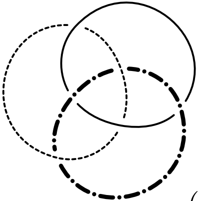
The embedded connected sum and the second Kirby move for higher-dimensional links

arXiv:2406.15367, A. Skopenkov



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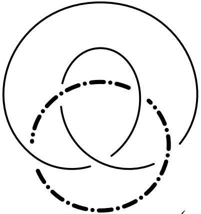

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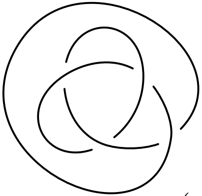

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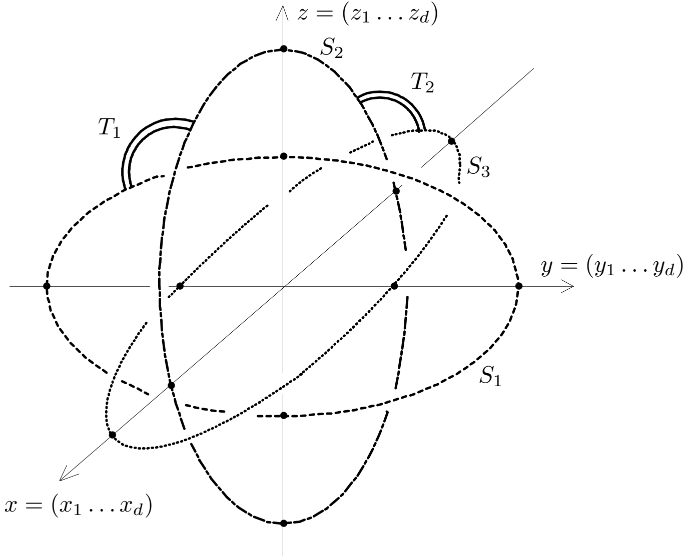
Embedded connected sum



We work in the smooth category.
Take an (ordered oriented) link, i.e. an embedding $f : S^q \sqcup S^q \to S^m$.
Up to isotopy this is equivalent to taking two numbered oriented q-spheres in S^m.
Make **embedded connected sum** of the components of f along some tube (=band) joining them. We obtain a knot $\#f : S^q \to S^m$.

Description of the first main result

- For $m \geqslant q+3$ the isotopy class $\#[f]$ of $\#f$ is independent of the choices of the tube, and of the link f within its isotopy class $[f]$.
- This is not so for $m = q+2 = 3$. Then this multivalued operation is called *band connected sum* of the components of the link. Unlike in this paper, this operation was mostly studied for links whose components are contained in disjoint cubes.

How does the isotopy class $\#[f]$ depend on f?

- For $2m \geqslant 3q+4$ every two embeddings $S^q \to S^m$ are isotopic (Haefliger, 1961).
- We give the answer for the 'first non-trivial case' $2m = 3q+3$ (for m is even).

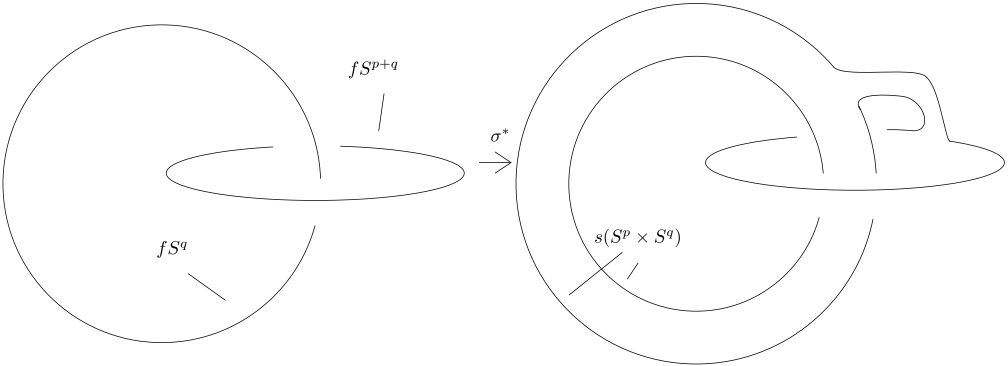
This answer was used for classification of linked 3-manifolds in S^6 (Avvakumov, 2010s). This answer gives an alternative construction of a generator in the group of knots $S^{4k-1} \to S^{6k}$ for $k = 1, 2, 4$ (first constructed by Haefliger in 1962).
It would be interesting to obtain analogues of this and other our results for m odd.

Take a 3-component link, i.e. an embedding $g : S^q_1 \sqcup S^q_2 \sqcup S^q_3 \to S^m$.
Make embedded connected sum of the second and the third components along some tube joining them. We obtain a link $\#_{23}g : T^{0,q} \to S^m$.
For $m \geqslant q + 3$ the isotopy class $\#_{23}[g]$ of $\#_{23}g$ is independent of the choices of the tube, and of the link g within its isotopy class $[g]$.

How does the isotopy class $\#_{23}[g]$ depend on $[g]$?

- For $2m \geqslant 3n + 4$ the answer is simple and so essentially known.
- We give the answer for the 'first non-trivial case' $2m = 3q + 3$ (for m is even).

The unframed second Kirby move



Assume that the first component of a link $f : S^q \sqcup S^q \to S^m$ is unknotted.
Define a link $\sigma f : S^q \sqcup S^q \to S^m$ (**see the figure for** $p = 0$) by taking

- the first component to be the 'standardly shifted' first component of f
- the second component to be the embedded connected sum of the components of f

Concerning low-dimensional version of $\#$ and σ see Remark 5.6 of arXiv:2406.15367.

Description of the third main result

For $m \geqslant q+3$ the isotopy class $\sigma[f]$ of σf is independent of the choices of the tube, and of the link f within its isotopy class $[f]$ (M. Skopenkov, 2011).

How does the isotopy class $\sigma[f]$ depend on $[f]$?

- $2m \geqslant 3q+4$ we have $\sigma[f]=[f]$.
- We give the answer for the 'first non-trivial case' $2m=3q+3$ (for m even).

This is used to show that $\sigma \neq \pm id$.
This is used to obtain an alternative classification of links $S^{4k-1} \sqcup S^{4k-1} \to S^{6k}$.
This is interesting because this rules out a natural inductive proof of classification of embeddings $S^p \times S^q \to S^m$ (conjecture of S, 2015).
These resuts are particularly interesting because theey are essentially piecewise linear (PL) results proved using differential topology. Indeed, the linking coefficients and the second Kirby move can be defined in the PL category.

Main notation

For a manifold N denote by $E^m(N)$ the set of embeddings $N \to \mathbb{R}^m$ up to isotopy.
By $[\cdot]$ we denote the isotopy class of an embedding or the homotopy class of a map.

We assume that $m \geqslant q+3$, unless indicated otherwise.

The sum operations on $E^m(S^0 \times S^q)$ and on $E^m(S^q)$ are 'embedded connected sums of two embeddings whose images are contained in disjoint cubes' (Haefliger, 1966).
Identify $E^{6k}(S^{4k-1})$ with $\mathbb{Z}$ by the isomorphism of Haefliger, 1962.
`http://www.map.mpim-bonn.mpg.de/Knots,_i.e._embeddings_of_spheres`

In which terms the answers are given?

Let us define and use the following diagram of groups and homomorphisms.

$$E^{6k}(S^{4k-1}) \overset{r_\pm}{\underset{\#}{\longleftarrow}} E^{6k}(S^0 \times S^{4k-1}) \xrightarrow[\lambda_\pm]{} \pi_{4k-1}(S^{2k}) \xrightarrow{H} \mathbb{Z}, \qquad \zeta : \pi_{4k-1}(S^{2k}) \to E^{6k}(S^0 \times S^{4k-1}).$$

- $S^0 = \{+1, -1\}$; definition of the Zeeman map ζ is postponed.
- The 'embedded connected sum' map $\#$ defined above is clearly a homomorphism.
- Let $r_\pm$ be 'the knotting class of the $\pm$-component',
- Let $\lambda_\pm$ be the linking coefficient, i.e. the homotopy class of $f|_{\pm 1 \times S^{4k-1}}$ in the complement to the other component.
- Let H be the Hopf invariant, i.e. the linking number of preimages of two regular points under a smooth or a PL approximation of a map $S^{4k-1} \to S^{2k}$.

The first main result

Theorem (Connected Sum)

$$\# = r_+ + r_- \pm \frac{H\lambda_+ + H\lambda_-}{2}$$

- The sign in this formula could depend on k.
- The integers $H\lambda_+$ and $H\lambda_-$ have the same parity by (b) below.

Recall known result used in our proof.

Theorem (Haefliger 1960s, for $k=1$ M. Skopenkov 2009)

(a) The following map has a finite kernel:

$$H\lambda_+ \oplus H\lambda_- \oplus r_+ \oplus r_- : E^{6k}(S^0 \times S^{4k-1}) \to \mathbb{Z}^4.$$

(b) The image of this map is $\{(a,b,c,d) \ : \ a \equiv b \mod 2\}$ *for* $k = 1, 2, 4$, *and is* $\{(a,b,c,d) \ : \ a \equiv b \equiv 0 \mod 2\}$ *otherwise.*

Clearly, $\#, r_+, r_-, \lambda_+, \lambda_-$ are homomorphisms.
The group $E^{6k}(S^0 \times S^{4k-1})$ is generated by (isotopy classes of) links whose components are contained in disjoint balls, and by $K_0 := \ker(r_+ \oplus r_-)$. We have $\# = r_+ + r_-$ for the former links. Hence it suffices to prove the theorem for links in K_0.
By (a) above the map $H\lambda_+ \oplus H\lambda_- : K_0 \to \mathbb{Z}^2$ has finite kernel. Any homomorphism from a finite group to $\mathbb{Z}$ is zero. So this kernel goes to 0 under the map $\#$. Hence

$$\#|_{K_0} = n \circ (H\lambda_+ \oplus H\lambda_-) \quad \text{for some homomorphism} \quad n = n_k : \mathrm{im}(H\lambda_+ \oplus H\lambda_-) \to \mathbb{Z}.$$

So by (b) above

$$\#|_{K_0} = n_+ H\lambda_+ + n_- H\lambda_- \quad \text{for some} \quad n_\pm = n_{\pm,k} \in \mathbb{Q}.$$

Analogously to the commutativity of summation on $E^m(S^q)$, the map $\#$ is invariant under exchange of the components. Hence $n_+ = n_-$.
So by the following Whitehead Link Lemma $n_+ = \pm 1/2$.

Lemma (Whitehead Link)

For any $l \geqslant 2$ there is an embedding $\omega : S^0 \times S^{2l-1} \to S^{3l}$ such that

$$r_+\omega = r_-\omega = 0, \quad \lambda_-\omega = 0, \quad \text{and, for } l \text{ even}, \quad H\lambda_+\omega = \pm 2, \quad \#\omega = 1.$$

This is proved using the following interpretation of linking coefficients λ via Pontryagin isomorphism between the group $\pi_q(S^n)$ and the set of framed cobordism classes of framed $(q-n)$-submanifolds of S^q.

Lemma

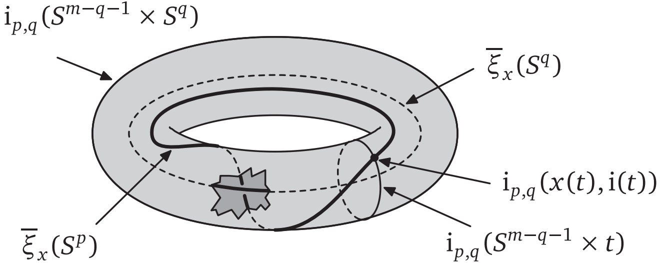
Let $f : S^0 \times S^q \to S^m$ be a link such that $r_+f = r_-f = 0$. Then λ_+f is equal to the framed intersection of an arbitrarily framed $f(1 \times S^q)$ and a general position arbitrarily framed $(q+1)$-disk spanned by $f(-1 \times S^q)$.

The natural normal framing on the inclusion $S^q \to S^m$ defines **the standard embedding**

$$\mathrm{i}_{m,q} : D^{m-q} \times S^q \to S^m.$$

For a map $x : S^q \to S^{m-q-1}$ define an embedding ($\overline{\xi}_x$ on the picture; $p = q$)

$$\zeta x : S^0 \times S^q \to S^m \quad \text{by} \quad (\zeta x)(t,a) := \mathrm{i}_{m,q}\left(\frac{1+t}{2}x(a), a\right).$$



Define **the Zeeman map** ζ by $\zeta[x] := [\zeta x]$.
Clearly, ζ is well-defined, is a homomorphism, and $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$ (Haefliger-Zeeman, 1960s).
We do not assert that $\lambda_-\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$.

An alternative construction of a generator $S^{4k-1} \to S^{6k}$

Corollary

(a) We have $\#\zeta = \pm H$ on $\pi_{4k-1}(S^{2k})$.
(b) For any $k = 1, 2, 4$ let $\eta \in \pi_{4k-1}(S^{2k})$ be the homotopy class of the Hopf map. The embedded connected sum $\#\zeta\eta$ is a generator of $E^{6k}(S^{4k-1}) \cong \mathbb{Z}$.

Part (b) follows from (a) because $H\eta = 1$.
Part (a) follows by the Connected Sum Theorem since $r_\pm\zeta = 0$, $\lambda_+\zeta = \mathrm{id}\,\pi_q(S^{m-q-1})$, and $H\lambda_-\zeta = H$.
Denote by A_q the symmetry of S^q w.r.t the origin.
Proof of $H\lambda_-\zeta = H$. We have $\lambda_- f = \lambda_+ \widehat{f}$, where the link $\widehat{f}$ is obtained from f by exchange of the components.
For a map $x : S^{4k-1} \to S^{2k}$ the link $\widehat{\zeta x}$ is isotopic to $\zeta(A_{2k} \circ x)$.
Then $\lambda_-\zeta x = \lambda_+\widehat{\zeta x} = \lambda_+\zeta(A_{2k} \circ x) = [A_{2k} \circ x]$.
So $H\lambda_-\zeta x = H(A_{2k} \circ x) = Hx$, where the latter equality is easy and well-known.

Take an embedding $g : S^q_1 \sqcup S^q_2 \sqcup S^q_3 \to S^m$. Denote $Q := m - q - 1$

- Let $r_i = r_i(g) \in E^m(S^q)$, $i \in [3]$, be the isotopy classes of the **restrictions** of g to the components.
- Let $\lambda_{ij} = \lambda_{ij}(g) \in \pi_q(S^Q)$, $(i,j) \in [3]^2$, $i \neq j$, be the **pairwise linking coefficients** of the components.
- For $m = 6k$ and $q = 4k - 1$ denote $h_{ij} = h_{ij}(g) = \frac{1}{2}(H\lambda_{ij} + H\lambda_{ji})$.
- Let $\lambda^1(g) \in \pi_q(S^Q \vee S^Q)$ be homotopy class of $g|_. : S^q_1 \to S^m - g(S^q_2 \sqcup S^q_3)$. For $3m \geqslant 4q + 6$ let the **triple linking coefficient** $\mu = \mu(g)$ be the image of $\lambda^1(g)$ under the composition

$$\pi_q(S^Q_2 \vee S^Q_3) \to \frac{\pi_q(S^Q_2 \vee S^Q_3)}{i_{2*}\pi_q(S^Q_2) \oplus i_{3*}\pi_q(S^Q_3)} \to \pi_q(S^{2m-2q-3}) \overset{\Sigma^\infty}{\to} \pi^S_{3q-2m+3}$$

of the projection from the Hilton theorem and the stable suspension.

Theorem

(a) $\lambda_-\#_{23} = \lambda_{21} + \lambda_{31}$ *and* $r_+\#_{23} = r_1$.
(b) $r_-\#_{23} = r_2 + r_3 \pm h_{23}$.
(c) $H\lambda_+\#_{23} = 2\mu + H\lambda_{12} + H\lambda_{13}$.
(d) $\#\#_{23} = r_1 + r_2 + r_3 \pm (\mu + h_{12} + h_{23} + h_{31})$.

- Part (a) is obvious (and holds whenever $m - q \geqslant 3$).
- Part (b) holds by the Connected Sum Theorem.
- In parts (c,d) $q = 4k - 1$, $m = 6k$, so $\mu(g) \in \pi_0^S \cong \mathbb{Z}$. These parts are non-trivial. They are proved using the interpretation of linking coefficients via Pontryagin construction. Parts (c) and (d) are equivalent (modulo (a,b)), but they are proved together, not deduced one from the other.

The following map is injective:

$$H\lambda_+ \oplus \lambda_- \oplus r_+ \oplus r_- : E^{6k}(S^0 \times S^{4k-1}) \to \mathbb{Z} \oplus \pi_{4k-1}(S^{2k}) \oplus \mathbb{Z}^2.$$

(Haefliger 1960s, for $k = 1$ M. Skopenkov 2009)
So the following theorem describes the above self-map σ of $\ker r_+$.

Theorem (on σ)

(a) We have $r_-\sigma = \#$ and $\lambda_-\sigma = \lambda_-$.
(b) For $q = 4k - 1$ and $m = 6k$ we have $H\lambda_+\sigma = H\lambda_+ + 2H\lambda_-$.

- The formula for $r_-\sigma$ is obvious.
- The formula for $\lambda_-\sigma$ holds since in the definition of σ the restrictions of σf and f to the second component $-1 \times S^q$ are homotopic as maps to $S^m - f(1 \times S^q)$;
- Part (b) is non-trivial, and is proved below using the Symmetry Lemma.

Lemma (Symmetry)

(a) For any embedding $g : S^{4k-1} \to S^{6k}$ the composition with the reflection-symmetry of S^{6k} is isotopic to g.
Or, equivalently, for any embedding $g : S^{4k-1} \to S^{6k}$ the composition with the reflection-symmetry of S^{4k-1} represents a knot $-[g] \in E^{6k}(S^{4k-1})$.
(b) Let ψ_+ be the 'change of the orientation of $+1 \times S^q$' self-map of $E^m(S^0 \times S^q)$. Then

$$r_-\psi_+ = r_-, \quad r_+\psi_+ = -r_+, \quad \lambda_+\psi_+ = -\lambda_+ \quad \text{and} \quad H\lambda_-\psi_+ = H\lambda_-.$$

Part (a) follows by definition of the Haefliger isomorphism $E^{6k}(S^{4k-1}) \to \mathbb{Z}$. Cf. analogous result (S, 2005) on embeddings $S^4 \to S^7$, where situation is 'the opposite'.
In (b) the equation $r_-\psi_+ = r_-$ is clear, the equation $r_+\psi_+ = -r_+$ holds by (a), and we have

$$\lambda_+\psi_+ = \lambda_+ \circ A_{4k-1} = -\lambda_+ \quad \text{and} \quad H\lambda_-\psi_+ = H(A_{2k} \circ \lambda_-) = H\lambda_-.$$

Proof of Theorem on σ, part (b)

Theorem on σ, part (b) follows because on $\ker r_+$ we have

$$2r_- \overset{(1)}{=} 2\#\psi_+\sigma \overset{(2)}{=} (2r_- \pm H\lambda_+ \pm H\lambda_-)\psi_+\sigma \overset{(3)}{=} (2r_- \mp H\lambda_+ \pm H\lambda_-)\sigma \overset{(4)}{=}$$

$$= 2\# \mp H\lambda_+\sigma \pm H\lambda_- \overset{(5)}{=} (2r_- \pm H\lambda_+ \pm H\lambda_-) \mp H\lambda_+\sigma \pm H\lambda_-, \quad \text{where}$$

- equality (1) holds because two copies of the first component having opposite orientations ‘cancel’;
- equality (2) holds by the Connected Sum Theorem because $r_+\psi_+\sigma = 0$;
- equality $r_+\psi_+\sigma = 0$ holds because change of the orientation of the standard embedding $S^q \to \mathbb{R}^m$ gives embedding $S^q \to \mathbb{R}^m$ isotopic to the standard one;
- equality (3) holds by the Symmetry Lemma ($r_-\psi_+ = r_-$, $\lambda_+\psi_+ = -\lambda_+$, $H\lambda_-\psi_+ = H\lambda_-$);
- equality (4) holds by Theorem on σ, part (a) ($r_-\sigma = \#$ and $\lambda_-\sigma = \lambda_-$);
- equality (5) holds by the Connected Sum Theorem because $r_+ = 0$ on $\ker r_+$.