.

The following observation is behind the description of "one-dimensional" hemispaces in Proposition 1 and Figure 3.

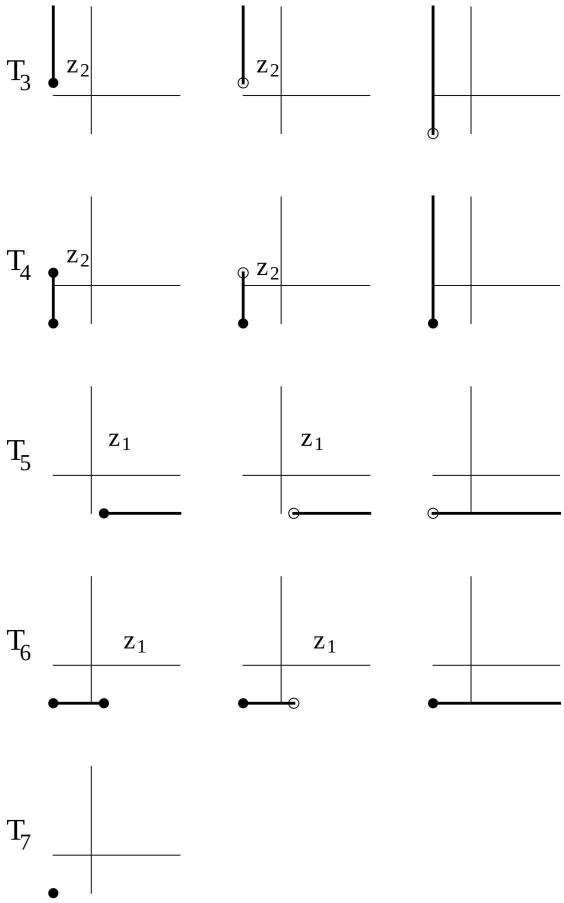

Fig. 3 Hemispaces in dimension 2, II

Proposition 3 *Let $I \neq \emptyset$ be a set, let $z^j = (z_1^j, z_2^j) \in R^2_{\max}, j \in I$ be finite, let $3 \leq k \leq 7$, and let F be the max-plus convex hull of $\cup_{j\in I} \complement S_k(z^j)$.*

(3) Let $k = 3$, $z_2 = \inf_{k\in I} z_2^k$, $\quad z = (-\infty, z_2)$.

(i) If $-\infty < z_2$, then $F = T_3(z, (\ell))$. The label ℓ is 1 if the minimum in $\inf_{k\in I} z_2^k$ is reached, and 0 otherwise.

(ii) If $z_2 = -\infty$ then $F = T_3((-\infty, -\infty))$.

(4) Let $k = 4$, $z_2 = \sup_{k\in I} z_2^k$, $\quad z = (-\infty, z_2)$.

(i) If $z_2 < \infty$, then $F = T_4(z, (\ell))$. The label ℓ is 1 if the maximum in $\sup_{k\in I} z_2^k$ is reached, and 0 otherwise.

(ii) If $z_2 = \infty$ then $F = T_4((-\infty, \infty))$.

(5) Let $k = 5$, $z_1 = \inf_{k\in I} z_1^k, z = (z_1, -\infty)$.

(i) *If* $-\infty < z_1$, *then* $F = T_5(z, (\ell))$. *The label* ℓ *is* 1 *if the minimum in* $\inf_{k \in I} z_1^k$ *is reached, and* 0 *otherwise.*

(ii) *If* $z_1 = -\infty$ *then* $F = T_5((-\infty, -\infty))$.

(6) *Let* $k = 6$, $z_1 = \sup_{k \in I} z_1^k, \quad z = (z_1, -\infty)$.

(i) *If* $z_1 < \infty$, *then* $F = T_6(z, (\ell))$. *The label* ℓ *is* 1 *if* $\sup_{k \in I} z_1^k$ *is reached, and* 0 *otherwise.*

(ii) *If* $z_1 = \infty$ *then* $F = T_6((\infty, -\infty))$.

(7) *Let* $k = 7$. *Then* $F = T_7((-\infty, -\infty)) = \{(-\infty, -\infty)\}$.

Theorem 3 *Let* $H \subset R_{\max}^2$ *be a hemispace. Then* H *is either one of the hemispaces listed in Proposition 1, or one of their complements.*

Propositions 1 and 2 exhibit a kind of "rotational" symmetry, which is due to the fact that each hemispace can be represented as a section of a homogeneous (or conical) hemispace, via the homogenization $x \mapsto \{(\lambda, \lambda x) \mid \lambda \in R_{\max}\}$. To this end, Propositions 1 and 2 could be written in the homogeneous setting, assuming $z_0 = 0$ and redefining ξ_1 and ξ_2 accordingly.

General treatment of max-plus hemispaces in higher dimensions will be presented in a forthcoming joint work with Ricardo D. Katz (Rosario, Argentina).

References

1. W. Briec, C.D. Horvath, Halfspaces and Hahn-Banach like properties in B-convexity and Max-Plus convexity, Pacific J. Optim. 2, 2008, 293–317.
2. W. Briec, C.D. Horvath, A.M. Rubinov, Separation in $\mathbb{B}$-convexity, Pacific J. Optim. 1 (2005), 13–30.
3. M. Develin, B. Sturmfels, Tropical convexity. Documenta Math., 9 (2004), 1–27.
4. M. Joswig, Tropical halfspaces. Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., vol. 52, Cambridge Univ. Press, Cambridge (2005), 409–431.
5. V. Nitica, I. Singer, Max-plus convex sets and max-plus semispaces. I. Optimization 56 (2007), 171–205.
6. V. Nitica, I. Singer, Max-plus convex sets and max-plus semispaces. II. Optimization 56 (2007), 293–303.
7. V. Nitica, I. Singer, The structure of max-plus hyperplanes. Lin. Alg. Appl. 426 (2007), 382–414.
8. K. Zimmermann, A general separation theorem in extremal algebras. Ekonom.-Mat. Obzor 13 (1977), 179–201.
9. K. Zimmermann, Convexity in semimodules. Ekonom.-Mat. Obzor 17 (1981), 199–213.

The work is supported by EPSRC grant RRAH15735, joint RFBR-CNRS grant 11–01–93106-a, RFBR grant 12–01–00886-a (S. Sergeev), and Simons Foundation grant 208729 (both authors).

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Semispaces in the max-min convexity

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1 Introduction

The max-min semiring is defined on the unit interval $\mathcal{B} = [0, 1]$ with arithmetics $a \oplus b := \max(a, b)$ and $ab := \min(a, b)$. These arithmetics can be naturally extended to matrices and vectors leading to the max-min (fuzzy) linear algebra of [1, 4]. The max-min vectors belong to $\mathcal{B}^n$, the Cartesian product of n copies of $\mathcal{B}$. The **max-min interval** between $x, y \in \mathcal{B}^n$ is defined as

$$\begin{aligned}&\operatorname{conv}(\{x, y\}) = \{\alpha x \oplus (\beta y) \mid \alpha \oplus \beta = 1\}, \text{ or}\\ &\operatorname{conv}(\{x, y\}) = \{\max(\min(\alpha, x_i), \min(\beta, y_i)) \; \forall i \mid \max(\alpha, \beta) = 1\}.\end{aligned} \tag{1}$$

A set $C \subseteq \mathcal{B}^n$ is called **max-min convex**, if it contains, with any two points x, y, the interval (1) between them. For a general subset $\mathcal{X} \subseteq \mathcal{B}^n$, define its **convex hull** $\operatorname{conv}(\mathcal{X})$ as the smallest max-min convex set containing $\mathcal{X}$, i.e., the smallest set containing $\mathcal{X}$ and stable under taking intervals (1).

A **semispace** at $x \in \mathcal{B}^n$ is defined as a maximal max-min convex set not containing x. A strightforward application of Zorn's Lemma shows that if $x \notin C$ and C is convex, then x can be separated from C by a semispace. It follows that the semispaces constitute the smallest intersectional basis of max-min convex sets (and more generally in abstract convexity).

The max-min segments and semispaces were described, respectively, in [8,11] and in [9]. In [6, 7] the authors made a further progress in the study of max-min convexity focusing on the role of semispaces. Being motivated by the Hahn-Banach separation theorems in the tropical (max-plus) convexity [12] and extensions to functional and abstract idempotent semimodules [2, 5, 13], we compared semispaces to max-min hyperplanes in [6], and developed an interval extension of separation by semispaces in [7]. These results are summarized below in Section 3. New results and conjectures about the Carathéodory Theorem and its colorful extensions are given in Section 4, inspired by Gaubert and Meunier [3].

2 Description of semispaces

For any point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathcal{B}^n$ we define a family of sets $S_0(x^0), \ldots, S_n(x^0)$ in $\mathcal{B}^n$. These sets were shown to be semispaces in [9, Proposition 4.1]. Recall that x^0 is called *finite* if it has all coordinates different from zeros and ones. Without loss of generality we may assume that: $x_1^0 \geq \cdots \geq x_n^0$. Writing this more precisely we have

$$\begin{aligned} x_1^0 = \cdots = x_{k_1}^0 > \cdots > x_{k_1+l_1+1}^0 = \cdots = x_{k_1+l_1+k_2}^0 > \cdots \\ > x_{k_1+l_1+k_2+l_2+1}^0 = \cdots = x_{k_1+l_1+k_2+l_2+k_3}^0 > \cdots \\ > x_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+1}^0 = \cdots = x_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+k_p}^0 \\ > \cdots > x_{k_1+l_1+\cdots+k_p+l_p}^0 (= x_n^0). \end{aligned} \tag{2}$$

Let us introduce the following notations:

$$\begin{aligned} &L_0 = 0, K_1 = k_1, L_1 = K_1 + l_1 = k_1 + l_1, \\ &K_j = L_{j-1} + k_j = k_1 + l_1 + ... + k_{j-1} + l_{j-1} + k_j \quad (j = 2, ..., p), \\ &L_j = K_j + l_j = k_1 + l_1 + ... + k_j + l_j \quad (j = 2, ..., p); \end{aligned}$$

we observe that $l_j = 0$ if and only if $K_j = L_j$.

We are ready to define the sets. We need to distinguish the cases when the sequence (2) ends with zeros or begin with ones, since some sets S_i become empty in that case.

Definition 1 a) If x^0 is finite, then:

$$\begin{aligned} S_0(x^0) =& \{x \in \mathcal{B}^n | x_i > x_i^0 \text{ for some } 1 \leq i \leq n\}, \\ S_{K_j+q}(x^0) =& \{x \in \mathcal{B}^n | x_{K_j+q} < x_{K_j+q}^0, \text{ or } x_i > x_i^0 \\ & \text{for some } K_j + q + 1 \leq i \leq n\} (q = 1, ..., l_j; j = 1, ..., p) \text{ if } l_j \neq 0, \\ S_{L_{j-1}+q}(x^0) =& \{x \in \mathcal{B}^n | x_{L_{j-1}+q} < x_{L_{j-1}+q}^0, \text{ or } x_i > x_i^0 \\ & \text{for some } K_j + 1 \leq i \leq n\} \\ & (q = 1, ..., k_j; j = 1, ..., p \text{ if } k_1 \neq 0, \text{ or } j = 2, ..., p \text{ if } k_1 = 0). \end{aligned}$$

b) If there exists an index $i \in \{1, ..., n\}$ such that $x_i^0 = 1$, but no index j such that $x_j^0 = 0$, then the sets are $S_1, ..., S_n$ of part a).

c) If there exists an index $j \in \{1, ..., n\}$ such that $x_j^0 = 0$, but no index i such that $x_i^0 = 1$, then the sets are $S_0, S_1, ..., S_{\beta-1}$ of part *a)*, where $\beta := \min\{1 \leq j \leq n |\ x_j^0 = 0\}$.

d) If there exist an index $i \in \{1, ..., n\}$ such that $x_i^0 = 1$, and an index j such that $x_j^0 = 0$, then the sets are $S_1, ..., S_{\beta-1}$.

Proposition 1 ([9]) *For any $x^0 \in \mathcal{B}^n$ the sets $S_i(x^0), 1 \leq i \leq n$, are max-min convex.*

3 Separation and non-separation

In the tropical convexity, all semispaces are open halfspaces expressed as solution sets to a strict two-sided max-linear inequality. The closures of semispaces are hyperplanes. In the case of max-min convexity, hyperplane can be defined as the solution set to

$$\begin{aligned} &\max(\min(a_1, x_1), \ldots, \min(a_n, x_n), a_{n+1}) = \\ &\max(\min(b_1, x_1), \ldots, \min(b_n, x_n), b_{n+1}). \end{aligned} \tag{3}$$

In [6] we investigated the relation between the max-min hyperplanes and the closures of semispaces (in the usual topology), which we denote by $\overline{S}_i(x)$.

Theorem 1 ([6], Theorem 3.1) *A closure of semispace is a hyperplane if and only if it can be represented as $\overline{S}_i(y)$ for some y belonging to the* **diagonal** $\mathcal{D}_n = \{(a, \ldots, a) \mid a \in \mathcal{B}\}$.

This theorem shows exactly when the classical separation by hyperplanes is always possible.

Corollary 1 ([6], Coro. 3.3 and 3.4) *Let $x \in \mathcal{B}^n$, then any closed max-min convex set $C \subseteq \mathcal{B}^n$ not containing x can be separated from x by a hyperplane if and only if x lies on the diagonal.*

A geometric idea of the proof of Theorem 1 is to construct examples of non-separation in $\mathcal{B}^2$ (if a point does not belong to the diagonal), and to extend it cylindrically to higher dimensions.

Below $[a, c]$ denotes the ordinary interval on the real line $\{b \colon a \leq b \leq c\}$, provided $a \leq c$ (and possibly $a = c$).

In [7], we found a way to enhance separation by semispaces showing that a point can be replaced by a box, i.e., a Cartesian product of closed intervals. Namely, we investigated the separation of a box $B = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$

from a max-min convex set $C \subseteq \mathcal{B}^n$, by which we mean that there exists a set S described in Definition 1, which contains C and avoids B.

Assume that $\overline{x}_1 \geq \ldots \geq \overline{x}_n$ and suppose that $t(B)$ is the greatest integer such that $\overline{x}_{t(B)} \geq \underline{x}_i$ for all $1 \leq i \leq t(B)$. We will need the following condition:

$$\begin{gathered} \text{If } (\overline{x}_1 = 1) \ \& \ (y_l \geq \underline{x}_l, 1 \leq l \leq n) \ \& \\ (\overline{x}_l < y_l \text{ for some } l \leq t(B)), \quad \text{then } y \notin C. \end{gathered} \tag{4}$$

Note that if the box is reduced to a point and if $\overline{x}_1 = 1$, then $\overline{x}_l = 1$ for all $l \leq t(B)$ so that $\overline{x}_l < y_l$ is impossible. So (4) always holds in the case of a point.

Theorem 2 ([7], Theorem 1) *Let $B = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$, and let $C \subseteq \mathcal{B}^n$ be a max-min convex set avoiding B. Suppose that B and C satisfy* (4)*. Then there is a set S described by Definition 1, which contains C and avoids B.*

The box B can be a point and in this case condition (4) always holds. Therefore, some results on max-min semispaces [9] can be deduced from Theorem 2. The following statement is an immediate corollary of Theorem 2 and Proposition 1.

Corollary 2 ([9]) *Let $x \in \mathcal{B}^n$ and $C \subseteq \mathcal{B}^n$ be a max-min convex set avoiding x. Then C is contained in one $S_i(x), 1 \leq i \leq n$, as in Definition 1. Consequently these sets are indeed the family of semispaces at x.*

However, separation by semispaces is impossible when B and C do not satisfy (4).

Theorem 3 ([7], Theorem 2) *Suppose that $B = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n]$ and max-min convex set $C \subseteq \mathcal{B}^n$ are such that $B \cap C = \emptyset$ but the condition* (4) *does not hold. Then there is no semispace that contains C and avoids B.*

In [7] we also investigate the separation of max-min convex sets by a box, and by a box and a semispace. We show that both kinds of separation are always possible $n = 2$, but they do not work in higher dimensions.

4. Carathéodory Theorem. Denote by $\complement S_i(x^0)$ the complement of $S_i(x^0)$. As only a finite number of semispaces at a given point exist, the max-min convexity can be regarded as a multiorder convexity [8, 9].

Lemma 1 (Multiorder Principle) *For $\mathcal{X} \subseteq \mathcal{B}^n$ and $x \in \mathcal{B}^n$, the following are equivalent:*

a) $x \in \operatorname{conv}(\mathcal{X})$;

b) For all i with $x_i \neq 0$ there exists $x^i \in \mathcal{X} \cap \complement S_i(x)$.

Like its tropical (max-plus) analogue, the max-min Carathéodory Theorem can be easily derived either from the multiorder principle above, or from the mere fact that the max-min semiring is linearly (totally) ordered.

Theorem 4 (Carathéodory) *Let $X \subseteq \mathcal{B}^n$ and $x \in \operatorname{conv}(\mathcal{X})$. Then there exist $x^1, \ldots, x^{n+1} \in \mathcal{X}$ such that $x \in \operatorname{conv}(x^1, \ldots, x^{n+1})$*

Inspired by [3], let us formulate the following colourful version of Theorem 4 as a conjecture.

Conjecture 1 (Colorful Carathéodory) Let $C \subseteq \mathcal{B}^n$ have a non-empty intersection with $\operatorname{conv}(\mathcal{X}_1), \ldots, \operatorname{conv}(\mathcal{X}_{n+1})$, where $\mathcal{X}_1, \ldots, \mathcal{X}_{n+1}$ are subsets of $\mathcal{B}^n$. Then there exist $x^i \in \mathcal{X}_i$, $1 \leq i \leq n+1$ such that $C \cap \operatorname{conv}(\{x^1, \ldots, x^{n+1}\})$.

There are two special cases when we know that the conjecture is true: 1) when C reduces to a point, 2) when all points in $\mathcal{X}^1, \ldots, \mathcal{X}^n$ have a fixed ordering of coordinates, say $x_1 \geq x_2 \geq \ldots \geq x_n$. If the ordering of coordinates is not fixed, then the sudden changes of the shape of semispaces become a major obstacle to a direct combinatorial proof. Trying to solve the problem by means of topological arguments, we also observed the following.

Lemma 2 (Internal Separation) *Assume that the max-min convex hull of $\mathcal{X} = \{x^1, \ldots, x^{n+1}\} \subseteq \mathcal{B}^n$ has a non-empty interior. Then for any point x from this interior, if all coordinates of x are different, we have $x^i \in \complement S_i(x)$, for $1 \leq i \leq n$, up to a permutation.*

The proof of the Tropical Colorful Carathéodory Theorem [3] is based on the fact that each convex hull of n points contains a set, where each point has the internal separation property of Conjecture 2. This set can be found algebraically, see [3] or [10], and it is also non-empty when the convex hull does not have interior.

Conjecture 2 For each $\mathcal{X} = \{x^1, \ldots, x^{n+1}\} \subseteq \mathcal{B}^n$, the max-min convex hull of $\mathcal{X}$ contains a point x with **internal separation property**:

- up to a permutation, each semispace $S_i(x)$ corresponds to $x^i \in \complement S_i(x)$.

If Conjecture 2 holds, then so does Conjecture 1. With the assumption that each n-tuple of points $x^i \in \operatorname{conv} \mathcal{X}^i$, for $i = 1, \ldots, n$ can be approximated by points y^i with different coordinates such that $\text{conv}(\{y^1, \ldots, y^n\})$ has a non-empty interior, Conjecture 1 can be proved by means of standard arguments based on compactness and Lemma 2.

It is generally not true that if all entries of a matrix $A \in \mathcal{B}^{n\times(n+1)}$ are different, then the max-min convex hull of the columns of A has non-empty interior.

For example, consider the following two matrices

$$A_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}, \tag{5}$$

The max-min convex hull of the first one has a non-empty interior, while the max-min convex hull of the second one may have only subsets of dimension two. Generalizing this example we obtained the following

Proposition 1 *Let* $A = (a_{ij}) \in \mathcal{B}^{n\times n+1}$ *satisfy* $\max_k a_{ki} \leq \min_k a_{k,i+1}$. *Then convex hull of the columns of* A *does not have any interior.*

References

1. K. Cechlárová. On the powers of matrices in bottleneck/fuzzy algebra. *Linear Alg. Appl.*, 246:97–111, 1996.
2. G. Cohen, S. Gaubert, J. P. Quadrat, and I. Singer. Max-plus convex sets and functions. In G. Litvinov and V. Maslov, editors, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics*, pages 105–129. AMS, Providence, 2005. E-print *http://www.arXiv.org/abs/math.FA/0308166*.
3. S. Gaubert and F. Meunier. Carathéodory, helly and the others in the max-plus world. *Discrete and Computational Geometry*, 43(3):648–652, 2010. E-print *http://www.arXiv.org/abs/0804.1361*.
4. M. Gavalec. *Periodicity in Extremal Algebras.* Gaudeamus, Hradec Králové, 2004.
5. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz. Idempotent functional analysis: An algebraic approach. *Math. Notes (Moscow)*, 69(5):758–797, 2001.
6. V. Nitica and S. Sergeev. On hyperplanes and semispaces in max-min convex geometry. *Kybernetika*, 46(3):548–557, 2010.
7. V. Nitica and S. Sergeev. An interval version of separation by semispaces in max-min convexity. *Linear Alg. Appl.*, 435(7):1637–1648, 2011.
8. V. Nitica and I. Singer. Contributions to max-min convex geometry. i: Segments. *Linear Alg. Appl.*, 428(7):1439–1459, 2008.
9. V. Nitica and I. Singer. Contributions to max-min convex geometry. ii: Semispaces and convex sets. *Linear Alg. Appl.*, 428(8-9):2085–2115, 2008.
10. S. Sergeev. Multiorder, Kleene stars and cyclic projectors in the geometry of max cones. In G. L. Litvinov and S. N. Sergeev, editors, *Tropical and Idempotent Mathematics*, volume 495 of *Contemporary Mathematics*, pages 317–342. AMS, Providence, 2009. E-print *http://www.arXiv.org/abs/0807.0921*.
11. S. N. Sergeev. Algorithmic complexity of a problem of idempotent convex geometry. *Math. Notes (Moscow)*, 74(6):848–852, 2003.
12. K. Zimmermann. A general separation theorem in extremal algebras. *Ekonom.-Mat. Obzor (Prague)*, 13:179–201, 1977.
13. K. Zimmermann. Convexity in semimodules. *Ekonom.-Mat. Obzor (Prague)*, 17:199–213, 1981.

The work is supported by EPSRC grant RRAH15735, joint RFBR-CNRS grant 11–01–93106-a, RFBR grant 12–01–00886-a (S. Sergeev), and Simons Foundation grant 208729 (both authors).

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Spectral approach to composition formulas

Michael Pevzner

Abstract We shall discuss an alternative approach to composition formulas (*-products) of quantized operators based on the representation theory of underlying Lie groups. Two examples, leading to interactions with number theory, will be presented.

1 Weyl quantization

The Weyl symbolic calculus gives a rigorous mathematical framework for the quantization procedure. It defines a correspondence that associates to a function (classical observable) $\mathfrak{S} = \mathfrak{S}(x, \xi)$ of $n+n$ variables, lying in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, the operator (quantum observable) $Q(\mathfrak{S})$, called the operator with symbol $\mathfrak{S}$, defined by the equation:

$$(Q(\mathfrak{S})\, u)(x) = \hbar^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{S}(\frac{x+y}{2}, \eta)\, e^{\frac{2i\pi}{\hbar}\langle x-y,\, \eta\rangle}\, u(y)\; dy\, d\eta\,. \tag{1}$$

Such a linear operator extends as a continuous operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ while, in the case when $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, one can still define $Q(\mathfrak{S})$ as a linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$; also, Q sets up an isometry from $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. The composition $\mathfrak{S}_1 \,\#\, \mathfrak{S}_2$ of two symbols, lying for instance in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, is defined by the formula

$$Q(\mathfrak{S}_1)\, Q(\mathfrak{S}_2) = Q(\mathfrak{S}_1 \,\#\, \mathfrak{S}_2)\,, \tag{2}$$

where the left-hand side denotes the usual composition of operators. The main concern of this note is to compare two different approaches to the analysis of the $\sharp$-product: the asymptotic and spectral ones.

The celebrated deformation quantization method developed in [1] suggests to avoid the Heisenberg formulation of Quantum mechanics and proposes to consider instead a quantized mechanical system as a “deformation" or a “perturbation" of the classical one.

According to this theory one should interpret quantum observables $Q(f)$ as formal power series in the Planck's constant $\hbar$ on the phase space constructed in such a way that letting $\hbar$ tend to zero one should recover the corresponding classical observable f. The algebraic structure underlying the quantization map Q, *i.e.* the way that the composition of operators $Q(f)$ is reflected on the level of their symbols f, and mechanical data, encoded by the Poisson bracket on the phase space, should also be respected.

In other words one should interpret the quantization procedure as a construction of an associative multiplication law on the space of formal power series in $\hbar$ that would "deform" the usual point-wise product of functions, say f and g, in the "direction" of the Poisson bracket $\{\ \}$:

$$f\sharp g = f \cdot g + \frac{\hbar}{2}\{f, g\} + ... + \frac{\hbar}{n!}B_n(f, g) + ..., \tag{3}$$

where B_n's with $n > 1$ stand for suitable bi-differential operators determined by the associativity condition.

At a first glance such a requirement might look extravagant but the formula (3) may be derived from the asymptotic expansion with respect to $\hbar$ of the composition of two operators within Weyl calculus (1). In fact, let Λ be the canonical Poisson structure on $\mathbb{R}^{2n}$ given by $\Lambda = \sum \Lambda^{ij}\partial_i \wedge \partial_j$, with $\Lambda^{ij} = -\Lambda^{ji} \in \{0, \pm 1\}$. Then the composition of operators with symbols f and g respectively gives rise to a non-commutative product $f\sharp g$ defined by:

$$Q(f) \circ Q(g) = Q(f\sharp g).$$

Naively, a *composition formula* for a quantization map Q is a (not necessarily unique) expression for $f\sharp g$ involving f, and g.

The best known composition formula for the Weyl calculus, usually called *Moyal product*, depends on the Poisson structure Λ and is given by:

$$f\sharp g\,(z) = \exp(i\pi\hbar\,\Lambda^{rs}\partial_{x_r}\partial_{y_s})(f(x)g(y))_{|x=y=z} = f \cdot g + i\pi\hbar\{f, g\} + ...$$

On the other hand it turns out that the above composition formula may also be understood by use of the spectral theory of appropriate invariant differential operators.

Let us observe that the formula (1) actually defines a representation of the convolution algebra $L^1(H^{2n+1})$ of the Heisenberg group H^{2n+1}. Indeed, the image of the Heisenberg representation is the group of unitary transformations $\exp(2i\pi(\langle\eta, q\rangle - \langle y, p\rangle - t))$ of $L^2(\mathbb{R}^n)$, as made meaningful by Stone's theorem, where the j-th component of the vector $q = (q_1, \dots, q_n)$ is the multiplication by the j-th coordinate x_j, $p = (p_1, \dots, p_n)$ with $p_j = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}$, and $y, \eta \in \mathbb{R}^n$, $t \in \mathbb{R}$. Introducing on $(\mathbb{R}^n \times \mathbb{R}^n)^2$ the symplectic form $[\,,\,]$ associated with the canonical Poisson structure Λ, which we can write as

$$[(x, \xi), (y, \eta)] = -\langle x, \eta\rangle + \langle y, \xi\rangle, \tag{4}$$

let us use on $\mathbb{R}^n \times \mathbb{R}^n$ the *symplectic* Fourier transformation $\mathcal{F}$ defined by the equation

$$(\mathcal{F}\,\mathfrak{S})(X) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathfrak{S}(Y)\, e^{-2i\pi\,[X,\,Y]}\, dY, \tag{5}$$

which commutes with all symplectic linear transformations of the variable in $\mathbb{R}^n \times \mathbb{R}^n$. Another, fully equivalent, way to define the Weyl calculus (1) is by means of the equation

$$Q(\mathfrak{S}) = \hbar^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n} (\mathcal{F}\,\mathfrak{S})(y, \eta)\, \exp\left(\frac{2i\pi}{\hbar}\,(\langle\eta, q\rangle - \langle y, p\rangle)\right) dy\, d\eta. \tag{6}$$

In the sequel, in order to make the presentation lighter we normalize the Planck constant by $\hbar = 1$.

The correspondence (1) has two types of symmetries. The first covariance rule of the Weyl calculus is the observation that

$$\begin{aligned}\exp(2i\pi(\langle\eta, q\rangle - \langle y, p\rangle))\; Q(\mathfrak{S})\; \exp(-2i\pi(\langle\eta, q\rangle - \langle y, p\rangle))\\ = Q((x, \xi) \mapsto \mathfrak{S}(x - y, \xi - \eta)).\end{aligned} \tag{7}$$

One way to emphasize this action on symbols of the group of translations of $\mathbb{R}^{2n}$ is to decompose in a systematic way the space of symbols $L^2(\mathbb{R}^{2n})$ with respect to this action. Now, the operators which commute with it are just the partial differential operators with constant coefficients: the generalized joint eigenfunctions of these are exactly the exponentials $X = (x, \xi) \mapsto e^{2i\pi\,[A,\,X]}$ with $A \in \mathbb{R}^{2n}$, and the sought-after decomposition of a symbol is provided by the symplectic Fourier transformation. On the other hand, if $A = (y, \eta)$, the operator with symbol $e^{2i\pi\,[A,\,X]}$ is none other than the operator $\exp(2i\pi(\langle\eta, q\rangle - \langle y, p\rangle))$, so that Heisenberg's commutation relation, expressed in Weyl's exponential version, takes the form

$$e^{2i\pi\,[A^1,\,X]}\,\#\,e^{2i\pi\,[A^2,\,X]} = e^{i\pi\,[A^1,\,A^2]}\,e^{2i\pi\,[A^1+A^2,\,X]}\,. \tag{8}$$

Let us recall some consequences of this relation. First, one has (say, when $\mathfrak{S}_1$ and $\mathfrak{S}_2$ lie in $\mathcal{S}(\mathbb{R}^{2n})$), using (6), the integral composition formula

$$(\mathfrak{S}_1\,\#\,\mathfrak{S}_2)(X) = 2^{2n}\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \mathfrak{S}_1(Y)\,\mathfrak{S}_2(Z)\,e^{-4i\pi\,[Y-X,\,Z-X]}\,dY\,dZ \tag{9}$$

or (a fully equivalent one)

$$(\mathfrak{S}_1\,\#\,\mathfrak{S}_2)(X) = [\,\exp(i\pi L)\,(\mathfrak{S}_1(Y)\,\mathfrak{S}_2(Z))\,]\;(Y=Z=X) \tag{10}$$

with (setting $Y=(y,\,\eta),\ Z=(z,\,\zeta)$)

$$i\pi L = \frac{1}{4i\pi}\sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial y_j\,\partial\zeta_j}+\frac{\partial^2}{\partial z_j\,\partial\eta_j}\right). \tag{11}$$

Expanding the exponential into a series, one obtains the Moyal formula:

$$(\mathfrak{S}_1\,\#\,\mathfrak{S}_2)(x,\,\xi) = \sum\frac{(-1)^{|\alpha|}}{\alpha\,!\,\beta\,!}\left(\frac{1}{4i\pi}\right)^{|\alpha|+|\beta|}\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial\xi}\right)^{\beta}\mathfrak{S}_1(x,\,\xi)\left(\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\mathfrak{S}_2(x,\,\xi)\,. \tag{12}$$

This formula is an exact one in the case when the two operators under consideration are differential operators, which means exactly that their symbols (of course, not in $\mathcal{S}(\mathbb{R}^{2n})$) are polynomials with respect to the variables ξ, with coefficients depending on x in a smooth, but otherwise fairly arbitrary way; it is also exact when one of the two symbols is a polynomial in $(x,\,\xi)$.

As it turns out, this version of the composition formula is the only universally known one. Indeed, it has considerable importance in applications of pseudodifferential analysis to partial differential equations: classes of symbols for which the above formula, without being an exact one, still has some asymptotic value, provide a good proportion of the auxiliary operators needed for the solution of P.D.E. problems.

Our derivation of (9) was obtained as the result of pairing the concept of sharp composition of symbols with the decomposition of symbols according to the action by translations of the group $\mathbb{R}^{2n}$: the success of this point of view was essentially dependent on the fact that this action is an ingredient of the covariance formula (7). This takes us to the aim of the present note: to take advantage of the other covariance property of the Weyl calculus – to be recalled now – and follow the same policy.

Recall that the metaplectic representation Met on $L^2(\mathbb{R}^n)$ is a certain unitary representation of the twofold covering of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, which consists of all linear transformations g of $\mathbb{R}^n \times \mathbb{R}^n$ such that $[gX,\, gY] = [X,\, Y]$ for every pair $(X,\, Y)$ of points of $\mathbb{R}^n \times \mathbb{R}^n$: it acts irreducibly on each of the two subspaces of $L^2(\mathbb{R}^n)$ consisting of functions with a given parity. Unitary transformations in the image of the metaplectic representation also act as automorphisms of the space $\mathcal{S}(\mathbb{R}^n)$ or of the space $\mathcal{S}'(\mathbb{R}^n)$: moreover, if such a unitary transformation U lies above $g \in \mathrm{Sp}(n,\mathbb{R})$, and if $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^{2n})$, one has the covariance formula

$$U\, Q(\mathfrak{S})\, U^{-1} = Q(\mathfrak{S} \circ g^{-1})\,. \tag{13}$$

In full analogy with the procedure adopted above in connection with the Heisenberg representation, we start from a decomposition of the phase space representation $(g,\, \mathfrak{S}) \mapsto \mathfrak{S} \circ g^{-1}$ of $\mathrm{Sp}(n,\mathbb{R})$ in $L^2(\mathbb{R}^{2n})$ into irreducibles: this is just the same as decomposing functions in $L^2(\mathbb{R}^{2n})$ as integral superpositions of functions homogeneous of a given degree, and with a given parity.

In this setting the formula which takes the place of (8) corresponds to the decomposition of the $\sharp$-product of two symbols h_1 and h_2, homogeneous of degrees $-n - i\lambda_1$ and $-n - i\lambda_2$ and with parities characterized by indices δ_1 and δ_2, as an integral superposition of functions homogeneous of degrees $-n - i\lambda$, with the parity $\delta \equiv \delta_1 + \delta_2$ (see Theorem 6.1 [5]). It involves the integral kernel

$$|\,[Y,\, X]\,|_{\varepsilon_2}^{\frac{-n-i\lambda+i\lambda_1-i\lambda_2}{2}}\; |\,[X,\, Z]\,|_{\varepsilon_1}^{\frac{-n-i\lambda-i\lambda_1+i\lambda_2}{2}}\; |\,[Z,\, Y]\,|_{\varepsilon}^{\frac{-n+i\lambda+i\lambda_1+i\lambda_2}{2}}\,, \tag{14}$$

a product of three *signed* powers, obtained from the decomposition into homogeneous components with respect to the three variables of the integral kernel which occurs in the composition formula (9). Some preparation is needed in order to give this kernel a genuine meaning as a distribution, not only as a partially defined function. The principle of the proof of the new composition formula is simple, and relies on the decomposition of symbols into hyperplane waves, and the dual notion of rays. Its main difficulty lies in the singular nature of such distributions, which are nevertheless the only ones, sufficiently general, for which explicit computations are possible.

Triple integrals associated with the singular kernel (14), which are sometimes referred to as generalized Bernstein-Reznikov integrals, were explicitly computed by use of the representation theory of compact groups in [3]. Final formulas for these integrals are closed and involve particular values of higher order hypergeometric functions.

2 Rankin-Cohen quantization

Another interesting case concerns the covariant quantization of the one-sheeted hyperboloid seen as a coadjoint orbit of the Lie group $SL(2,\mathbb{R}) \simeq Sp(1,\mathbb{R})$.

We shall apply the same philosophy as before: find eigenfunctions of corresponding Casimir operators and establish their multiplication table in spirit of an appropriate operator calculus.

More precisely, A. & J. Unterberger developed such a calculus for the one-sheeted hyperboloid in $\mathbb{R}^3$ which may be identified with the symmetric space $SL(2,\mathbb{R})/SO(1,1)$ [7]. Let $\mathcal{X}$ be the set of pairs $(s,t) \in \mathbb{RP}^1 \times \mathbb{RP}^1$ such that $s \neq t$. The group $G = SL(2,\mathbb{R})$ acts on $\mathcal{X}$ by fractional-linear transformations. This set can be seen as a coadjoint orbit of G. It admits an invariant measure $d\mu(s,t) = (s-t)^{-2} ds dt$ and an invariant differential operator

$$\square = (s-t)^2 \frac{\partial^2}{\partial s \partial t}.$$

Moreover, this symmetric space carries an equivariant causal structure.

It turns out that one can build up a symbolic calculus on $L^2(\mathcal{X})$, *i.e.* associate to every function $f \in L^2(\mathcal{X})$ a Hilbert-Schmidt operator $\mathrm{Op}(f)$ acting on $L^2(\mathbb{R})$. This operator is defined by

$$(\mathrm{Op}(f)u)(s) = c_\lambda \int f(s,t)|s-t|^{-1-i\lambda}(\theta u)(t)dt, \qquad \forall u \in L^2(\mathbb{R}),$$

where θ is an operator of convolution with a particular distribution (kernel of some intertwining operator) and c_λ is some coefficient. This symbolic calculus is covariant:

$$\pi_\lambda(g)\mathrm{Op}(f)\pi_\lambda(g^{-1}) = \mathrm{Op}(\rho(g)f), \qquad g \in G,$$

where π_λ is a unitary maximally degenerate principal series representation of the group G on $L^2(\mathbb{R})$ and ρ is the quasi-regular representation of G on $L^2(\mathcal{X})$.

The composition of operators on $L^2(\mathbb{R})$ gives rise to an associative product $\sharp$ on $L^2(\mathcal{X})$:

$$\mathrm{Op}(f)\mathrm{Op}(g) = \mathrm{Op}(f \sharp g).$$

The main result in [7] concerns the existence of algebras of functions on $\mathcal{X}$ with respect to the non commutative product $\sharp$.

In fact, the space $L^2(\mathcal{X})$ decomposes in a direct sum of invariant subspaces for the G-action (this is the spectral decomposition of the operator $\Box$). It is known that the spectrum of the operator $\Box$ contains a continuous and a discrete part of the form $\{-n(n+1), n \in \mathbb{N}\}$. The eigenspace corresponding to the eigenvalue $-n(n+1)$ decomposes as a direct sum $E_n^+ \oplus E_n^-$ of two spaces of representations of the holomorphic and anti-holomorphic discrete series of G. It turns out that the closure, in the space of Hilbert-Schmidt operators, of the vector space $\{\mathrm{Op}(f) \mid f \in \sum_n^{\oplus} E_n^+\}$, is an algebra.

The key point of the proof is the fact that the product $\sharp$ of two functions $f \in E_m^+$ and $g \in E_n^+$ is expressed as a series of terms from the subspaces E_k^+, each of which is given, up to an explicit constant, by a corresponding Rankin-Cohen bracket of f and g :

$$F_k(f,g) = \sum_{\ell=0}^{k} (-1)^\ell \begin{pmatrix} m+k-1 \\ \ell \end{pmatrix} \begin{pmatrix} n+k-1 \\ k-\ell \end{pmatrix} f^{(k-\ell)} g^{(\ell)}. \tag{15}$$

In the case of a general causal symmetric space of Cayley type G/H a similar phenomenon holds (see [4]). Namely, the set of discrete series representations of the symmetric space G/H coming from holomorphic discrete series of G has a non-commutative ring structure.

The tangent space to G/H at the origin admits a G-invariant polarization $T_o(G/H) = V \oplus V$, where V is a real vector space (actually a Euclidean Jordan algebra) of dimension $\frac{1}{2}dim(G/H)$. Using this splitting of the tangent bundle of G/H one develops on it, following [7], a covariant symbolic calculus. The latter is based on maximally degenerate principal series representations of the conformal Lie group G. Let $\pi_\lambda^\pm$ with $\lambda \in i\mathbb{R}$ be such a unitary representation.

There exists a G-equivariant embedding of the space of square integrable functions on the causal space G/H into the algebra (for composition) of Hilbert-Schmidt operators :

$$L^2(G/H) \hookrightarrow \pi_\lambda^+ \otimes \pi_\lambda^- \hookrightarrow \pi_\lambda^+ \otimes \overline{\pi_\lambda^+} \simeq \mathrm{HS}(L^2(V, dx)),$$

where dx is the Lebesgue measure on V.

The first arrow is of geometric nature and its corresponds to the fact that the symmetric space G/H is an open dense subset in $G/P \cap G/\bar{P}$ where P is a maximal parabolic subgroup of G. The last isomorphism is given by

$$L^2(V, dx) \otimes \overline{L^2(V, dx)} \simeq HS(L^2(V, dx)).$$

The composition of operators gives rise therefore to a non-commutative product $\sharp$ on the space $L^2(G/H)$.

The link between this covariant calculus and the holomorphic discrete series of G becomes visible when one deals with the spectral analysis of the above mentioned space $L^2(G/H)$.

One says that a symmetric space G/H has discrete series if the set of representations of G on minimal closed invariant subspaces of $L^2(G/H)$ is non empty. According to a fundamental result of Flensted-Jensen such representations exist if $rank(G/H) = rank(K/K \cap H)$. For a causal symmetric space of Cayley type this condition is satisfied (this fact explains the choice of such a particular geometric setting), and moreover, a part of the discrete spectrum can be realized on the representation spaces of holomorphic discrete series of the group G itself.

Denote by $L^2(G/H)_{\mathrm{hol}}$ the part of the discrete spectrum of G/H coming from holomorphic discrete series representations of G.

Using recent results by T. Kobayashi [2] on tensor products of Harish-Chandra highest weight modules one has (see Theorem 4.5 [4]) :

Theorem 1 *Let π and π' be two representations of holomorphic discrete series of G, and H_π and $H_{\pi'}$ be the corresponding closed irreducible subspaces of $L^2(G/H)_{\mathrm{hol}}$. Then*

$$f \sharp g \in L^2(G/H)_{\mathrm{hol}}, \quad \forall f \in \mathcal{H}_\pi,\ g \in \mathcal{H}_{\pi'}.$$

Therefore, in the case when G is the transformation group of a causal symmetric space of Cayley type, it is natural to define generalized Rankin-Cohen brackets as orthogonal projectors of the tensor product of two holomorphic discrete series representations of the conformal Lie group G onto its, necessarily discrete, irreducible components. More general account on various aspects of the theory of Rankin-Cohen brackets may be found, for instance, in [6].

This work is supported by the RFBR-CNRS grant 11–01–93106.

References

1. F.Bayen, M. Flato, C. Fronsdal, A. Lischnerowicz, D. Sternheimer, Deformation theory and Quantization. *Ann. Phys.* **111**, (1978), pp. 61–110.
2. T. Kobayashi, Discrete Series Representations for the Orbit Spaces Arising from Two Involutions of Real Reductive Lie Groups, *J. Funct. Anal.* **152** (1998), pp. 100–135.
3. J.-L. Clerc, T. Kobayashi, B. Ørsted, M. Pevzner, Generalized Bernstein-Reznikov integrals, to appear in Math. Annalen (2010). E-print: arXiv:0906.2874.
4. G. van Dijk, M. Pevzner, Ring structures for holomorphic discrete series and Rankin-Cohen brackets. *J. Lie Theory* **17** (2007), no. 2, 283–305.
5. T. Kobayashi, B. Ørsted, M. Pevzner, A. Unterberger, Composition formulas in the Weyl calculus, *J. Funct. Anal.* **257**, (2009), pp. 948–991.

6. M. Pevzner, Covariant quantization: spectral analysis versus deformation theory. *Jpn. J. Math.* **3** (2008), no. 2, 247–290.
7. A. Unterberger, J. Unterberger, Algebras of symbols and modular forms. *J. Anal. Math.* **68**, (1996), pp. 121–143.
8. A. Unterberger, Automorphic pseudodifferential analysis and higher level Weyl calculi. Progress in Mathematics, **209**. Birkhäuser Verlag, Basel, 2003.

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Tropicalization of systems biology models

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Abstract Systems biology use networks of biochemical reactions as models for cellular process. The dynamics of reaction networks with many well separated time scales, is well captured by asymptotic models obtained by tropicalization of the smooth dynamics, via the Litvinov-Maslov correspondence principle. The tropicalized models can be used to check the global stability and to identify sensitive parameters and rapid variables of the original models.

1. Introduction

In the last decade, systems biology became the playground of several mathematical fields of study, among which algebraic geometry is one of the most important. Cellular biochemistry can be suitably modelled by networks of reactions with rational and polynomial rate functions. The dynamics of these networks can be described by rational or polynomial ODEs, for which some results exist concerning the type and the complexity of the solutions.

Most of the previous algebraic work on reaction networks was dedicated to the study of steady states [CTF06, Sou03, Son05, RLS+06, RSP+11]. This issue is important, because networks with multiple stable steady states control biological cell fate decisions in development and differentiation [T+98, Del49].

However, biological cell physiology relies preeminently upon network dynamics. To interpret stimuli, adapt to environmental changes, make decisions, the cells lean on the rich dynamical possibilities of regulatory networks. We say that regulatory networks are flexible, because they can support, in principle, any type of attractors and spatial-temporal patterns [VR12]. Networks are also robust, because these patterns resist to perturbations and are maintained for wide ranges of parameter values [GR07, RGZL08]. To achieve robust and flexible functioning, biological networks employ hierarchies of biochemical processes with well separated time scales.

Tropical geometry is well adapted for studying robust and flexible, multi-scale, biochemical networks. In [NGVR12] we used the Litvinov-Maslov correspondence principle to tropicalize rational or polynomial ODE models of biochemical networks. The tropicalization make it possible to develop geometrical methods for critical parameter identification and for studying the qualitative dependence of the model dynamics on these parameters. These methods are based on arrangements of tropical manifolds, reminding combinatorial methods such as polyhedral complexes used in tropical convexity [AD09], or geometric analysis of S-systems proposed by [SCF$^+$09] in relation to biochemical network steady states design. Another application of the tropicalization is the detection of quasi-steady and quasi-equilibrium conditions, that are very useful for model reduction.

2. Settings

In chemical kinetics, the reagent concentrations satisfy ordinary differential equations:

$$\frac{dx_i}{dt} = F_i(\boldsymbol{x}),\ 1 \leq i \leq n. \tag{1}$$

Rather generally, the rates are rational functions of the concentrations and read

$$F_i(\boldsymbol{x}) = P_i(\boldsymbol{x})/Q_i(\boldsymbol{x}), \tag{2}$$

where $P_i(\boldsymbol{x}) = \sum_{\alpha \in A_i} a_{i,\alpha}\boldsymbol{x}^\alpha$, $Q_i(\boldsymbol{x}) = \sum_{\beta \in B_i} b_{i,\beta}\boldsymbol{x}^\beta$, are multivariate polynomials. Here $\boldsymbol{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\boldsymbol{x}^\beta = x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}$, $a_{i,\alpha}, b_{i,\beta}$, are nonzero real numbers, and A_i, B_i are finite subsets of $\mathbb{N}^n$.

Special case are represented by

$$F_i(\boldsymbol{x}) = P_i^+(\boldsymbol{x}) - P_i^-(\boldsymbol{x}), \tag{3}$$

where $P_i^+(\boldsymbol{x})$, $P_i^-(\boldsymbol{x})$ are Laurent polynomials with positive coefficients, $P_i^\pm(\boldsymbol{x}) = \sum_{\alpha \in A_i^\pm} a_{i,\alpha}^\pm \boldsymbol{x}^\alpha$, $a_{i,\alpha}^\pm > 0$, $A_i^\pm$ are finite subsets of $\mathbb{Z}^n$. Real powers $A_i^\pm \subset \mathbb{R}^n$ are sometimes used for the so-called S-systems [SCF$^+$09].

Litvinov and Maslov [LMS01, LM96] proposed a heuristic (correspondence principle) allowing to transform mathematical objects (integrals, polynomials) into their quantified (tropical) versions. According to this heuristic, to a Laurent polynomial with positive real coefficients $\sum_{\alpha \in A} a_\alpha \boldsymbol{x}^\alpha$, where $A \subset \mathbb{Z}^n$ is the support of the polynomial, one associates the max-plus polynomial $max_{\alpha \in A}\{log(a_\alpha) + < log(\boldsymbol{x}), \alpha >\}$. This heuristic can be used to associate a piecewise-smooth hybrid model to the system of rational ODEs (1), in two different ways.

The first method was proposed in [NGVR12] and can be applied to any rational ODE system defined by (1),(2):

Definition 1 We call complete tropicalization of the smooth ODE system (3) the following piecewise-smooth system:

$$\frac{dx_i}{dt} = DomP_i(\boldsymbol{x})/DomQ_i(\boldsymbol{x}), \tag{4}$$

where $Dom\{a_{i,\alpha}\boldsymbol{x}^\alpha\}_{\alpha \in A_i} = sign(a_{i,\alpha_{max}})exp[max_{\alpha \in A_i}\{log(|a_{i,\alpha}|) + < \boldsymbol{u}, \alpha >\}]$. $\boldsymbol{u} = (logx_1, \ldots, logx_n)$, and $a_{i,\alpha_{max}}$, $\alpha_{max} \in A_i$ denote the coefficient of the monomial for which the maximum is attained.

The second method,proposed in [SCF+09], applies to the systems (1),(3).

Definition 2 We call two terms tropicalization of the smooth ODE system (1) the following piecewise-smooth system:

$$\frac{dx_i}{dt} = DomP_i^+(\boldsymbol{x}) - DomP_i^-(\boldsymbol{x}), \tag{5}$$

The two-terms tropicalization was used in [SCF+09] to analyse the dependence of steady states on the model parameters. The complete tropicalization was used for the study of the model dynamics and for the model reduction [NGVR12].

For both tropicalization methods, for each occurrence of the Dom operator, one can introduce a tropical manifold, defined as the subset of $\mathbb{R}^n$ where the maximum in Dom is attained at least twice. For instance, for $n = 2$, such tropical manifold is made of points, segments connecting these points, and half-lines. The tropical manifolds in such an arrangement decompose the space into sectors, inside which one monomial dominates all the others in the definition of the reagent rates. The combinatorial study of the arrangement give hints on the possible steady states and attractors, as well as on their bifurcations.

3. Justification of the tropicalization and some estimates

In the general case, the tropicalization heuristic is difficult to justify by rigorous estimates, however, this is possible in some cases. We state here some results in this direction. Let us consider the class of polynomial systems, corresponding to mass action law chemical kinetics:

$$\frac{dx_i}{dt} = F_i(\boldsymbol{x}, \epsilon) = \sum_{j=1}^{M} F_{ij}(\boldsymbol{x}, \epsilon), \quad F_{ij} = P_{ij}(\epsilon)\boldsymbol{x}^{\alpha_{ij}} \tag{6}$$

where α_{ij} are multi-indices, and ϵ is a small parameter. So, the right hand side of (6) is a sum of monomials. We suppose that coefficients P_{ij} have different orders in ϵ:

$$P_{ij}(\epsilon) = \epsilon^{b_{ij}} \hat{P}_{ij}, \tag{7}$$

where $b_{ij} \neq b_{i'j'}$ for $(i,j) \neq (i',j')$.

We also suppose that the cone $\mathbf{R}_{>} = \{x : x_i \geq 0\}$ is invariant under dynamics (6) and initial data are positive:

$$x_i(0) > \delta > 0.$$

The terms (7) can have different signs, the ones with $\hat{P}_{ij} > 0$ are production terms, and those with $\hat{P}_{ij} < 0$ are degradation terms.

From the biochemical point of view, the choice (7) is justified by the multiscaleness of the biochemical processes. Furthermore, we are interested in biochemical circuits that can function "stably" even in extremal conditions. More precisely, we use the permanence concept, borrowed from the theory of species coexistence (the Lotka -Volterra model, see for instance [Tak96]).

Definition 3 The system (6) is permanent, if there are two constants $C_- > 0$ and $C_+ > 0$ such that

$$C_- < x_i(t) < C_+, \quad \textit{for all } t > T_0(x(0)) \textit{ and for every } i. \tag{8}$$

We assume that $C_\pm$ and T_0 are uniform in ϵ as $\epsilon \to 0$.

This means that concentrations of all the reagents cannot vanish or become too big, even in extremal conditions. Biological oscillators, such as the circadian clock and the cell cycle, satisfy this condition. We can also consider systems (6) that become permanent after rescaling of the concentrations, $x_i = \hat{x}_i \epsilon^{a_i}$, such as systems with quasi-stationary, low concentration, reagents.

For permanent systems, we can obtain some results justifying the two procedures of tropicalization. The complete tropicalization reads

$$\frac{d\bar{x}_i}{dt} = Dom(F_i(\bar{x})), \tag{9}$$

where $Dom(F_i) = F_{ik(i)}(\boldsymbol{x}, \epsilon)$, $|F_{ik(i)}(\boldsymbol{x}, \epsilon)| > |F_{ij}(\boldsymbol{x}, \epsilon)|$, $j \neq k(i)$ is the dominant term.

Notice that, because of the changing sign, (9) has discontinuous right hand side, therefore it is a differential inclusion. We assume here that there is no sliding motion. The situation with sliding motion needs special treatment and is discussed in the last section.

The two terms tropicalization reads

$$\frac{d\bar{x}_i}{dt} = Dom(F_i^+(\bar{x})) - Dom(F_i^-(\bar{x})) = Dom_2(F_i(\bar{x})), \tag{10}$$

where F_i^+, F_i^- gather the positive and negative terms of F_i, respectively.

Proposition 1 *Assume that system* (6) *is permanent. Let us consider the Cauchy problem for (6) and (9) (or (10)), with the same initial data:*

$$x(0) = \bar{x}(0).$$

Then the difference $y(t) = x(t) - \bar{x}(t)$ satisfies the estimate

$$|y(t)| < C_1 \epsilon^{\gamma} \exp(bt), \quad \gamma > 0, \tag{11}$$

positive constant C_1, b is uniform in ϵ. If the original system (6) is structurally stable in the domain $\Omega_{C_-,C_+} = \{x : C_- < |x| < C_+\}$, then the corresponding tropical systems (9) and (10) are also permanent and there is a orbital topological equivalency h_ϵ between the trajectories $x(t)$ and $\bar{x}(t)$ of the corresponding Cauchy problems close to the identity as $\epsilon \to 0$.

The proof of estimate (11) follows immediately by the Gronwall lemma. The second assertion follows from the definition of structural stability.

Permanency property is not easy to check. One of the possible methods is to find an invariant domain I in $\mathbf{R}^n$ such that the vector field $\boldsymbol{F}$ is directed inward I at the boundary ∂I. It is clear that to find such a domain I is simpler for tropicalizations than for the original system (for instance, when dominant monomial ODEs can be integrated). Furthermore, if x is a solution of (6), then

$$\frac{dx_i}{dt} \leq K(\epsilon)|Dom(F_i(x, \epsilon))|, \tag{12}$$

where $K(\epsilon) = 1 + O(\epsilon^{\gamma})$ for small ϵ and $K(1) = M$. Similarly, for the two term tropicalization, we have

$$\frac{dx_i}{dt} \leq K(\epsilon) Dom(F_i^+(x,\epsilon)) - Dom(F_i^-(x,\epsilon)). \tag{13}$$

In general, the inequalities (13) and (12) say nothing about $x(t)$. However, if the family of systems depending on a parameter K

$$\frac{dx_i}{dt} \leq K Dom(F_i^+(x,\epsilon)) - Dom(F_i^-(x,\epsilon)). \tag{14}$$

defines monotone semiflows then the permanency of $\bar{x}(t)$ can be used to obtain permanency of $x(t)$. In practice, the monotonicity condition is rarely satisfied globally (global validity is incompatible with the possibility of oscillations, and can not be satisfied by biological clocks), but can be satisfied locally, on some sectors bounded by tropical manifolds. Then, one can combine conditions on $\bar{x}(t)$ and conditions on $x(t)$ piecewisely, in order to prove permanency of $x(t)$.

4. Tropical sliding motions and model reduction

Biologists are attached to details. For the sake of completeness, systems biologists generate large, complex models. However, many details of these models are not important and can be simplified to facilitate model analysis. The model reduction problem is to find a simpler system, whose dynamics approximates the dynamics of the complex system [RGZL08]. Current model reduction techniques use quasi-equilibrium and quasi-steady state approximations. In both situations, the trajectories of some fast species satisfy approximate algebraic conditions imposed by the slow species. Given the trajectories $\boldsymbol{x}(t)$ of all species, we call imposed trajectory of the i-th species a real, positive, and stable solution $x_i^*(t)$ of the polynomial equation

$$P_i(x_1(t), \ldots, x_{i-1}(t), x_i^*(t), x_{i+1}(t), \ldots, x_n(t)) = 0, \tag{15}$$

We say that a species i is slaved if the distance between the trajectory $x_i(t)$ and some imposed trajectory $x_i^*(t)$ is small for some time interval I, $sup_{t\in I}|log(x_i(t)) - log(x_i^*(t))| < \delta$, for some $\delta > 0$ sufficiently small. The remaining species, that are not slaved, are called slow species.

Identifying slaved species is a first step of model reduction algorithms. Tropical geometry can be used for identification of slaved species without having to simulate the system and compute the trajectories. As first proposed in [NGVR12], the existence of slaved species implies the existence of attractive sliding modes of the complete tropicalization, defined as stable motions on the tropical manifold. Attractive sliding modes are possible in the theory of piecewise-smooth systems [FA88] provided that the following condition is satisfied, for $\boldsymbol{x}$ on the tropical manifold:

$$< n_+(x), f_+(x) >< 0, \quad < n_-(x), f_-(x) >< 0 \tag{16}$$

where f_+, f_- are the dominant vector fields on the two sides of a tropical hypersurface and $n_+ = -n_-$ are the normals to the interior faces.

References

[AD09] F. Ardila and M. Develin. Tropical hyperplane arrangements and oriented matroids. *Mathematische Zeitschrift*, 262(4):795–816, 2009.

[CTF06] G. Craciun, Y. Tang, and M. Feinberg. Understanding bistability in complex enzyme-driven reaction networks. *Proceedings of the National Academy of Sciences*, 103(23):8697–8702, 2006.

[Del49] M. Delbrück. Discussion: Unitées biologiques douées de continuité génétique. In *Actes du colloque international du CNRS*, pages 33–3, Paris, 1949. Editions du CNRS.

[FA88] A.F. Filippov and FM Arscott. *Differential equations with discontinuous righthand sides*, volume 18. Springer, 1988.

[GR07] A. N. Gorban and O. Radulescu. Dynamical robustness of biological networks with hierarchical distribution of time scales. *IET Systems Biology*, 1:238–246, 2007.

[LM96] G.L. Litvinov and V.P. Maslov. Idempotent mathematics: a correspondence principle and its applications to computing. *Russian Mathematical Surveys*, 51(6):1210–1211, 1996.

[LMS01] G.L. Litvinov, V.P. Maslov, and G.B. Shpiz. Idempotent functional analysis: an algebraic approach. *Mathematical Notes*, 69(5):696–729, 2001.

[NGVR12] V. Noel, D. Grigoriev, S. Vakulenko, and O. Radulescu. Tropical geometries and dynamics of biochemical networks. Application to hybrid cell cycle models. *Electronic Notes in Theoretical Computer Science*, 284:75–91, 2012.

[RGZL08] O. Radulescu, A. N. Gorban, A. Zinovyev, and A. Lilienbaum. Robust simplifications of multiscale biochemical networks. *BMC Systems Biology*, 2(1):86, 2008.

[RLS+06] O. Radulescu, S. Lagarrigue, A. Siegel, P. Veber, and M. Le Borgne. Topology and static response of interaction networks in molecular biology. *Journal of The Royal Society Interface*, 3(6):185–196, 2006.

[RSP+11] O. Radulescu, A. Siegel, E. Pécou, C. Chatelain, and S. Lagarrigue. Genetically regulated metabolic networks: Gale-Nikaido modules and differential inequalities. *Transactions on computational systems biology XIII*, pages 110–130, 2011.

[SCF+09] M.A. Savageau, P.M.B.M. Coelho, R.A. Fasani, D.A. Tolla, and A. Salvador. Phenotypes and tolerances in the design space of biochemical systems. *Proceedings of the National Academy of Sciences*, 106(16):6435, 2009.

[Son05] E.D. Sontag. Molecular systems biology and control. *European Journal of Control*, 11:396–435, 2005.

[Sou03] C. Soulé. Graphic requirements for multistationarity. *Complexus*, 1(123-133), 2003.

[T+98] R. Thomas et al. Laws for the dynamics of regulatory networks. *International Journal of Developmental Biology*, 42:479–485, 1998.

[Tak96] Y. Takeuchi. *Global dynamical properties of Lotka-Volterra systems*. World Scientific, Singapore, 1996.

[VR12] S. Vakulenko and O. Radulescu. Flexible and robust patterning by centralized gene networks. *Fundamenta Informaticae*, 118:345–369, 2012.

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Layered tropical algebras, applied to tropical algebraic geometry

Louis Rowen

Tropical mathematics is defined over an ordered cancellative monoid $\mathcal{M}$, usually taken to be $(\mathbb{R}, +)$ or $(\mathbb{Q}, +)$. An algebraic description of the max-plus algebra: We say that a semiring is **bipotent** if $a+b \in \{a, b\}$ for all a, b. There is a 1:1 correspondence between ordered monoids $(\mathcal{M}, <)$ and bipotent semirings A, where $A = \mathcal{M}$ as a monoid. Namely, multiplication in A is the given monoid operation, and addition in A is defined by $a + b = \max\{a, b\}$. Every bipotent semiring is idempotent, since $a + a \in \{a, a\} = \{a\}$.

Customarily, tropical curves have been defined either combinatorially in terms of two monomials having equal (leading) values, or synthetically as domains of non-differentiability of polynomials over the max-plus algebra satisfying the "balancing condition." Also tropical mathematics has been viewed in terms of valuation theory applied to curves over Puiseux series.

Although there is a rich theory arising from this viewpoint, cf. [7], idempotent semirings possess a restricted algebraic structure theory, and also do not reflect certain valuation-theoretic properties, thereby forcing researchers to rely often on combinatoric techniques.

Example 1 Consider the polynomial $f(a) = \lambda^3 + a\lambda^2 + a^3$, viewed tropically (over a bipotent algebra A). Note that $f(b) = b^3$ when $b > a$ and $f(b) = a^3$ when $b < a$. Likewise, $f(a) = a^3$ has three equal values for the monomials, so a is a tropical root, and there are no others, although one could not detect this directly from the value $f(a) = a^3$.

The object of this talk is to describe an alternative structure studied over the past few years by Izhakian, Knebusch, and Rowen [4] (with a more categorical context given in [5,6]) that permits fuller use of the algebraic theory especially in understanding the underlying tropical geometry. We replace the bipotent algebra A of an ordered monoid $\mathcal{M}$ by $R := L \times \mathcal{M}$, where L is a given indexing semiring (not necessarily with 0). Rewriting (ℓ, a) as ${}^{[\ell]}a$ for $\ell \in L,\ a \in \mathcal{M}$, we define multiplication componentwise, i.e.,

$$ {}^{[k]}a\ {}^{[\ell]}b \;=\; {}^{[k\ell]}(ab)\,, \tag{1}$$

but addition is a bit trickier:

$$ {}^{[k]}a \,+\, {}^{[\ell]}b = \begin{cases} {}^{[k]}a & \text{if } a > b, \\ {}^{[\ell]}b & \text{if } a < b, \\ {}^{[k+\ell]}a & \text{if } a = b. \end{cases} \tag{2}$$

Define the projection $s : L \times \mathcal{M} \to L$ by $s(\,{}^{[\ell]}a\,) = \ell$. Identifying A with $\{1\} \times \mathcal{M} \subseteq R$, we see for example that $a + a = {}^{[2]}a$, so $s(a + a) = 2$. We say $r \in R$ is a **ghost** if $s(r) > 1$. Now we say $a \in A$ is a **tropical root** of the polynomial f when $s(f(a))$ is ghost. In Example 1, $s(f(a)) = 3$, whereas $s(f(b)) = 1$ for all $b \neq a$. From this point of view, a is a tropical root of f.

When L is trivial, i.e, $L = \{1\}$, R is the usual bipotent max-plus algebra. When $L = \{1, \infty\}$ we recover the "standard" supertropical structure of [1]. When $L = \{\mathbb{N}\}$ we can describe multiple roots (such as a in Example 1), and we "almost" have unique factorization of polynomials. (Furthermore, the counterexamples have tropical geometric explanations.)

Likewise, one can define $s : R^{(n)} \to L^{(n)}$ componentwise; vectors $v_1, \dots, v_m$ are called **tropically dependent** iff each component of some nontrivial linear combination $w = \sum a_i v_i$ is a ghost. An $n \times n$ matrix has tropically dependent rows iff its permanent is a ghost, cf. [2].

This more algebraic formulation of roots enables one to transfer much of the standard algebraic-geometric theory directly to the tropical environment. For example, two polynomials in one indeterminate have a common root iff the permanent of their Sylvester matrix is a ghost. This enables one to describe multiple roots of f in terms of its tropical derivative f'.

One defines a **tropical variety** Z as the set of simultaneous tropical roots of a collection of polynomials. Its **coordinate semiring** is the semiring of polynomial functions from Z to R. Although this definition permits degenerate intersections which would not arise in the more familiar definitions, they can be separated out easily from the others by means of the layered theory. The algebraic properties of the coordinate semiring permit a direct algebraic description of basic geometric properties such as dimension.

Factorization of polynomials now corresponds to decompositions of varieties, enabling us to study irreducible varieties in an algebraic context, together with basic notions such as the prime spectrum.

References

1. Z. Izhakian and L. Rowen, *Supertropical algebra*, Adv. in Math. **225** (2010), 2222-2286.
2. Z. Izhakian and L. Rowen, *Supertropical matrix algebra*, Israel J. Math. **182** (2011), 383-424.
3. Z. Izhakian and L. Rowen, *Supertropical Resultants*, Journal of Algebra **324** (2010), 1860-1886.
4. Z. Izhakian, M. Knebusch, and L. Rowen. *Layered tropical mathematics* (submitted).
5. Z. Izhakian, M. Knebusch, and L. Rowen, *Categorical layered mathematics* (submitted)
6. Z. Izhakian, M. Knebusch, and L. Rowen, *Categories of layered semirings* Preprint, 2011.
7. G. Litvinov, *The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. J. of Math. Sciences*, 140(3):426–444, 2007.

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On idempotents in compact left topological universal algebras

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Abstract A standard fact important for applications is that any compact left topological semigroup has an idempotent. We extend this to certain compact left topological universal algebras.

A well-known fact is that any compact left topological semigroup has an idempotent, i.e. an element forming a subsemigroup. This firstly was established for compact topological semigroups independently by Numakura [1] and Wallace [2, 3], and in the final form (perhaps) by Ellis in [4]. This fact, despite of its easy proof, is fundamental for Ramsey-theoretic applications in number theory, algebra, topological dynamics, and ergodic theory. Hindman's Finite Products Theorem, van der Waerden's and Szemerédy's Arithmetic Progressions Theorems, and Furstenberg's Multiple Recurrence Theorem can be mentioned as widely known examples. Most of such applications have no (known) alternative proofs. The crucial fact for all them is the existence of idempotent ultrafilters over semigroups.

Let us shortly recall what are *idempotent ultrafilters*. The set $\boldsymbol{\beta}X$ of ultrafilters over a set X with a natural topology generated by basic (cl)open sets $\{u \in \boldsymbol{\beta}X : S \in u\}$, $S \subseteq X$, forms the largest (Stone–Čech or Wallman) compactification of the discrete space X. If $\cdot$ is a binary operation on X, it extends to a binary operation on $\boldsymbol{\beta}X$ by letting for all $u, v \in \boldsymbol{\beta}X$

$$uv = \big\{S \subseteq X : \{a \in X : \{b \in X : ab \in S\} \in u\} \in v\big\}.$$

The extended operation is continuous in any first argument, i.e. the groupoid $(\boldsymbol{\beta}X, \cdot\,)$ is left topological, and in any second argument whenever it is in X. Not many algebraic properties are stable under this extension, but associativity is. Hence any semigroup X extends to a compact left topological semigroup $\boldsymbol{\beta}X$, and therefore, there exists an ultrafilter $u \in \boldsymbol{\beta}X$ that is an idempotent of the extended operation. The book [5] is a comprehensive treatise on ultrafilter extensions of semigroups and various applications, with some historical remarks.

The ultrafilter extension actually is a general construction. As we shown in [6], arbitrary first-order model on X, i.e. a set X with operations and relations on it, canonically extends to the model on $\boldsymbol{\beta}X$ such that its model-theoretic properties are, in a sense, completely analogous to the topological properties of $\boldsymbol{\beta}X$. Certainly, not all extended models contain idempotents, associativity is essential in Ellis' result. In this note, we replace it by other, much wider algebraic conditions thus showing that compact left topological universal algebras satisfying these conditions have single-point subalgebras. We also mention applications using idempotent ultrafilters over such algebras. For simplicity, we shall consider only algebras with one or two binary operations; this suffices however to demonstrate related ideas. General results in this direction can be found in [7].

Fix some terminology. An *algebra* is a universal algebra, i.e. a set with arbitrary operations of any arities on it. A *groupoid* is an algebra with one binary operation. As a rule, we call the operation a multiplication and write rather xy than $x \cdot y$. If F is an n-ary operation on X, its *idempotent* is an $a \in X$ such that $F(a, \ldots, a) = a$. An *idempotent of an algebra* is a common idempotent of all its operations, i.e. an element forming a subalgebra.

In the sequel, all topological spaces are assumed to be Hausdorff. An algebra endowed with a topology is *left topological* iff for any its operation F, the unary map

$$x \mapsto F(a_1, \ldots, a_n, x)$$

is continuous, for any fixed $a_1, \ldots, a_n \in X$. *Right topological* algebras are defined dually. An algebra is *semitopological* iff all unary maps obtained from any of its operation by fixing all but one arguments are continuous, and *topological* iff all its operations are continuous. In particular, a groupoid is semitopological iff it is left and right topological simultaneously, and topological if its multiplication is continuous. This hierarchy does not degenerate, even for compact semigroups (see e.g. [5]).

An algebra is *minimal* iff it includes no proper subalgebras, and *minimal compact* iff it carries a compact topology and includes no proper compact subalgebra. Clearly, an algebra may include no minimal subalgebra. Contrary to this, any compact algebra does include a minimal compact subalgebra (apply Zorn's Lemma to the family of compact subalgebras ordered by inclusion). As we shall see, certain algebraic properties restrict possible size of minimal and minimal compact algebras.

An occurrence of a variable x into a term $t(x,\ldots)$ is *right-most* iff whenever

$$t_1(x,\ldots)\cdot t_2(x,\ldots)$$

is a subterm of t, then x occurs into $t_2(x,\ldots)$ but not $t_1(x,\ldots)$. E.g. all the occurrences of the variable x into the terms x, vx, $v(vx)$, $(v_1v_2)(v_3x)$ are right-most, while all its occurrences into the terms v, xv, $x(vx)$, $(v_1x)(v_2x)$ are not. A *left-most* occurrence is defined dually. Clearly, if the occurrence of x into t is right-most (or left-most), then x occurs there exactly once.

It easy to see that if X is a left topological algebra and $t(v_1,\ldots,v_n,x)$ a term with the right-most occurrence of the last argument, then the map

$$x\mapsto t(a_1,\ldots,a_n,x)$$

is continuous, for any fixed $a_1,\ldots,a_n\in X$.

The following theorem generalizes Ellis' result to certain groupoids.

Theorem 1 *Let X be a compact left topological groupoid, $r(v_1)$, $s(v_1,v_2)$, and $t(v_1,v_2,v_3)$ some terms, and let $s(v_1,v_2)$ have the right-most occurrence of the last argument. If X satisfies*

$$s(x,y)\cdot s(x,z)=s(x,t(x,y,z))\quad \textit{and}$$

$$s(x,y)=s(x,z)=r(x)\ \rightarrow\ s(x,yz)=r(x),$$

then it has an idempotent.

Proof (Scetch of proof) The conditions of Theorem 1 are universal formulas, so any subgroupoid of X should satisfy them. By Zorn's Lemma, isolate a minimal compact subgroupoid A and show that A consists of a single point. Pick any $a\in A$. The map $x\mapsto s(a,x)$ is continuous since the occurrence of x in $s(v,x)$ is right-most. Hence the first condition implies that $s(a,A)=\{s(a,b):b\in A\}$ is a compact subgroupoid of A, whence $s(a,A)=A$, and so $B=\{b\in A: s(a,b)=r(a)\}$ is nonempty. Now the second condition implies that B is a compact subgroupoid of A, whence $B=A$, and then $aa=a$ by the first condition.

Although the conditions of Theorem 1 look technical, they follow from various easy particular identities. Thus any compact left topological groupoids satisfying such identities does have an idempotent. Let us give some examples.

First af all, the associativity law implies the conditions, with $r(v) = v$, $s(v_1, v_2) = v_1v_2$, $t(v_1, v_2, v_3) = v_2v_1v_3$. Thus Ellis' result follows from Theorem 1.

Next, let us call the following identity

$$x(yz) = (xz)y$$

left skew associativity and groupoids satisfying it *left skew semigroups*. The identity clearly follows from conjunction of associativity and commutativity but implies neither of them. Left skew associativity also implies the conditions of Theorem 1, with $r(v) = v$, $s(v_1, v_2) = v_1v_2$, $t(v_1, v_2, v_3) = (v_1v_3)v_2$. Thus we see: *Any compact left topological left skew semigroup has an idempotent.*

There are many identities strictly weaker than associativity that imply the conditions of Theorem 1. E.g., so is the identity (of Bol–Moufang type)

$$(xx)(yz) = ((xx)y)z.$$

Here $r(v) = vv$, $s(v_1, v_2) = (v_1v_1)v_2$, $t(v_1, v_2, v_3) = v_2((v_1v_1)v_3)$. Examples of such kind can be easily multiplied.

Using of the conditions of Theorem 1 is essential; in general, neither minimal compact left topological groupoids, neither minimal groupoids, when the latter exist, need consist of a single point. E.g. there exist countable minimal quasigroups (see [7]). To mention a topological counterpart, consider the identity

$$x(yz) = (xy)(xz),$$

called *left distributivy*. Thus a groupoid $(X, \cdot)$ satisfies it iff the map $x \mapsto ax$ is its endomorphism, for any fixed $a \in X$. *Right distributivity* is defined dually, and *distributivity* is the conjunction of left and right versions.

Such groupoids arise in knot theory, where they usually are idempotents, and also in set theory, where, as Laver shown, nontrivial elementary embeddings of V_δ into itself with their *application* operation $f \cdot g = \bigcup_{\alpha<\delta} f(g \restriction V_\alpha)$ form a free left distributive groupoid without minimal subgroupoids. The existence of such embeddings is an extremely large cardinal axiom, and it is still a major open problem whether the axiom is necessary to prove a purely algebraic fact about certain finite left distributive groupoids, so-called Laver's tables (see [8] and references there).

Ježek and Kepka shown that in distributive groupoids, all $(wx)(yz)$, and so all terms with more than 2 occurrences, are idempotents (see [9]). The one-sided case differs; in [10] we shown: *All minimal left distributive groupoids are finite, and for any finite n there exists exactly one (up to isomorphism) such groupoid of cardinality n. There exists a minimal compact topological left distributive groupoid of cardinality* $2^{2^{\aleph_0}}$. (The proof of the latter fact uses algebra of ultrafilters.)

Let us discuss an application of Theorem 1. As mentioned, any groupoid uniquely extends to the compact left topological algebra of ultrafilters over it; moreover, associativity is stable under this extension, so any semigroup extends to the semigroup of ultrafilters over it. One can ask about more stable identities. Elsewhere we prove: *Let an identity $s_1 = s_2$ be equivalent to some identity $t_1 = t_2$ such that the common variables of t_1 and t_2 appear in these terms in the same ordering, and any common variable occurs in each of the terms only once. Then the identity $s_1 = s_2$ is stable under $\boldsymbol{\beta}$.* In particular, identities that follow from associativity are stable under $\boldsymbol{\beta}$ whenever one of its terms is *repeatless*, i.e. each variable occurs in it at most once. (On the other hand, it can be shown that e.g. neither commutativity, nor idempotency is stable.)

An interesting case is when some identities are stable under $\boldsymbol{\beta}$ and in the same time imply the conditions of Theorem 1. If a groupoid satisfies such identities, one can apply Theorem 1 to the groupoid of ultrafilters over it, thus obtaining an idempotent ultrafilter. The identity

$$(wx)(yz) = ((wx)y)z$$

is an example. It is stable under $\boldsymbol{\beta}$ (by the criterion above) and implies the weaker identity $(xx)(yz) = ((xx)y)z$, which in turn implies the conditions of Theorem 1 (as noted above). Therefore, it provides an idempotent ultrafilter. (Note that we could not use the identity $(xx)(yz) = ((xx)y)z$, which is not repeatless and actually is *not* stable under $\boldsymbol{\beta}$.)

As mentioned, such ultrafilters allow to obtain significant combinatorial results. E.g. the following version of Hindman's Finite Products Theorem holds: *If a groupoid X satisfies $(wx)(yz) = ((wx)y)z$, then any of its finite partitions has a part containing a countable sequence $a_0, \ldots, a_n, \ldots$ together with finite products*

$$a_{n_0}(a_{n_1} \ldots (a_{n_k} a_{n_{k+1}}) \ldots)$$

for all $n_0 < n_1 < \ldots < n_k < n_{k+1}$. Moreover, if X does not have idempotents or is right cancellative, one can find such a sequence consisting of pairwise distinct elements. For the proof and various refinements, see [11].

Theorem 1 can be generalized to the case of one operation of arbitrary arity, in the expected way. Let us now pass to algebras with many operations. For simplicity we consider only the case of two binary operations, denoted as addition and multiplication, although it is possible to establish a general result about arbitrary algebras, for which we refer to [7].

Theorem 2 *Let* $(X, +, \cdot\,)$ *be a compact left topological algebra such that any compact subgroupoid of its additive groupoid* $(X, +)$ *has an idempotent. Let* $q_1(v)$*,* $q_2(v)$*,* $r(v)$*,* $s(v_1, v_2)$*,* $t_1(v_1, v_2, v_3)$*,* $t_2(v_1, v_2, v_3)$ *be some terms, where* $q_1(v)$*,* $q_2(v)$ *are additive, and* $s(v_1, v_2)$ *has the right-most occurence of the last argument. If* X *satisfies*

$$x + x = x \to r(x) + r(x) = r(x),$$

$$\begin{aligned} s(x, y) + s(x, z) &= s(x, t_1(x, y, z)), \\ s(x, y) \cdot s(x, z) &= s(x, t_2(x, y, z)), \end{aligned}$$

and

$$\begin{aligned} s(x, y) = s(x, z) = r(x) &\to s(x, y + z) = q_1(r(x)), \\ s(x, y) = s(x, z) = r(x) &\to s(x, yz) = q_2(r(x)), \end{aligned}$$

then it has an idempotent.

Proof (Scetch of proof) Let A be a minimal compact subalgebra of X, $a \in A$ an additive idempotent. The map $x \mapsto s(a, x)$ is continuous, hence the conditions imply firstly that $s(a, A)$ is a compact subalgebra of A, and secondly that $B = \{b \in A : s(a, b) = r(a)\}$ is a compact subalgebra of A, so $B = A$. Then $aa = a$ follows.

Theorem 2 extends Theorem 1 since any groupoid $(X, \cdot\,)$ satisfying the conditions of Theorem 1, with some r, s, and t, can be turned into an algebra $(X, +, \cdot\,)$ satisfying the conditions of Theorem 2 by defining an extra operation $+$ as the projection onto the first argument: $x + y = x$ for all $x \in X$. In result, $(X, +)$ is a left-zero semigroup, and one can put $q_1(v) = q_2(v) = v$, $t_1(v_1, v_2, v_3) = v_2$, the same r, s, and t as t_2.

Let us consider some examples of identities implying the conditions of Theorem 2. The identity

$$x(y + z) = xy + xz$$

is *left distributivy* of $\cdot$ w.r.t. $+$. Thus an algebra $(X, +, \cdot\,)$ satisfies it iff the map $x \mapsto ax$ is an endomorphism of $(X, +)$, for any fixed $a \in X$. If $+$ and $\cdot$ coincide, this gives left distributive groupoids mentioned above. *Right distributivity* of $\cdot$

w.r.t. $+$ is defined dually, and *distributivity* is the conjunction of left and right versions.

An algebra $(X, +, \cdot)$ is a *left semiring* iff both its groupoids are semigroups and $\cdot$ is left distributive w.r.t. $+$. *Right semirings* are defined dually, and *semirings* are algebras that are left and right semirings simultaneously. E.g. $(\mathbb{N}, +, \cdot)$ is a semiring, ordinals with their usual addition and multiplication form a left semiring, and if $(X, +)$ is a semigroup then $(X^X, +, \circ)$ is a left semiring where $f \circ g\,(x) = g(f(x))$.

Left semirings satisfy the conditions of Theorem 2, with $q_1(v_1) = v_1 + v_1$, $q_2(v_1) = r(v_1) = v_1$, $s(v_1, v_2) = v_1 v_2$, $t_1(v_1, v_2, v_3) = v_2 + v_3$, $t_2(v_1, v_2, v_3) = v_2 v_1 v_3$. One gets: *Any compact left topological left semiring has an idempotent*. Thus if it is minimal compact then it consists of a single point, and an interesting question is about an algebraic counterpart of this, i.e. whether any minimal left semiring consists of a single point. This is indeed the case for *finite* left semirings (since their discrete topology is compact), and we also were able to establish the following: *Any minimal semiring consists of a single point* (see [12]).

Extending $(\mathbb{N}, +, \cdot)$ to ultrafilters, one gets the algebra $(\boldsymbol{\beta}\mathbb{N}, +, \cdot)$ with two semigroups which however satisfies neither left nor right distributivity. The set $\mathbb{N}^* = \boldsymbol{\beta}\mathbb{N} \setminus \mathbb{N}$ of nonprincipal ultrafilters is its compact subalgebra. A long-standing problem is whether some three particular $a, b, c \in \mathbb{N}^*$ satisfy $a(b+c) = ab + ac$ or $(a+b)c = ac + bc$. As van Douwen shown (see [13]), such ultrafilters, if exist at all, are topologically rare. We can take another step in negative direction (see [12]): *Neither closed subalgebra of* $(\mathbb{N}^*, +, \cdot)$ *is a left semiring.* This follows from the fact that the algebra has no common idempotents (actually, no $a \in \mathbb{N}^*$ with $a + a = aa$, see [5]), despite of the existence of additive idempotents as well as multiplicative ones.

As in the case of one operation, it is not difficult to provide other identities that imply the conditions of Theorem 2. E.g. let us generalize the concept of left semirings by preserving left distributivity of $\cdot$ w.r.t. $+$ but weakening both associativity laws to

$$((w + x) + y) + z = (w + x) + (y + z),$$
$$((wx)y)z = (wx)(yz).$$

These algebras yet satisfy the conditions of Theorem 2 (required terms can be obtained from the terms for left semirings if one takes rather xx than x), so any such compact left topological algebra has an idempotent. Both identities are stable under $\boldsymbol{\beta}$, so any algebra satisfying them carries additively idempotent ultrafilters as well as multiplicatively idempotent ones. Furthermore, it can be shown that under left distribtivity of + w.r.t. $\cdot$ some multiplicatively idempotent ultrafilters are in the closure of the set of additively idempotent ultrafilters. This fact leads to the following result (established by Hindman and Bergelson for the semiring of natural numbers, see [5]): *If* $(X,+,\cdot)$ *satisfies two identities above and left distribtivity of + w.r.t.* $\cdot$*, then any of its finite partitions has a part containing countable sequences* $a_0,\ldots,a_n,\ldots$ *and* $b_0,\ldots,b_n,\ldots$ *together with finite sums*

$$a_{n_0}+(a_{n_1}+\ldots+(a_{n_k}+a_{n_{k+1}})\ldots)$$

and products

$$b_{n_0}(b_{n_1}\ldots(b_{n_k}b_{n_{k+1}})\ldots)$$

for all $n_0<n_1<\ldots<n_k<n_{k+1}$*. Moreover, if each of the operations does not have idempotents or is right cancellative, one can find such sequences consisting of pairwise distinct elements.* (See [11].)

Finally, let us consider algebras $(X,\circ,\cdot)$ satisfying the following identities:

$$\begin{aligned}(x\circ y)\circ z&=x\circ(y\circ z),\\(x\circ y)z&=x(yz),\\x(y\circ z)&=xy\circ xz,\\x\circ y&=xy\circ x.\end{aligned}$$

It follows that $(X,\cdot)$ is a left distributive groupoid (and conversely, it can be shown that any left distributive groupoid extends to such an algebra). As Laver established, elementary embeddings with their application $\cdot$ and composition $\circ$ form algebras satisfying these identities (see [8]). Unlike the case of one left distributive operation, any such compact left topological algebra does have an idempotent: as $(X,\circ)$ is a semigroup, it has an idempotent a, then it easily follows from the identities that aa is a common idempotent. We think that a study of ultrafilter extensions of these algebras could throw light upon the problem of Laver's tables.

References

1. K. Numakura. *On bicompact semigroups.* Math. J. Okayama Univ., 1 (1952), 99–108.
2. A. Wallace. *A note on mobs, I.* An. Acad. Brasil. Ciênc., 24 (1952), 329–334. *A note on mobs, II.* An. Acad. Brasil. Ciênc., 25 (1953), 335–336.
3. A. Wallace. *The structure of topological semigroups.* Bull. Amer. Math. Soc., 61 (1955), 95–112.
4. R. Ellis. *Lectures on topological dynamics.* Benjamin, N.Y., 1969.
5. N. Hindman, D. Strauss. *Algebra in the Stone–Čech compactification.* W. de Gruyter, Berlin–N.Y., 1998.
6. D. I. Saveliev. *Ultrafilter extensions of models.* Lecture Notes in AICS, Springer, 6521 (2011), 162–177. An extended version in: S.-D. Friedman et al. (eds.). *The Infinity Proceedings.* CRM Documents 11, Barcelona, 2012.

7. D. I. Saveliev. *On the existence of idempotents in compact left topological universal algebras.* Manuscript, 2008.
8. P. Dehornoy. *Elementary embeddings and algebra*. In: M. Foreman, A. Kanamori (eds.). *Handbook of set theory*. Springer, Berlin–N.Y., 2010.
9. D. Stanovský. *Distributive groupoids are symmetyric-by-medial: an elementary proof.* Comment. Math. Univ. Carolinae, 49:4 (2008), 541–546.
10. D. I. Saveliev. *Minimal and minimal compact left topological left distributive groupoids.* www.karlin.mff.cuni.cz/∼ical/presentations/ saveliev.pdf.
11. D. I. Saveliev. *On Hindman sets.* Manuscript, 2008.
12. D. I. Saveliev. *Common idempotents in compact left topological left semirings.* arxiv:1002.1599.
13. E. K. van Douwen. *The Stone–Čech compactification of a discrete groupoid.* Topol. Appl., 39 (1991), 43–60.

Partly supported by RFBR grants 11-01-00958 and 11-01-93107, NSh grant 5593-2012-1, and INFTY grant of ESF.

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Injectivity Modules of a Tropical Map: Extended Abstract

Edouard Wagneur

1 Introduction

A tropical torsion module M is an idempotent commutative semimodule over the idempotent commutative extended semiring $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. Endowed with the max operator (written $\vee$) as first composition law, and classical addition (written $\cdot$ which will usually be omitted when no confusion arises), with the (torsion) property that, for any two generators, x, y, there exist $\lambda_{xy} = \inf\{\xi \in \underline{\mathbb{R}} | x \leq \xi y\}$ and $\lambda_{yx} = \inf\{\xi \in \underline{\mathbb{R}} | y \leq \xi x\}$. Moreover, the product $\lambda_{xy} \cdot \lambda_{yx}$ in $\underline{\mathbb{R}}$ is an invariant of the isomorphy class of M, called the *torsion* of M.

We write $\mathbf{0}$ and $\mathbb{1}$ for the neutral elements of $\vee$ and $\cdot$ respectively.

In [3], we show that any m-dimensional tropical torsion module can be embedded in $\underline{\mathbb{R}}^d$, with $d \leq m(m-1)$, and that m-dimensional tropical torsion modules are classified by a p-parameter family, with $p \leq (m-1)[m(m-1)-1]$.

The aim of the paper is to revisit and extend some of these results by showing that – at least in the 3-dimensional case – the two upper bounds are tight. More precisely, we show that for $m = 3$, we can find tropical torsion modules which cannot be embedded in $\underline{\mathbb{R}}^d$ for $d < 6$, and that all the $p = 2 \cdot (2 \cdot 3 - 1) = 10$ parameters required for the unambiguous specification of the 3 generators of M are necessary for the characterization of M.

Also, the concept of injectivity set (or injectivity tropical module) briefly dealt with in [3] is further investigated. in particular, we show the counterintuitive result that, for a given tropical map $\varphi \colon M \to N$, the quotient $M_{|\varphi}$ defined by the equivalence $\sim$ given by $x \sim y \iff \varphi(x) = \varphi(y)$ is not isomorphic to $\mathrm{Im}\varphi$.

The paper is organized as follows. In Section 2, we briefly recall some of the results of [3] which will be used in the paper. in Section 3, we state the main result of the paper, related to the injectivity modules of a tropical map. then these results are illustrated in Section 4, by way of two examples, where $m < n$ and $n < m$, respectively. The first one with a tropical map in $\mathrm{Hom}(\underline{\mathbb{R}}^3, \underline{\mathbb{R}}^6)$, the second with a map in $\mathrm{Hom}(\underline{\mathbb{R}}^4, \underline{\mathbb{R}}^3)$. In both cases, (some of) the injectivity modules are exhibited.

2 The main results of [3]

In this section we briefly recall the main results of [3] which will be used in this paper.

1. The canonical form of the torsion matrix:

$$A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & a_{13} & \dots & a_{1m} \\ \mathbb{1} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \tag{1}$$

with $\mathbb{1} = a_{12} \leq a_{22} \leq \dots \leq a_{n2}\, a_{ij} \leq a_{ij+1}$, $i = 1, \dots, n$, $j = 2, \dots, m$, and $\tau(x_{j-1}, x_j) \leq \tau(x_j, x_{j+1})$, $j = 2, \dots, m-1$, where x_j stands for column j of A.

This canonical form also defines the canonical basis of M_A.

2. $\forall j (1 \leq j \leq m-1)$, $\exists i (1 \leq i \leq n)$ such that $a_{ij+1} = a_{ij}$ (hence $\lambda_{jj+1} = \mathbb{1}$).

3. The λ_{ij} (from which we readily get the τ_{ij}) are given by the matrix

$$\Lambda_A = A^t \cdot A^- = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \lambda_{13} & \dots & \lambda_{1m-1} & \lambda_{1m} \\ \tau_{12} & \mathbb{1} & \mathbb{1} & \dots & \lambda_{2m-1} & \lambda_{2m} \\ \lambda_{31} & \tau_{23} & \mathbb{1} & \mathbb{1} & \dots & \lambda_{3m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \mathbb{1} & \mathbb{1} \\ \lambda_{m1} & \lambda_{m2} & \dots & \lambda_{mm-2} & \tau_{m-1m} & \mathbb{1} \end{bmatrix} \tag{2}$$

where A^t, and A^- stand for the transpose of A and for the matrix with entries the inverses of those of A.

4. The Whitney embedding theorem and the classification of tropical modules have been recalled in Section 1 above.

3 The injectivity modules of a tropical map

In this section, we investigate some properties of INJ_A for a tropical torsion matrix (TTM) A.

Let M, N be two tropical modules of dimension m, n respectively, $\varphi \in \mathrm{Hom}(M, N)$, and π the canonical projection $M \to M|_\sim$, defined by the equivalence relation $x \sim y \iff \varphi(x) = \varphi(y)$.

Definition 1 We say that $\mathrm{INJ}_\varphi = \{\xi \in M | \; \forall \lambda \in M, \, \lambda \neq \xi \Rightarrow \varphi(\lambda) \neq \varphi(\xi)\}$ is the **injectivity set** of φ.

In [3], we proved the following statement for $M = N = \underline{R}^n$ (then φ may be written as a tropical torsion square matrix A).

Proposition 1 *For any square tropical square matrix $A \in \mathrm{Hom}(I\!R^n, I\!R^n)$ of maximal column rank, there is a unique permutation $\sigma \in \mathcal{S}_n$ such that*

$$\mathrm{INJ}_A = \{\xi \in I\!R^n \; s.\, t. \text{ for } k = 1, \ldots, n \,, \bigvee_{j=1, j\neq k}^{n} a_{\sigma(k)j}\xi_j \leq a_{\sigma(k)k}\xi_k\}. \tag{3}$$

It is easy to see that the injectivity set of A satisfying (3) is a tropical module.

Clearly, for any $n \times n$ permutation matrix $P\mathrm{INJ}_{PA} = \mathrm{INJ}_A$, and, by Proposition 1, there exists a unique permutation matrix P such that, for $B = PA$, (3) is equivalent to

$$\mathrm{INJ}_A = \{\xi \in \mathrm{I\!R}^n \; s.\, t. \text{ for } k = 1, \ldots, n \,, \bigvee_{j=1, j\neq k}^{n} b_{kj}\xi_j \leq b_{kk}\xi_k\}. \tag{4}$$

Let $\tilde{A} = (\mathrm{diag}(b_{ii}^{-1}))B$.

As a straightforward application of a well-known result (cf [1] for instance), we have the following statement.

Proposition 2 INJ_A *is generated by the columns of* $\tilde{A}^*$.

Theorem 1 *Let A be a TTM $m \times n$, then there are $\binom{\max\{m,n\}}{\min\{m,n\}}$ tropical modules where A is injective. Each of these injectivity modules is generated by the Kleene star of some square matrix derived from A.*

Proposition 3 *The tropical modules* $\mathrm{Im}A$ *and* INJ_A *are not isomorphic in general.*

Proposition 4 *If A is a rectangular $n \times m$ matrix with $m \neq n$, then* INJ_A *is not a tropical module.*

Remark. The statement in Proposition 3 differ from that in Propostion 4, since INJ_A is a TTM in Proposition 3.

4 Examples

The first two examples illustrate the statement in Theorem 1. In addition, our first example shows that the bound given in [3] for the Whitney embedding is tight, i.e., there exists a 3-dimensional tropical module which cannot be embedded in $\mathbb{R}^d$ for $d < 6 = m(m-1) = 6$. Also, as a complement to the classification theorem of the same reference, this example will be used to show that all the $p = (m-1)[m(m-1)-1]$ parameters are needed for the classification of M_A.

Our third example shows that we can find n-dimensional tropical modules with $m \leq n$ generators with equal torsion coefficients.

Example 1. Let

$$A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \\ \mathbb{1} & a & a \\ \mathbb{1} & 8 & 15 \\ \mathbb{1} & 9 & 11 \end{bmatrix},$$

with $5 < a < 8$. We have

$$\Gamma_A = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 1 & 2 & a & 8 & 9 \\ 5 & 4 & 14 & a & 15 & 11 \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5^{-1} \\ \mathbb{1} & 1^{-1} & 4^{-1} \\ \mathbb{1} & 2^{-1} & 14^{-1} \\ \mathbb{1} & a^{-1} & a^{-1} \\ \mathbb{1} & 8^{-1} & 15^{-1} \\ \mathbb{1} & 9^{-1} & 11^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 4^{-1} \\ 9 & \mathbb{1} & \mathbb{1} \\ 15 & 12 & \mathbb{1} \end{bmatrix},$$

with $\lambda_{12} = \mathbb{1}$, $\lambda_{21} = 9$, $\lambda_{13} = 4^{-1}$, $\lambda_{31} = 15$, $\lambda_{23} = \mathbb{1}$ and $\lambda_{32} = 12$ given by rows $1, 6, 2, 5, 4$ and 3, respectively.

We have $\tau_{12} = \lambda_{12} \cdot \lambda_{21} = 9 < \tau_{13} = \lambda_{13} \cdot \lambda_{31} = 11 < \tau_{23} = \lambda_{23} \cdot \lambda_{32} = 12$.

It follows that all six rows of A are required for the torsion of M_A. Hence, it A cannot be embedded into $\underline{\mathbb{R}}^d$ for $d < 6$. Note that the τ_{ij} are independent of a.

The tropical modules INJ_A

We compute the tropical modules $M_{ijk} = \mathrm{INJ}_{A_{ijk}}$ for $i = 1, j = 2, k = 3$, and for $i = 1, j = 2, k = 4$, where A_{ijk} is the map given by the square submatrix of A determined by rows i, j, k.

We have :

$$A_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & 2 & 14 \end{bmatrix},$$

then, since $\sigma = I$, we have

$$\tilde{A}_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ 1^{-1} & \mathbb{1} & 3 \\ 14^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix},$$

and

$$\tilde{A}^*_{123} = \tilde{A}^2_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ 1^{-1} & \mathbb{1} & 4 \\ 13^{-1} & 12^{-1} & \mathbb{1} \end{bmatrix}.$$

Hence M_{123} is generated by $\begin{bmatrix} \mathbb{1} \\ 1^{-1} \\ 13^{-1} \end{bmatrix}$, $\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ 12^{-1} \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \\ \mathbb{1} \end{bmatrix}$.

$$A_{124} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 5 \\ \mathbb{1} & 1 & 4 \\ \mathbb{1} & a & a \end{bmatrix},$$

with $\sigma = (123)$, and

$$M_{124} = \{\xi |\ 1\xi_2 \le \xi_1 \le a\xi_2\,,\ 4\xi_3 \le \xi_1 \le 5\xi_3\,,\ \xi_2 \le 5\xi_3 \le 5\xi_2\},$$

its generators are generated by the columns of

$$\tilde{A}^*_{124} == \begin{bmatrix} \mathbb{1} & 1 & 4 \\ 5^{-1} & \mathbb{1} & \mathbb{1} \\ 5^{-1} & 5^{-1} & \mathbb{1} \end{bmatrix}.$$

This example also illustrates the fact that the domain of a tropical map $\varphi\colon M \to N$ splits into two parts:

- INJ_φ, every point of which is an equivalence class of "$\sim$" defined by $x \sim y \iff \varphi(x) = \varphi(y)$,

- $M \setminus \mathrm{INJ}_\varphi$ where the equivalence classes contain more than one point of M.

Moreover: as easily seen from the torsion coefficients between generators, the M_{ijk} are neither isomorphic to $\mathrm{Im}A$, nor isomorphic to one another in general.

Our next example, which first appeared in [2] has been shortly examined in [3]. It is revisited here for an illustration of the case $m > n$ in Theorem 1.

Example 2. Let $x_i = \begin{bmatrix} \mathbb{1} \\ i \\ i^2 \end{bmatrix}$, $i = 1, 2, \ldots, m$, with $i = i^2 = \mathbb{1}$ for $i = 0$, and $A = [x_1|x_2|\cdots|x_m|]$.

The tropical submodule M_A of $\mathbb{R}^3$ can be made infinite dimen sional by letting $m \to \infty$.

It is not difficult to see that A is injective on $\bigcup_{0\le i<j<k} M_{ijk}$, where

$$M_{ijk} = \{\xi | \bigvee_{\ell\ge 1, \ell\neq i} \xi_\ell \le \xi_i\,, \bigvee_{\ell\ge 1, \ell\neq j} \ell\xi_\ell \le j\xi_j\,, \bigvee_{\ell\ge 1, \ell\neq k} \ell^2\xi_\ell \le k^2\xi_k\}.$$

For instance, with $m = 4$, we have:

$$M_{123} = \{\xi \in \mathbb{R}^4 | \ \xi_i \le \xi_1\,, i = 2,3,4\,,\ \xi_1 \vee 2\xi_3 \vee 3\xi_4 \le 1\xi_2\,,\ \xi_1 \vee 2\xi_2 \vee 6\xi_4 \le\ 4\xi_3\},$$

$$M_{124} = \{\xi \in \mathbb{R}^4 | \ \xi_i \le \xi_1\,, i = 2,3,4\,,\ \xi_1 \vee 2\xi_3 \vee 3\xi_4 \le 1\xi_2\,,\ \xi_1 \vee 2\xi_2 \vee 4\xi_3 \le 6\xi_4\},$$

$$M_{134} = \{\xi \in \mathbb{R}^4 | \ \xi_i \le \xi_1\,, i = 2,3,4\,,\ \xi_1 \vee 1\xi_2 \vee 3\xi_4 \le\ 2\xi_3\ \ \xi_1 \vee 2\xi_2 \vee 4\xi_3 \le 6\xi_4\},$$

$$M_{234} = \{\xi \in \mathbb{R}^4 | \ \xi_1 \vee \xi_3 \vee \xi_4 \le \xi_2\,,\ \xi_1 \vee 1\xi_2 \vee 3\xi_4 \le\ 2\xi_3\,,\ \xi_1 \vee 2\xi_2 \vee 4\xi_3 \le 6\xi_4\}.$$

The method described in Theorem 1 is illustrated as follows for the generators of the M_{ijk}, where the i (resp. j , k] stands for the rank of the column which dominates row 1 (resp. $2, 3$) of $A\xi$.

For M_{123}, define $\hat{A}_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & 1 & 2 & 3 \\ \mathbb{1} & 1^2 & 2^2 & 3^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} \end{bmatrix}$.

Then from $\hat{A}_{123}\xi = \begin{bmatrix} \xi_1 \\ 1\xi_2 \\ 2^2\xi_3 \\ \xi_4 \end{bmatrix}$, we get

$$\tilde{\hat{A}}_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ 1^{-1} & \mathbb{1} & 1 & 2 \\ 2^{-2} & 2^{-1} & \mathbb{1} & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} \end{bmatrix} \text{ and } \tilde{\hat{A}}^*_{123} = \begin{bmatrix} \mathbb{1} & \mathbb{1} & 1 & 3 \\ 1^{-1} & \mathbb{1} & 1 & 3 \\ 3^{-1} & 2^{-1} & \mathbb{1} & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1} \end{bmatrix}.$$

Clearly: $\begin{bmatrix} \mathbb{1} \\ 1^{-1} \\ 3^{-1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ 2^{-1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \mathbb{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \\ \mathbb{1} \end{bmatrix} \in M_{123}$.

For a straightforward verification, let

$$u = x_1 \begin{bmatrix} 1\!\!1 \\ 1^{-1} \\ 3^{-1} \\ \mathbf{0} \end{bmatrix} \vee x_2 \begin{bmatrix} 1\!\!1 \\ 1\!\!1 \\ 2^{-1} \\ \mathbf{0} \end{bmatrix} \vee x_3 \begin{bmatrix} 1 \\ 1 \\ 1\!\!1 \\ \mathbf{0} \end{bmatrix} \vee x_4 \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1\!\!1 \end{bmatrix} = \begin{bmatrix} x_1 \vee x_2 \vee 1x_3 \vee 3x_4 \\ 1^{-1}x_1 \vee x_2 \vee 1x_3 \vee 3x_4 \\ 3^{-1}x_1 \vee 2^{-1}x_2 \vee x_3 \vee 2x_4 \\ x_4 \end{bmatrix}.$$

We leave it to the reader to check that

$$\begin{bmatrix} 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 \\ 1\!\!1 & 1 & 2 & 3 \\ 1\!\!1 & 1^2 & 2^2 & 3^2 \end{bmatrix} u = \begin{bmatrix} u_1 \\ 1u_2 \\ 2^2u_3 \end{bmatrix}, \text{ i.e. } u \in M_{123}.$$

Example 3 This last example shows that we can find $n-1$ torsion elements in $\underline{\mathbb{R}}^n$ exhibiting two by two the same torsion.

Let

$$A = \begin{bmatrix} 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 \\ 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 & \tau \\ 1\!\!1 & 1\!\!1 & 1\!\!1 & 1\!\!1 & \tau & \tau \\ 1\!\!1 & 1\!\!1 & 1\!\!1 & \tau & \tau & \tau \\ 1\!\!1 & 1\!\!1 & \tau & \tau & \tau & \tau \\ 1\!\!1 & \tau & \tau & \tau & \tau & \tau \end{bmatrix}.$$

References

1. L. Libeault, Sur l'utilisation des dioïdes pour la commande des systèmes à événements discrets, Thèse, Université de Nantes, 1996.
2. E. Wagneur, Torsion matrices in the max-algebra, WODES, Edinburgh, August 1996.
3. E. Wagneur, The Whitney embedding theorem for tropical torsion modules. Classification of tropical modules. Linear Algebra and its Applications, **435**, (2011) 1786–1795.

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Studying isometry groups using the horofunction boundary

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The horofunction boundary was introduced by Gromov [13] in the late 1970s but did not receive much study until recently. It has applications in studying isometry groups [16], random walks [15], quantum metric spaces [21], and is a good setting for Patterson–Sullivan measures [4].

To define this boundary for a metric space (X, d), one identifies each point x with the function $d(\cdot, x) - d(b, x)$, where b is some arbitrary basepoint, and then takes the closure of the set of these functions with respect to some function space topology, usually the topology of uniform convergence on bounded sets. Under appropriate conditions, this gives a compactification of the metric space, and one defines the horofunction boundary to be the closure minus the original set of functions. Elements of this boundary are called horofunctions.

This boundary is not the same as the better known Gromov boundary of a δ-hyperbolic space. For these spaces, it has been shown [5, 23, 32] that the horofunction boundary is finer than the Gromov boundary in the sense that there exists a continuous surjection from the former to the latter.

A particularly interesting subset of the horofunction boundary is the set of those horofunctions that are the limits of *almost-geodesics*. An almost-geodesic, as defined by Rieffel [21], is a map γ from an unbounded set $T \subset \mathbb{R}_+$ containing 0 to X, such that for any $\epsilon > 0$,

$$|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \epsilon$$

for all $t \in T$ and $s \in T$ large enough with $t \geq s$. Rieffel called the limits of such paths *Busemann points*.

As noted by Ballmann [2], the construction above is an additive analogue of the way the Martin boundary is constructed in Probabilistic Potential Theory. One may pursue the analogy further in the framework of max-plus algebra, where one replaces the usual operations of addition and multiplication by those of maximum and addition. Indeed, this approach has already provided inspiration for many results about the horofunction boundary [1, 29]. We mention, for example, the characterisation of Busemann points as the functions in the horofunction boundary that are extremal generators in the max-plus sense of the set of 1-Lipschitz functions. So the set of Busemann points is seen to be an analogue of the *minimal* Martin boundary. There is also a representation of 1-Lipschitz functions in terms of horofunctions analogous to the Martin representation theorem.

The first metric spaces for which the horofunction boundary was investigated were those of Hadamard manifolds [3] and Hadamard spaces [2], where the horofunction boundary turns out to be homeomorphic to the ray boundary and all horofunctions are Busemann points. The case of finite-dimensional normed spaces has also been studied, by Karlsson *et. al.* when the norm is polyhedral [14], and more generally by the present author [26]. More examples include various finitely-generated groups with their word metrics [7,28,31] and Finsler p–metrics on $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}_n$ [11, 12]. Webster and Winchester have some general results on when all horofunctions are Busemann points [33].

The action of the isometry group of a metric space extends continuously to an action by homeomorphisms on the horofunction boundary. Thus, the horofunction boundary is useful for studying groups of isometries of metric spaces. In particular, one of the tools it provides is the *detour metric*, which is a (possibly infinite valued) metric on the set of Busemann points. One may define it as the symmetrisation of the detour cost

$$H(\xi,\eta) = \inf_{\gamma} \liminf_{t\to\infty} \Big(d(b,\gamma(t)) + \eta(\gamma(t)) \Big).$$

Here, ξ and η are horofunctions, and the infimum is taken over all paths $\gamma : \mathbb{R}_+ \to X$ converging to ξ. This concept appears first in [1].

In the following sections, we describe in more detail how the horofunction boundary may be used to investigate the isometry groups of particular metric spaces.

1. Isometries of the Hilbert metric

Let x and y be distinct points in a bounded open convex subset X of $\mathbb{R}^n$, with $n \geq 1$. Define w and z to be the points in the Euclidean boundary of X such that w, x, y, and z are collinear and arranged in this order along the line in which they lie. The Hilbert distance between x and y is defined to be the logarithm of the cross ratio of these four points:

$$d(x, y) := \log \frac{|zx|\,|wy|}{|zy|\,|wx|}.$$

As Hilbert noted, if X is an ellipsoid, then (X, d) is a model for hyperbolic n-space. On the other hand, if X is a simplex, then (X, d) is isometric to a normed space.

De la Harpe [6] was the first to consider the isometry group of the Hilbert geometry. Let $\mathbb{P}^n = \mathbb{R}^n \cup \mathbb{P}^{n-1}$ be real n-dimensional projective space, and suppose that X is contained within the open cell $\mathbb{R}^n$ inside $\mathbb{P}^n$. Let $\mathrm{Coll}(X)$ be the set of collineations, that is elements of $\mathrm{PGL}(n+1, \mathbb{R})$ preserving X. As de la Harpe observed, each element of $\mathrm{Coll}(X)$ is an isometry since collineations preserve the cross-ratios.

However, not every isometry is a collineation, as can be seen by considering the case of the simplex. We think of the n-simplex as being a cross section of the positive cone $\mathbb{R}^{n+1}_+$. Consider the projective action of the coordinate-wise reciprocal map

$$\rho : \mathrm{int}\, \mathbb{R}^{n+1} \to \mathrm{int}\, \mathbb{R}^{n+1},\ (x_i)_i \mapsto \left(\frac{1}{x_i}\right)_i.$$

This action can be shown to be an isometry but is clearly not a collineation.

The isometry group of the simplicial Hilbert geometries was determined in [16] and independently in [10]. It turns out to be generated by (the projective action of) ρ and the collineation group, so that the latter is a subgroup of index two of the isometry group.

General polyhedral Hilbert geometries were considered in [16], where the following theorem was proved.

Theorem 1 ([16]) *If (X, d) is a polyhedral Hilbert geometry, then* $\mathrm{Isom}(X)$ *differs from* $\mathrm{Coll}(X)$ *if and only if X is an open n-simplex, with $n \geq 2$.*

The proof involves studying the action of an isometry on the set of Busemann points of the horofunction boundary, endowed with the detour metric. This set of Busemann points had previous been determined in [27].

The above theorem verifies, for the case of polyhedral Hilbert geometries, some conjectures of de la Harpe, namely that Isom(X) is a Lie group, that its connected component coincides with that of Coll(X), and that Isom(X) acts transitively on X if and only if Coll(X) does.

What happens outside the polyhedral case? Recall [9] that a proper open cone C in a finite dimensional real vector space V is called *symmetric* if it is *homogeneous*, meaning that its group of linear automorphisms acts transitively on it, and *self dual*, meaning that it equals its own dual

$$C^* := \Big\{ y \in V^* \mid \langle x, y \rangle > 0 \text{ for all } x \in \overline{C} \backslash \{0\} \Big\}.$$

The symmetric cones have been classified, and the positive cone is one of them. On symmetric cones, there exists Vinberg's $*$-*map*, which is involutive, homogeneous of degree -1, and order-reversing for the natural order arising from the cone structure. From these properties, one may deduce that its projective action on a cross section of the cone is an isometry of the Hilbert metric on the cross section. This isometry is not a collineation except when the symmetric cone is a *Lorentz cone*,

$$\Lambda_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0 \text{ and } x_1^2 - x_2^2 - \dots - x_n^2 > 0\},$$

for some $n \geq 2$. It was conjectured in [16] that Isom(X) and Coll(X) differ if and only if the cone over X is symmetric and not Lorentzian, in which case the isometry group was conjectured to be generated by the collineations and the isometry coming from the $*$-map. This is conjecture is known to hold for the cone of positive-definite Hermitian matrices [17].

2. Isometries of the Lipschitz metric on Teichmüller space

Let S be a connected oriented surface of negative Euler characteristic. One way of defining the Teichmüller space $\mathcal{T}(S)$ of S is as the space of complete finite-area hyperbolic metrics on S up to isotopy. Here, by hyperbolic metric we mean a Riemannian metric of constant curvature -1.

Thurston [25] defined the distance from one hyperbolic metric x to another y to be the logarithm of the smallest Lipschitz constant over all homeomorphisms from (S, x) to (S, y) that are isotopic to the identity. In symbols,

$$L(x, y) := \log \inf_{\phi \approx \mathrm{Id}} \sup_{p \neq q} \frac{d_y(\phi(p), \phi(q))}{d_x(p, q)}, \qquad \text{for } x, y \in \mathcal{T}(S).$$

Thurston showed that this is indeed a metric, although in general it is asymmetric, in other words, $L(x, y)$ does not necessarily equal $L(y, x)$. In the same paper, he showed that this distance can be written

$$L(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{\ell_y(\alpha)}{\ell_x(\alpha)},$$

where $\mathcal{S}$ is the set of isotopy classes of non-peripheral simple closed curves on S, and $\ell_x(\alpha)$ denotes the shortest length in the metric x of a curve isotopic to α.

Thurston's Lipschitz metric has not been as intensively studied as the Teichmüller metric or the Weil–Petersson metric, although it has started to attract more interest recently [18, 19, 24].

In [30], we showed that the horofunction compactification of Teichmüller space with the Lipschitz metric is isomorphic to the well-known Thurston compactification, and we gave an explicit expression for the horofunctions.

Theorem 2 ([30]) *A sequence x_n in $\mathcal{T}(S)$ converges in the Thurston compactification if and only if it converges in the horofunction compactification. If the limit in the Thurston compactification is the projective class $[\mu] \in \mathcal{PML}$, then the limiting horofunction is*

$$\Psi_\mu(x) = \log \left(\sup_{\eta \in \mathcal{ML}} \frac{i(\mu, \eta)}{\ell_x(\eta)} \Big/ \sup_{\eta \in \mathcal{ML}} \frac{i(\mu, \eta)}{\ell_b(\eta)} \right).$$

Here, b is a base-point in $\mathcal{T}(S)$, and $i(\cdot, \cdot)$ denotes the geometric intersection number. Recall that the latter is defined for pairs of curve classes $(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}$ to be the minimum number of transverse intersection points of curves α' and β' with $\alpha' \in \alpha$ and $\beta' \in \beta$. This minimum is realised if α' and β' are closed geodesics. The geometric intersection number extends to a continuous symmetric function on $\mathcal{ML} \times \mathcal{ML}$.

It is known that geodesics always converge to a point in the horofunction boundary. Hence, an immediate consequence of the above theorem is the following.

Corollary 1 *Every geodesic of Thurston's Lipschitz metric converges in the forward direction to a point in the Thurston boundary.*

This generalises a result of Papadopoulos [18], which states that every member of a special class of geodesics, the *stretch lines*, converges in the forward direction to a point in the Thurston boundary.

As before one can learn much about the isometry group by studying its action on the horofunction boundary. Denote by Mod_S the extended mapping class group of S, that is, the group of isotopy classes of homeomorphisms of S. It is easy to see that Mod_S acts by isometries on $\mathcal{T}(S)$ with the Lipschitz metric. We used the horofunction boundary to prove the following.

Theorem 3 ([30]) *If S is not a sphere with four or fewer punctures, nor a torus with two or fewer punctures, then every isometry of $\mathcal{T}(S)$ with Thurston's Lipschitz metric is an element of the extended mapping class group* Mod_S.

This answers a question in [19, §4].

It is well known that the subgroup of elements of Mod_S acting trivially on $\mathcal{T}(S)$ is of order two if S is the closed surface of genus two, and is just the identity element in the other cases considered here.

Theorem 3 is an analogue of Royden's theorem concerning the Teichmüller metric, which was proved by Royden [22] in the case of compact surfaces and analytic automorphisms of $\mathcal{T}(S)$, and extended to the general case by Earle and Kra [8]. Our proof is inspired by Ivanov's proof of Royden's theorem, which was global and geometric in nature, as opposed to the original, which was local and analytic.

The following theorem shows that distinct surfaces give rise to distinct Teichmüller spaces, except possibly in certain cases. Denote by $S_{g,n}$ a surface of genus g with n punctures.

Theorem 4 ([30]) *Let $S_{g,n}$ and $S_{g',n'}$ be surfaces of negative Euler characteristic. Assume $\{(g,n),(g',n')\}$ is different from each of the three sets*

$$\{(1,1),(0,4)\}, \quad \{(1,2),(0,5)\}, \quad \textit{and} \quad \{(2,0),(0,6)\}.$$

If (g,n) and (g',n') are distinct, then the Teichmüller spaces $\mathcal{T}(S_{g,n})$ and $\mathcal{T}(S_{g',n'})$ with their respective Lipschitz metrics are not isometric.

This is an analogue of a theorem of Patterson [20]. In the case of the Teichmüller metric it is known that one has the following three isometric equivalences: $\mathcal{T}(S_{1,1}) \cong \mathcal{T}(S_{0,4})$, $\mathcal{T}(S_{1,2}) \cong \mathcal{T}(S_{0,5})$, and $\mathcal{T}(S_{2,0}) \cong \mathcal{T}(S_{0,6})$. It would be interesting to know if these equivalences still hold when one takes instead the Lipschitz metric.

It would also be interesting to work out the horofunction boundary of the reversed Lipschitz metric, that is, the metric $L^*(x,y) := L(y,x)$. Since L is not symmetric, L^* differs from L, and one would expect their horofunction boundaries to also differ.

References

1. Marianne Akian, Stéphane Gaubert, and Cormac Walsh. The max-plus Martin boundary. *Doc. Math.*, 14:195–240, 2009.
2. Werner Ballmann. *Lectures on spaces of nonpositive curvature*, volume 25 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin.
3. Werner Ballmann, Mikhael Gromov, and Viktor Schroeder. *Manifolds of nonpositive curvature*, volume 61 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.
4. M. Burger and S. Mozes. CAT(-1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.*, 9(1):57–93, 1996.
5. Michel Coornaert and Athanase Papadopoulos. Horofunctions and symbolic dynamics on Gromov hyperbolic groups. *Glasg. Math. J.*, 43(3):425–456, 2001.
6. Pierre de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory, Vol. 1 (Sussex, 1991)*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. Cambridge Univ. Press, Cambridge, 1993.
7. Mike Develin. Cayley compactifications of abelian groups. *Ann. Comb.*, 6(3-4):295–312, 2002.
8. Clifford J. Earle and Irwin Kra. On isometries between Teichmüller spaces. *Duke Math. J.*, 41:583–591, 1974.
9. Jacques Faraut and Adam Korányi. *Analysis on symmetric cones.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
10. Stefano Francaviglia and Armando Martino. The isometry group of Outer Space, 2009. Preprint. arXiv:0912.0299.
11. Shmuel Friedland and Pedro J. Freitas. p-metrics on $\mathrm{GL}(n,\mathbb{C})/\mathrm{U_n}$ and their Busemann compactifications. *Linear Algebra Appl.*, 376:1–18, 2004.
12. Shmuel Friedland and Pedro J. Freitas. Revisiting the Siegel upper half plane. I. *Linear Algebra Appl.*, 376:19–44, 2004.
13. M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213, Princeton, N.J., 1981. Princeton Univ. Press.
14. A. Karlsson, V. Metz, and G. Noskov. Horoballs in simplices and Minkowski spaces. *Int. J. Math. Math. Sci.*, 2006. Art. ID 23656, 20 pages.
15. Anders Karlsson and François Ledrappier. On laws of large numbers for random walks. *Ann. Probab.*, 34(5):1693–1706, 2006.
16. Bas Lemmens and Cormac Walsh. Isometries of polyhedral Hilbert geometries. *J. Topol. Anal.*, 3(2):213–241, 2011.
17. Lajos Molnár. Thompson isometries of the space of invertible positive operators. *Proc. Amer. Math. Soc.*, 137(11):3849–3859, 2009.
18. Athanase Papadopoulos. On Thurston's boundary of Teichmüller space and the extension of earthquakes. *Topology Appl.*, 41(3):147–177, 1991.
19. Athanase Papadopoulos and Guillaume Théret. On Teichmüller's metric and Thurston's asymmetric metric on Teichmüller space. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 111–204. Eur. Math. Soc., Zürich, 2007.
20. David B. Patterson. The Teichmüller spaces are distinct. *Proc. Amer. Math. Soc.*, 35:179–182, 1972.
21. Marc A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651 (electronic), 2002.
22. H. L. Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 369–383. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.
23. Peter A. Storm. The barycenter method on singular spaces. *Comment. Math. Helv.*, 82(1):133–173, 2007.
24. Guillaume Théret. *À propos de la métrique asymétrique de Thurston sur l'espace de Teichmüller d'une surface.* Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 2005/8. Université Louis Pasteur. Institut de Recherche Mathématique Avancée, Strasbourg, 2005. Thèse, Université Louis Pasteur (Strasbourg I), Strasbourg, 2005.

25. William Thurston. Minimal stretch maps between hyperbolic surfaces. preprint, arXiv:math GT/9801039, 1986.
26. Cormac Walsh. The horofunction boundary of finite-dimensional normed spaces. *Math. Proc. Cambridge Philos. Soc.*, 142(3):497–507, 2007.
27. Cormac Walsh. The horofunction boundary of the Hilbert geometry. *Adv. Geom.*, 8(4):503–529, 2008.
28. Cormac Walsh. Busemann points of Artin groups of dihedral type. *Internat. J. Algebra Comput.*, 19(7):891–910, 2009.
29. Cormac Walsh. Minimum representing measures in idempotent analysis. In *Tropical and idempotent mathematics*, volume 495 of *Contemp. Math.*, pages 367–382. Amer. Math. Soc., Providence, RI, 2009.
30. Cormac Walsh. The horoboundary and isometry group of Thurston's Lipschitz Metric, 2010. arXiv:1006.2158v1.
31. Cormac Walsh. The action of a nilpotent group on its horofunction boundary has finite orbits. *Groups Geom. Dyn.*, 5(1):189–206, 2011.
32. Corran Webster and Adam Winchester. Boundaries of hyperbolic metric spaces. *Pacific J. Math.*, 221(1):147–158, 2005.
33. Corran Webster and Adam Winchester. Busemann points of infinite graphs. *Trans. Amer. Math. Soc.*, 358(9):4209–4224 (electronic), 2006.

This work is partially supported by the joint RFBR-CNRS grant 11-01—93106-a.

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Properties of systems of $(\max,+)-$ and $(\max,\min)-$linear equations and inequalities

Karel Zimmermann

1. Introduction

Properties of systems of $(\max,+)$-linear inequalities or equations were studied in the literature, e.g., in [2], [3], [5], [7], [8]. The authors studied mostly either equations and inequalities with variables on only one-side of the equations or inequalities (we will call them "one-sided") or systems with variables on both sides of the relations (we will call them "two-sided") with a special form of one of the sides (e.g. when problems of $(\max,+)$-eigenvalues and eigenvectors were studied). The aim of the contribution is a presentation of some useful properties of systems $(\max,+)$- and $(\max,\min)$-linear equations and inequalities, which can be used for a detailed analysis of solution sets of such systems. Both "one-sided" and "two-sided" systems will be considered.

Problems on algebraic structures, in which pairs of operations $(\max,+)$ or $(\max,\min)$ replace addition and multiplication of the classical linear algebra, have appeared in the literature approximately since the sixties of the last century (see, e.g., [5], [10]). A systematic theory of such algebraic structures was published, e.g., in [2], [5] [7], [8]. These works investigate, among other problems, systems of the so called $(\max,+)$- or $(\max,\min)$-linear equations with variables on only one side of the equations. Since operation " max " replacing addition is no more a group operation, but only a semigroup operation, there is a substantial difference between solving systems with variables on one side and systems with variables occuring on both sides of equations or inequalities. The former systems will be called "one-sided" and the latter systems "two-sided". Special two-sided systems were studied e.g. in [4], [5], [9] in connection with the so called $(\max, +)$- or $(\max, \min)$-eigenvalue problem. Some two-sided $(\max, +)$-linear systems were studied in [2], [3]. Two-sided systems with a more general structure, in which on both sides of the equations residual functions occur were investigated in [6], where a general iteration method for solving such systems was proposed. This iteration method can be applied also to $(\max, +)$- or $(\max, \min)$-linear equation systems. The aim of this contribution is to present some useful properties of both one-sided and two-sided $(\max, +)$- linear inequality systems. These properties can be used for solving some optimization problems, the set of feasible solutions of which is described by a system of $(\max, +)$- or $(\max, \min)$-linear inequalities and/or equations, further for a parametric analysis of such problems and formulating some sufficient solvability conditions.

2. Notations, Problem Formulation

Let us introduce the folowing notations:
$J = \{1, \ldots, n\}$, $I = \{1, \ldots, m\}$, $R = (-\infty, \infty)$,
$R^n = R \times \cdots \times R$ (n-times), $x^T = (x_1, \ldots, x_n)$, $y^T = (y_1, \ldots, y_n) \in R^n$,
$a_{ij}, b_{ij} \in R \; \forall i \in I$, $j \in J$ are given numbers,

$$a_i(x) \equiv \max_{j \in J}(a_{ij} + x_j) \quad \text{for all } \; i \in I\,,$$
$$b_i(y) \equiv \max_{j \in J}(b_{ij} + y_j) \quad \text{for all } \; i \in I\,,$$

Let us consider the following inequality system:

$$a_i(x) \geq b_i(y) \quad \forall i \in I. \tag{1}$$

We will assume that variables x and y are independent of each other (we can call such variables also “separated” from each other). Such inequality system will be called $(\max, +)$-linear inequality system with separated variables on the sides of inequalities.

3. Optimization Problems with constraints (1)

Let us consider the system

$$\max_{j \in J}(a_{ij} + x_j) \geq b_i(y), i \in I, \tag{2}$$

and assume that variables $x^T = (x_1, \ldots, x_n), y^T = (y_1, \ldots, y_n)$ are independent. Let $M(x.y)$ denote the set of all solutions of system (2).

We will investigate the following optimization problem:

$$f(x) \equiv \max_{j \in J}(f_j(x_j)) \longrightarrow \min \tag{3}$$

subject to

$$x \in M(x, y), \tag{4}$$

where we assume that y is fixed and $f_j,\ j \in J$ are continuous increasing functions. Let $x^{opt}(y)$ be the optimal solution of problem (3), (4). Using the results of [11] we obtain explicit formulas for the optimal solutions of the optimization problem above. For this purpose we will introduce the following notations for all $i \in I,\ j \in J$:

$$T_{ij} \equiv \{x_j\ ;\ a_{ij} + x_j \geq b_i(y)\},$$

$$f_j(x_j^{(i)}(y)) = \min_{x_j \in T_{ij}} f_j(x_j),$$

where we set the minimum equal to ∞ if $T_{ij} = \emptyset$. We set further:

$$\min_{j \in J}(f_j(x_j^{(i)}(y)) = f_{j(i)}(x_{j(i)}^{(i)}(y)),$$

$$V_j \equiv \{i\ ;\ j(i) = j\},$$

Then the following theorem follows from the results contained in [11]:

Theorem 1 *Let*

$$x_j^*(y) = \max_{i \in V_j}(b_i(y) - a_{ij}) \text{ if } V_j \neq \emptyset$$

$$x_j^*(y) = -\infty \text{ if } V_j = \emptyset$$

Then $x^*(y)$ is the optimal solution of problem (3), (4).

Using this result we can derive explicit formulas for optimal solutions of the problem for any y. If we interpret y as parameters, on which the right hand sides of the constraint depend, post-optimal parametric analysis of the problems can be carried out. In the same way, problems, with $(\max, \min)$-linear inequalities will be analyzed. More general cases of dependence on y in the right hand sides will be investigated. Using Theorem 1 some sufficient conditions for solvability of two-sided $(\max, +)$- and $(\max, \min)$-linear equation and/or inequality systems will be presented.

4. The Reachable Right Hand Sides

Let us consider the system

$$\max(a_{ij} + x_j) = b_i, \quad i \in I, \tag{5}$$

where $b^T = (b_1, \ldots, b_m) \in R^m$. Let $M(b)$ denote the set of all solutions of system (5) for the given b. It is well-known that if $M(b) \neq \emptyset$, then there exists always the maximum element $x^{\max}(b)$ of set $M(b)$, i.e. element $x^{\max}(b) \in M(b)$ such that $x \leq x^{\max}(b), \quad \forall x \in M(b)$. The maximum element of $M(b)$ can be calculated as follows:

$$x_j^{\max} = \min_{i \in I}(b_i - a_{ij}) \quad \forall j \in J. \tag{6}$$

The reachability set $R(A)$ of a given matrix A with elements $a_{ij},\ i \in I,\ j \in J$ is defined as follows:

$$R(A) = \{b\ ;\ M(b) \neq \emptyset\}. \tag{7}$$

Let us consider the system of equations for unknown b of the form:

$$\min_{r \in I}(b_r - a_{rj}) = h_j, \quad j \in J, \tag{8}$$

where $h_j = \min_{i \in I}(-a_{ij})$. Let us set

$$S_j = \{k \in I\ ;\ h_j = -a_{kj}\} \quad \forall j \in J. \tag{9}$$

Theorem 2 *Let us set*

$$\hat{b}_k = 0 \quad \forall k \in \bigcup_{j \in J} S_j,$$

$$\hat{b}_k = \max_{j \in J}(h_j + a_{kj}) \quad \forall k \in I \setminus \bigcup_{j \in J} S_j.$$

Then $\hat{b} \in R(A)$.

Properties of $\hat{b}$ and of the corresponding maximum solution $x^{\max}(\hat{b})$ will be studied. Some implications of the properties for solving two-sided $(\max, +)$- and $(\max, \min)$-linear equation systems will be discussed.

We illustrate Theorem 2 by the following small numerical example.

Example 1 Let us consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & -5 & 0 \\ 8 & 7 & 1 \end{pmatrix}$$

$$A^* \equiv -A^T = \begin{pmatrix} -4 & -2 & -8 \\ -3 & 5 & -7 \\ -1 & 0 & -1 \end{pmatrix}$$

We have:

$$S_1 = \{3\},\ S_2 = \{3\},\ S_3 = \{1,3\}, \bigcup_{j \in J} S_j = \{1,3\},\ 2 \notin \bigcup_{j \in J} S_j$$

Following Theorem 2 we obtain:

$$\hat{b}_1 = \hat{b}_3 = 0, \quad \hat{b}_2 = \max(-6, -12, -1) = -1.$$

$$\hat{b} \in R(A), \quad x^{\max}(\hat{b}) = (-8, -7, -1)^T.$$

References

1. Baccelli, F.L., Cohen, G., Olsder, G.J., Quadrat, J.P.: Synchronization and Linearity, An Algebra for Discrete Event Systems, Wiley, Chichester, 1992.
2. Butkovič, P.: Max-linear Systems: Theory and Algorithms, Monographs in Mathematics, Springer Verlag 2010, 271 p.
3. Butkovič, P., Hegedüs, G.: An Elimination Method for Finding All Solutions of the System of Linear Equations over an Extremal Algebra, Ekonomicko - matematický obzor 20 (1984), No. 2, pp. 203-215.
4. Cechlárová, K.: Efficient Computation of the Greatest Eigenvector in Fuzzy Algebra, Tatra Mt. Math. Publications, 1997, pp. 73-79.
5. Cuninghame-Green, R.A.: Minimax Algebra, Lecture Notes in Economics and Mathematical Systems, 166, Springer Verlag, Berlin, 1979.
6. Cuninghame-Green, R.A., Zimmermann, K.: Equation with Residual Functions, Comment. Math. Univ. Carolinae 42(2001), 4, pp. 729-740.
7. G.L. Litvinov, V.P. Maslov, S.N. Sergeev (eds.): Idempotent and Tropical Mathematics and Problems of Mathematical Physics, vol. I and II, Independent University, 2007.
8. V.P. Maslov, S.N. Samborskiĭ (eds.): Idempotent Analysis, Advances in Soviet Mathematics, 13, AMS, Providence 1992.
9. Sanchez, E.: Inverses of Fuzzy Relations. Applications to Possibility Distributions and Medical Diagnosis, Fuzzy Sets and Systems, vol. 2, No 1, 1979, pp. 75 - 86.
10. Vorobjov, N.N.: Extremal Algebra of positive Matrices, Elektronische Informationsverarbeitung und Kybernetik, 3, 1967, pp. 39-71 (in Russian).
11. Zimmermann, K.: Solution of Some max-separable Optimization Problems with Inequality Constraints, Contemporary Mathematics, 377, Idempotent Mathematics and Mathematical Physics pp. 363 - 370, (ed. G. L. Litvinov, V. P. Maslov) American Mathematical Society, Providence, Rhod Island, 2003.

Supported by GA ČR grant #402/09/0405.

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Семейство распараллеленных алгоритмов распознавания изображений, оптимизированных для реализации на многоядерных центральных процессорах и спецвычислителях

Д.С. Аксенов
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Задача распознавания образов является одной из основных задач, решаемых системами компьютерного зрения и интеллектуального анализа данных. Несмотря на многократное увеличение быстродействия вычислительных ядер на сегодняшний день эта задача по-прежнему не имеет универсального решения, удовлетворительного одновременно как по скорости работы, так и по точности распознавания.

За последние 30 лет разработано множество подходов к этой задаче, причем ряд специализированных алгоритмов (например, распознавание рукописного письма) оказались весьма успешными, и нашли широкое применение в технике. Большинство методов распознавания образов можно условно разделить на четыре класса:

Геометрические методы нормализации и расчета расстояния до прототипа.

Статистические методы оценивания вероятностного распределения и классификация по правилу Байеса.

Нейросетевые методы построения нейронной сети с весами, полученными в ходе обучения на тренировочной выборке.

Структурные методы построения классификационных правил.

Эффективность методов распознавания оценивается как по скорости принятия решения, так и двумя вероятностями - вероятностью ошибочного распознавания и вероятностью нераспознавания. Уменьшение этих вероятностей обычно ведет к резкому увеличению вычислительных затрат и, следовательно, к увеличению времени принятия решения. Основной задачей предлагаемого семейства алгоритмов является компенсация увеличения вычислительных затрат за счет распараллеливания алгоритмов для эффективного использования многоядерных процессоров и спецвычислителей. В качестве основных методов для распараллеливания и оптимизации рассматриваются следующие алгоритмы распознавания образов:

1) Метод собственных изображений - разновидность метода главных компонент в задаче геометрического распознавания образов. В основе метода лежит математическая операция вычисления собственных значений и собственных векторов матрицы, составленной из изображений обучающей базы. Соответствующая оптимизированная система алгоритмов для параллельного вычисления ковариационной матрицы, для поворота матрицы методами Гаусса-Зейделя и Якоби, определения собственных значений и векторов позволяет до 25 раз ускорить реализацию метода собственных изображений применительно к большим базам изображений (до 10 тысяч изображений).
2) Метод сопоставления графов - параллельная реализация сорока фильтров Габора - свертка изображения с ядрами специального вида, позволяющая классифицировать графы, отвечающие наборам реперных точек изображения. Фильтры Габора обладают хорошей геометрической устойчивостью, а именно, они устойчивы по отношению к геометрическим поворотам, масштабированию, изменению яркости и контрастности.
3) Семейство нейросетевых методов распознавания изображений. Преимуществами таких методов является хорошая масштабируемость процесса распознавания, что позволяет использовать универсальные алгоритмы как для многопроцессорных систем, так и для высокопроизводительных одноядерных вычислителях. Стандартные библиотеки нейросетевых алгоритмов могут быть эффективно модифицированы для использования как в обычных многоядерных и многопроцессорных системах, так и в спецвычислителях с нейросетевой архитектурой.

В качестве примера приводится модификация алгоритма Якоби для параллельного вычисления собственных значений и собственных векторов, используемых для идентификации изображений методом главных компонент. Вычислительный алгоритм разделен на три последовательные стадии, каждая из которых в свою очередь распараллелена: вычисление ковариантной матрицы, модифицированный алгоритм Якоби и определение системы собственных векторов и собственных значений. За счет применения данной модификации алгоритма на видеокарте с поддержкой архитектуры универсальных параллельных вычислений CUDA (GeForce 9600GT производства компании NVIDIA) наблюдается значительное увеличение скорости распознавания изображений. По сравнению с выполнением на одном ядре CPU Pentium 4 с тактовой частотой 3 гигагерца выигрыш по скорости составляет 30-40 раз для видеокарты GeForce 9600GT. Например, время для распознавания изображения из базы в 4000 фотографий размером 92x112 сократилось с 86494 мс до 2338 мс.

Дальнейшее развитие параллельных алгоритмов распознавания изображений связано с увеличением универсальности использования отдельных алгоритмических блоков, которые с небольшой модификацией могут быть использованы как на высокопроизводительных неспециализированных рабочих станциях, так и на многоядерных вычислителях специальной архитектуры. Работа выполнена при поддержке грантов РФФИ 11-01-93106-НЦНИЛ-а, 12-01-00886-а.

Список литературы

1. Г.Л. Литвинов, В.П. Маслов. Идемпотентная математика: принцип соответствия и его компьютерные приложения// Успехи матем. наук, 1996, т. 51, N 6, с. 1210-1211.
2. G.L. Litvinov, V.P. Maslov, A.Ya. Rodionov, A.N. Sobolevski, Universal algorithms, mathematics of semirings and parallel computations. Lecture Notes in Computational Science and Engineering, 2011, vol. 75, p. 63-89. See also E-print arXiv: 1005.1252 (http://arXiv.org).
3. S. Singer, S. Singer, V. Novakovic, A. Uscumlic, and V. Dunnjko, Novel Modification of Parrallel Jacobi Algorithms, *arXie:1008.0201v2*,(2001).
4. П.П.Кольцов, Сравнительное изучение алгоритмов выделения и классификации текстур, *Ж. вычисл. матем. и матем. физ.*, 51:8 (2011), 1561-1568.
5. Самаль Д.И., Старовойтов В.В. Подходы и методы распознавания людей по фотопортретам, Минск, ИТК НАНБ, 1998.

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Оценки сложности алгоритма Григорьева для решения тропических линейных систем

А. П. Давыдов

1 Постановка задачи

Определение 1 Тропическая линейная система — *это прямоугольная матрица* $m \times n$. Решение тропической линейной системы — *это такая строка из n элементов, что при прибавлении ее к каждой из строк матрицы получится строка, в которой не будет строгого минимума, т.е. минимальный элемент в каждой строке должен встречаться хотя бы два раза. Тропическая линейная система называется* разрешимой*, если существует строка, являющаяся решением данной системы [3].*

В данной статье мы будем рассматривать задачу о разрешимости *целочисленных* тропических линейных систем. Хотя в частных случаях (например, на квадратных матрицах [1]) данная задача может быть решена эффективно, не известно ни одного алгоритма, для которого доказано полиномиальное время работы в худшем случае. Известно, что эта задача лежит в пересечении классов NP и $coNP$ [3].

Предложение 1 *Класс разрешимых тропических линейных систем инвариантен относительно прибавления произвольной константы ко всем числам в одной строке или в одном столбце. Более того, из решения системы после подобного преобразования можно получить решение исходной системы, прибавив к найденному решению разность первых строк матрицы до и после преобразования.*

Из этого предложения сразу же получается следующее замечание.

Замечание 1 *Не умаляя общности, можно считать, что все числа в матрице системы неотрицательны.*

Далее будут использоваться следующие обозначения:

– $n(A)$ — количество строк матрицы A,
– $m(A)$ — количество столбцов матрицы A,
– $M(A)$ — максимальное число в матрице A,
– $R(A)$ — множество строк матрцы A.

Для простоты, если понятно, о какой матрице идет речь, параметр в этих обозначениях будет опускаться.

2 Алгоритм Григорьева

Один из сравнительно новых алгоритмов для решения тропических линейных систем — это алгоритм Григорьева. Ключевая особенность этого алгоритма заключается в том, что в работе [3], в которой был предложен этот алгоритм, сразу же была показана как оценка, полиномиально зависящая от размеров матрицы (но при этом полиномиальная от M, а не от $\log M$), так и оценка, полиномиально зависящая от $\log M$ (но неполиномиально от размеров матрицы).

Это означает, что любая серия контрпримеров, на которой будет достигаться неполиномиальное время работы, должна состоять из матриц неограниченного размера с неограниченно большими элементами.

Приведем описание алгоритма Григорьева.

Для начала заметим, что ввиду предложения 1 можно искать не решение матрицы, а серию преобразований из добавления константы ко всем элементам в строке или столбце, которая приведет исходную матрицу к матрице, решением которой является нулевая строка. Далее решением системы мы будем называть именно такую матрицу.

Для того чтобы решить систему размера $m \times n$, решим систему размера $m \times (n-1)$, получаемую из первой системы удалением первой строки. С этого момента будем считать, что во всех строках матрицы, кроме, возможно, первой, строгих минимумов нет.

Далее определим операцию *спуска* матрицы следующим образом.

1. Заведем множество столбцов J. Изначально в нем будет только один столбец — тот, в котором достигается минимум в первой строке.
2. Если J содержит все столбцы, то спуск невозможен, и алгоритм завершает работу.

3. Если найдется строка, в которой лишь один минимум достигается на столбцах, не лежащих в J, то добавим столбец с этим минимумом в J и вернемся к шагу 2.
4. Вычтем из всех столбцов, не лежащих в J, максимальное число, для которого в каждой строке, все минимальные элементы которой лежат в J, эти элементы останутся минимальными и после вычитания.

Замечание 2 *Заметим, что хотя алгоритм спуска содержит некоторую недетерминированность в порядке прибавления столбцов к множеству J, в пункте 4 множество J определено однозначно, т.к. максимальное по включению множество, построенное по правилам 1–3, единственно.*

Предложение 2 *Если изначально строгого минимума во всех строках, кроме первой, не было, то и после спуска его не будет.*

Затем преобразуем матрицу в соответствии со следующим алгоритмом.

1. Если в первой строке нет строгого минимума, то задача решена.
2. Иначе — спустить матрицу. Если это невозможно, то решения нет, иначе перейти к пункту 1.

В [3] показано, что временная сложность данного алгоритма составляет $O(M \cdot \log M \cdot m^2 \cdot n^2)$. Позже также была показана оценка $O(2^{mn} \cdot \log M)$.

3 Улучшенная верхняя оценка

Теорема 1 (Верхняя оценка) *Алгоритм Григорьева завершает работу за $O(n \cdot \binom{m+n}{n})$ операций спуска, что дает оценку на сложность алгоритма в $O(\log M \cdot m \cdot n^2 \cdot \binom{m+n}{n})$.*

Чтобы доказать эту теорему, введем следующее определение.

Определение 2 *Назовем* расстоянием *до строки матрицы минимальную итерацию, на которой она может быть добавлена к множеству J в операции спуска матрицы. В случае, если строка не может быть добавлена к множеству J, расстояние до нее положим равным бесконечности. Здесь и далее мы будем говорить, что строка содержится в множестве столбцов J, если все столбцы, ы которых достигаются ее минимумы содержатся в множестве J.*

Обозначим расстояние до строки r через $d_0(r)$, а расстояние до строки r после i-го спуска алгоритма Григорьева — через $d_i(r)$. Обозначим мультимножество расстояний для всех строк матрицы A через $D_0(A)$, а соответствующее мультимножество после i-го спуска алгоритма Григорьева —

через $D_i(A)$. Также будем использовать обозначения $d(r)$ и $D(A)$ для $d_0(r)$ и $D_0(A)$ соответственно.

Для расстояний, определенных вышеописанным образом, верна следующая теорема.

Теорема 2 *Для любой матрицы A выполняются следующие свойства.*

1. $\forall_{n \in D(A)} (|\{x < n | x \in D(A)\}| \geq n - 1)$.
2. *После операции спуска алгоритма Григорьева расстояние до хотя бы одной из строк увеличится.*
3. *Обозначим через r_i строку с минимальным $d_{i+1}(r)$ среди всех строк, для которых $d_i(r) \neq d_{i+1}(r)$. Тогда*

$$\forall_{r \in R(A)}((d_i(r) < d_i(r_i)) \Rightarrow (d_{i+1}(r) = d_i(r)), \quad (1)$$

$$\forall_{r \in R(A)}((d_i(r) \geq d_i(r_i)) \Rightarrow (d_{i+1}(r) \geq d_{i+1}(r_i)). \quad (2)$$

Доказательство 1. Допустим, что данное утверждение неверно для расстояния n, соответствующего столбцу r. Рассмотрим операцию спуска, при которой столбец r добавляется к множеству J на итерации n. На этой итерации размер множества J равен $n - 1$, значит, найдется $n - 1$ столбец, расстояние до которых меньше n.

2. Пусть r — строка с минимальным расстоянием после спуска среди всех строк, у которых после операции спуска появился новый минимум. Несложно заметить, что расстояние до нее должно увеличиться.
3. Чтобы получить свойство (1) достаточно заметить, что невозможно уменьшить расстояние до строки, не добавив в J столбцов с новыми минимумами. Как мы уже отмечали в пункте 2, расстояние до строки, у которой появился новый минимум, и новое расстояние до которой минимально, изменилось. Свойство (2) следует по тем же соображениям, ведь если в оптимальном спуске для данной строки строка с добавленным минимумом не используется, то расстояние не уменьшится (раньше можно было использовать тот же спуск), в противном же случае расстояние не может быть меньше, чем расстояние до строки с добавленным минимумом.

Используя теорему 2, несложно заметить, что если записать элементы D в порядке возрастания, то подобная запись после спуска будет лексикографически больше записи до спуска, откуда и следует ограничение на число спусков (оно будет не больше, чем число отсортированных последовательностей длины n из чисел от 1 до m). Таким образом, теорема 1 доказана.

Одновременно с автором аналогичная нижняя оценка была получена В. Подольским [2].

4 Контрпример

Посмотрев на оценки сложности в частях 3 и 4, можно заметить, что если зафиксировать хотя бы одну из трех размерностей задачи (M, m, или n), то алгоритм Григорьева будет работать полиномиальное время. Этим и объясняется трудность конструкции контрпримера.

Теорема 3 (Нижняя оценка) *Существует последовательность матриц (с неограниченным количеством строк и столбцов), на которой алгоритм Григорьева работает, в худшем случае, за время $\Omega(n^{\frac{m}{6}} \log M)$. При этом $M = poly(n^{\frac{m}{6}})$.*

Доказательство Построим последовательность матриц $S_{k,i}$ со следующими свойствами:

- количество столбцов матрицы не зависит от i и равна $6k + C$, где C — константа;
- количество строк матрицы линейна по i при фиксированном k;
- максимальное число в матрице полиномиально по i^k;
- алгоритм Григорьева на матрице $S_{k,i}$ делает i^k спусков.

Эту последовательность мы будем строить индукционно, используя в конструкции для $S_{k,i}$ матрицу $S_{k-1,i}$. Не умаляя общности, будем считать, что все элементы $S_{k-1,i}$ неотрицательны, и минимальные элементы в каждой строке — нули.

Основная идея конструкции заключается в следующем: приводится конструкция подматрицы (будем называть ее *гаджет*), которая имеет константное количество строк (равное количеству столбцов $S_{k-1,i}$) и позволяет перезапустить алгоритм на $S_{k-1,i}$ заново. Соответственно, добавив к матрице $S_{k,i}$ i таких гаджетов мы и получим $S_{k,i}$. К сожалению конструкция гаджета весьма громоздка и нет возможности привести ее в формате данной статьи.

Список литературы

1. P. Butkovic and F. Hevery. A condition for the strong regularity of matrices in the minimax algebra. *Discr. Appl. Math*, 11:209–222, 1985.
2. D. Grigoriev and V. Podolskii. Complexity of tropical and min-plus linear prevarieties.
3. Dima Grigoriev. Complexity of solving tropical linear systems. Preprint MPIM 2010-60, Bonn, to appear in *Computational Complexity*

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Подписано в печать 16.08.2012 г. Формат 160 × 235
Гарнитура Times New Roman. Усл. печ. л. 23
Тираж 200 экз.

Отпечатано в Астраханской цифровой типографии
(ИП Сорокин Роман Васильевич)
414040, Астрахань, пл. К.Маркса, 33, 5-й этаж, 5-й офис
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