MATHEMATICS AND OLYMPIADS [1]

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These are preliminary notes of a course for Math in Moscow program, see http://www.mccme.ru/mathinmoscow/. Students attending this course solve interesting simply formulated ('olympic') problems. The problems are chosen in such a way that through their solution and discussion a student learns important mathematical *ideas* (without spending much time on learning mathematical *language*). See more in

N. N. Konstantinov and A. B. Skopenkov, Olympic and research problems, www.mccme.ru/circles/oim/oimpeng.pdf.

Part of the time is spent on solving problems from matematical olympiads (in particular, from Putnam competition, see http://www.maa.org/awards/putnam.html).

For MiM students who use this text in order to decide whether to attend the course. *(1) You could read one or two solutions in the references, but do not read many solutions. At the course the material would be presented as a sequence of problems, so reading solutions in advance will spoil your chance to learn how solutions are invented. (Reading solution in advance will also be harmful for exams, which would consist of problems requiring some experience of solving problems by oneself.) For this reason some solutions of problems below will only be available at the course.*

(2) It is advisable before enlisting to consult MiM students who earlier took such a course. E-mails are available from prof. Irina Paramonova upon request.

(3) If some student(s) wish to attend this course in Spring 2010 semester, the first class will take place on Thursday, February, 11, 2010, 17.30-19.00. If he/she/they persist, the classes will take place on Thursdays.

Part of the course is based on papers

A. Skopenkov, Borsuk's problem, Quantum, 7:1 (1996) 16–21, 63

A. Skopenkov, On the Kuratowski graph planarity criterion, Mat. Prosveschenie, 9 (2005), 116-128. http://arxiv.org/abs/0802.3820

P. Kozlov and A. Skopenkov, A la recherche de l'algèbre perdue: du cote de chez Gauss, Mat. Prosveschenie 12 (2008), 127–144. http://arxiv.org/abs/0804.4357

A. Skopenkov, Yet another proof from the book: the Gauss theorem on regular polygons, http://arxiv.org/abs/0908.2029

Another part involves 'Apply the Baire category theorem!' tutorial and olympic problems (both below), as well as Putnam competition problems. The following sources (or their English versions) would also be partly used (arxiv references contain English abstracts).

T. Andreescu, B. Enescu, Mathematical Olympiad Treasures, Birkhäuser, Boston-Basel-Berlin, 2004.

M. S. Clamkin, USA Mathematical Olympiads 1972-1986, MAA, 1988.

A. Oshemkov and A. Skopenkov, Olympiads in geometry and topology (in Russian), Mat. Prosveschenie, 11 (2007), 131–140.

A. Skopenkov, Basic Differential Geometry As a Sequence of Interesting Problems (in Russian), MCCME, Moscow, 2008. http://arxiv.org/abs/0801.1568

A. Skopenkov, Basic embeddings and Hilbert's 13th problem on superpositions (in Russian), Mat. Prosveschenie, 2010, to appear. http://arxiv.org/abs/1001.4011

In this text *if the statement of a problem is an assertion, then the problem is to prove this assertion.*

[1]Update version: www.mccme.ru/circles/oim/materials/momim.pdf. I would like to acknowledge N. Sheils and D. Yang for useful remarks. Readers are invited to send their critical remarks by e-mail. Use for non-commercial purposes is free upon acknowledgement.

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Apply the Baire category theorem!

By solving the following problems you will learn how to apply the Baire category theorem, a powerful tool for proving existence theorems. In analysis one uses it to prove the Banach inverse operator theorem, which in turn is applied for proving existence of sulutions of some equations. In topology the Baire category theorem is applied e.g. to embeddings of compacta and approximations of maps by homeomorphisms.

0. The union $U \subset \mathbf{R}$ of open intervals is unbounded. Prove that there exists x such that $nx \in U$ for an infinite number of integers n.

Hint. First prove that *there is* $x_1 \in (0,1)$ *such that* $n_1 x_1 \in U$ *for some* $n_1 > 1$.
Then *there is* $\varepsilon_1 > 0$ *such that* $n_1(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \subset U$.
Then prove that *there is* $x_2 \in (x_1 - \varepsilon_1, x_1 + \varepsilon_1)$ *such that* $n_2 x_2 \in U$ *for some* $n_2 > 2$.
And so on.

Such solutions based on 'included segments principle', are easy to invent and to write down using *Baire category theorem.* Recall that a subset $U \subset \mathbf{R}$ is called

- *open*, if for each $x \in U$ there is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset U$.
- *everywhere dense*, if for each $a, b \in \mathbf{R}$ we have $(a,b) \cap U \neq \emptyset$.

1. The Baire category theorem. The intersection of a countable number of open everywhere dense sets is everywhere dense (and, in particular, non-empty).

The solution of problem 0 given above can shortly be written as follows: by the Baire category theorem $\bigcap\limits_{n=1}^{\infty} \bigcup\limits_{k=n}^{\infty} \frac{1}{k} U \neq \emptyset$.

2. For functions $\mathbf{R} \to \mathbf{R}$ prove the following.

(a) A pointwise limit of a sequence of continuous functions has necessarily a continuity point.

Hint 1. The set of continuity points of f is

$$\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{x \ :\ |fy_1 - fy_2| < \frac{1}{n} \text{ when } x - \frac{1}{k} < y_1 < y_2 < x + \frac{1}{k}\}.$$

Hint 2. Fix $\varepsilon > 0$. Let $U_n := \bigcup\limits_{i,j \geq n} \{x \ :\ |f_i x - f_j x| > \varepsilon\}$. Then U_n is open, $U_n \supset U_{n+1}$ and $\bigcap_{n=1}^{\infty} U_n = \emptyset$. Hence for each $[a,b]$ by Baire category theorem there is n such that U_n is not everywhere dense on $[a,b]$. So there is $(c,d) \subset [a,b]$ disjoint with U_n. This means that $|f_i x - f_j x| \leq \varepsilon$ for each $x \in (c,d)$ and $i,j \geq n$. Hence $|fx - f_n x| \leq \varepsilon$ for each $x \in (c,d)$.

(b) The derivative of an everywhere differentiable function has a continuity point.

Hint: use (a) and $f'(x) = \lim\limits_{n\to\infty} n(f(x+\frac{1}{n}) - f(x))$.

(c)* Each monotonous function is differentiable at some point.

Hint: the function is differentiable at 'almost all' points by measure, not by Baire.

(d) Each Lipschitz function is differentiable at some point.

Hint. Use the previous problem and decomposition of a Lipschitz function into a difference of monotone functions.

3. Prove that if a function of two variables is continuous by each variable, then the function is continuous (as a function of two variables) at some point.

4.* Let $f : \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function. For each x and for each number $n > N_x$ we have $f^{(n)}(x) = 0$. Prove that f is a polynomial.

Some olympic problems

These problems form mathematical olympiad 16.05.2009 for students of Faculty of Mechanics and Mathematics, Moscow State University.

1. A linear self-map of the space of complex $n \times n$ matrices preserves the determinant. Does the map necessarily have an inverse?

2. For which dimensions n does there exist a hyperplane in $\mathbf{R}^n$ that

• intersects all closed $(n-1)$-dimensional faces of an n-dimensional cube but
• does not intersect the closed ball inscribed into the cube?

3. Let $m(k)$ be the maximal number of vectors from $\{-1,0,1\}^{2k}$ such that exactly k coordinates of each vector are 0, and no two vectors are orthogonal.

(a) Prove that $m(k) \geq 2^{2k-1}$ for odd k.

(b) Prove that $80 \leq m(4) \leq 140$.

4. Let n be an odd integer. Using the sides of an arbitrary ('the 1st') n-gon as bases one constructs in the exterior of the n-gon isosceles triangles of vertex angle $2\pi/n$. The 2nd n-gon is formed by the vertices of the triangles. Using its sides as bases one constructs isosceles triangles of vertex angle $4\pi/n$. The 3rd n-gon is formed by the vertices of the triangles. Analogously starting from the kth n-gon and angle $2k\pi/n$ one constructs the $(k+1)$th n-gon (the isosceles triangles are constructed in the exterior of the kth n-gon if $2k\pi/n < \pi$ and in the interior of the kth n-gon if $2k\pi/n > \pi$). Prove that the $(n-1)$th n-gon is regular.

5. For a continuos function $f : \mathbf{R}^2 \to [0,+\infty)$, each square K of edge 1 and each $x \in K$ we have $f(x) \leq 1 + \int_K f(y)\,dy$. Prove that $f(x) \leq Me^{\|x\|}$ for some number $M \geq 0$.

Hints to some olympic problems (in the form of new problems)

1. (a) If A is an $n\times n$-matrix and $\det A = 0$, then there exists an $n \times n$-matrix B such that $\det B = 0$ and $\det(A+B) \neq 0$.

2. (a) Let V_n be the volume of the n-dimensional ball inscribed into the unit n-dimensional cube. Prove that $\lim_{n\to\infty} V_n = 0$.

(b) A hyperplane $a_1x_1 + \dots + a_nx_n = b$ in $\mathbb{R}^n$ intersects the closed ball inscribed into the n-dimensional cube $\{(x_1,\dots,x_n) \in \mathbb{R}^n \ : \ |x_i| \leq 1\}$ if and only if $\sum a_i^2 < b^2$.

(c) Find at least one n for which there exist such a hyperplane in $\mathbb{R}^n$.

3. (α) $m(1) = 2$ and $m(2) = 6$.

Hint to $m(2) \leq 6$. Let $e_\pm = (\pm1,\pm1,0,0,0,0)$, $f_\pm = (\pm1,0,\pm1,0,0,0)$ $g_\pm = (\pm1,0,0,\pm1,0,0)$.

Hint to 3a. Let $H_1,\dots,H_{2^{k-1}}$ be all subsets of $\{1,2,\dots,k\}$ containing even number of elements. Let
$M_s = \{(a_1,b_1,\dots,a_k,b_k) \in \{-1,0,1\}^{2k} : (a_i,b_i) = (\pm1,0) \text{ if } i \in H_s,\ (a_i,b_i) = (0,\pm1) \text{ if } i \notin H_s\}$.
Prove that $\cup_{s=1}^{2^{k-1}} M_s$ is the required set of 2^{2k-1} vectors.

Beginning of the proof of the inequality $m(4) \leq 140$ *of 3b.* (This solution is due to V. Arutyunov; a similar solution was suggested by V. Vanovskiy.) Suppose that there exists such 140 vectors. Since no two of our vectors are orthogonal, for two our vectors their positions with zeros are not the 'opposite'. There are $\binom{8}{4} = 70$ variants for positions with zeros. If to the contrary there exist 141 such vectors, then some 5 our vectors have the same positions with zeros.

Beginning of the proof of the inequality $m(4) \geq 80$ *of 3b.* Take vectors

$e_1,\dots,e_5$ for which the first 3 coordinates are 1 and exactly one of the last 5 coordinates is 1;

$e_6,\dots,e_{10}$ for which the first 3 coordinates are 1 and exactly one of the last 5 coordinates is -1;

$f_1,\dots,f_{30}$ for which exactly two of the first 3 coordinates are 1 and exactly two of the last 5 coordinates are 1.

(The remaining coordinates are zeroes.)

Prove that no two of the 80 vectors $e_i, f_k, -e_i, -f_k$ are orthogonal.

4'. (a) Using the sides of an arbitrary quadrilateral as bases one constructs in the exterior of the quadrilateral isosceles triangles of vertex angle $\pi/2$. The 2nd quadrilateral is formed by the vertices of the triangles. Prove that the middle points of its sides form a square.

(b) *Napoleon problem.* In the exterior of triangle ABC three equilateral triangles ABC', BCA' and CAB' are constructed. Prove that the centroids of these triangles are the vertices of an equilateral triangle.

4. Denote by $z_{k1},\dots,z_{kn}$ the complex numbers corresponding to the vertices of the kth polygon. We can take a complex coordinate system so that $z_1 + \dots + z_n = 0$.

(a) The vertices of the 2nd polygon are linear functions of the vertices of the 1st polygon. In other words, there is a complex $n\times n$-matrix $A_1=(a_{1,st})$ such that $z_{2s}=\sum_t a_{1,st}z_{1t}$.

(b) For each k there exists a complex $n\times n$-matrix $A_k=(a_{k,st})$ such that $z_{k+1,s}=\sum_t a_{k,st}z_{kt}$.

(c) Coordinates of each eigenvector of A_1 form a geometric series in which the ratio of subsequent terms is certain n-th root of 1.

(d) The eigenvectors of A_1 are the same as the eigenvectors of A_2 and so on.

(e) For each k one eigenvalue of A_k is zero. In other words, multiplication by A_k defines the map $\mathbf{C}^n\to\mathbf{C}^n$ that is a projection onto a hyperplane.

(f) Denote by v_k the unit vector corresponding to the zero eigenvalue of A_k. Then vectors $v_1,\dots,v_{n-2}$ are different. In other words, the hyperplanes from (e) are different.

(g) $A_{n-2}A_{n-3}\dots A_2A_1\{(z_1,\dots,z_n)\in\mathbf{C}^n\ :\ z_1+\dots+z_n=0\}$ is the 1-dimensional space defining regular polygons.

5. (a) Let A and B be a unit square and a rectangular $1\times\frac12$ having a common edge. Then $f(x)\le 3+\max\limits_{y\in A}f(y)$ for each $x\in B$.

(b) For a continuos function $f:\mathbf{R}^2\to[0,+\infty)$ there is $C>0$ such that for each square K of edge 1 and each $x\in K$ we have $f(x)\le C+C\int_K f(y)\,dy$. Then $f(x)\le Me^{C\|x\|}$ for some $M\ge 0$.

More olympic problems

Problems 1–5 form mathematical olympiad 17.05.2008 for students of Faculty of Mechanics and Mathematics, Moscow State University.

1. Prove that for each integer $n>0$ there is a number $c_n>0$ such that the following holds: *each convex open figure of volume V inside the unit ball in $\mathbf{R}^n$ contains a ball of radius c_nV.*

2. For increasing functions $f,g:[0,\pi/2]\to\mathbf{R}$ prove that

$$\int_0^{\pi/2}f(x)g(x)\sin x\,dx\ge\int_0^{\pi/2}f(x)\sin x\,dx\int_0^{\pi/2}g(x)\sin x\,dx.$$

3. (a) For which elements $(b_1,b_2,c_1,c_2)\in\mathbf{Z}^4$ there is a subgroup D of $\mathbf{Z}^4$ such that $(b_1,b_2,c_1,c_2)\in D$ and $\mathbf{Z}^4=(\mathbf{Z}\oplus\mathbf{Z}\oplus0\oplus0)\oplus D$? Give your answer in terms of divisibility of numbers b_1,b_2,c_1,c_2 and the greatest common divisors of certain subsets of this set.

(b) Suppose that A is a direct sum of of free abelian finitely generated subgroups B and C. For which elements $(b,c)\in A$ there is a subgroup D of A such that $(b,c)\in D$ and $A=B\oplus D$?

4. Prove that there exists a convex bounded open figure F in the plane such that

• F has no symmetry center, and

• each line splitting the boundary of F into two parts of equal length splits F into two parts of equal area.

5. (a) Let W be a subset of $\mathbf{R}^n$ containing 4^n elements. Suppose that each subset of W containing 2^n elements contains two elements x,y at unit distance: $|x-y|=1$. Prove that for sufficiently large n the number of unit distances between points of W is at least $0.49\cdot8^n$:

$$\frac12\#\{(x,y)\in W\times W\colon\ |x-y|=1\}\ge0.49\cdot8^n.$$

Here $\#A$ is the number of elements in a finite set A.

(b) The same with $\ge0.99\cdot8^n$.

6. (a) Let B be a k-subspace of a vector space with a basis $\{e_1,\dots,e_n\}$. Then there exists a set $A\subset\{e_1,\dots,e_n\}$ consisting of $n-k$ elements such that $B\cap\operatorname{Lin}A=\{0\}$.

(b) Let B be a subspace of a vector space with a basis $\{e_1,\dots,e_n,f_1\dots,f_m\}$ such that $B\cap\operatorname{Lin}\{e_1,\dots,e_n\}=\{0\}$. Then at least one of the 8 subspaces

$$\operatorname{Lin}\{e_i-f_j,\ e_{3-i}-ke_i-f_{3-j},\ e_3,\dots,e_n\},\quad\text{where}\quad i,j\in\{1,2\},\ k\in\{0,1\}$$

has zero intersection with B. (There are in fact at most 7 different subspaces in the list, or less if the characteristic of the field is 2).

Hints to more olympic problems

4. To each direction (given by an angle ϕ) there corresponds exactly one line splitting the figure into two parts of equal area and perimeter. Denote by $2l(\phi)$ the length of the segment of this line inside our figure. Denote by $(x(\phi), y(\phi))$ the coordinates of the middle-point of this segment. Then one proves that l *is independent on* ϕ and that

(*) any line splitting the figure into two 'equal' parts is tangent to the curve $(x(\phi), y(\phi))$.

We shall find examples of functions x and y satisfying to (*) in the form $\sum\limits_{k=0}^{n}(a_k\cos k\phi + b_k\sin k\phi)$. So we reformulate (*) as an equation on a_k, b_k (with l a parameter). The boundary of our figure is given by parametric equations $X(\phi) = x(\phi) + l\cos\phi$, $Y(\phi) = y(\phi) + l\sin\phi$. For large enough l the obtained figure will be convex.

5a. Take the graph whose set of vertices is W, and two vertices are joined by an edge if the distance between them is 1. Prove that *if for each k vertices of a graph with v vertices there is an edge joining two of the k vertices, then the number of edges is at least $(k-1)q(q-1)/2$, where* $q := \left[\dfrac{v}{k-1}\right]$.

5b. The space $\mathbf{R}^n$ does not contain $n+2$ points of pairwise distances 1. This gives a special property of our graph allowing to improve the estimation of (a).

Solution of 6a. Use induction on n. We may assume that $e_1 \notin B$. Denote by V our vector space. Consider the projection $\pi : V \to V/e_1$. By inductive assumption there is $A' \subset \{e_2, \dots, e_n\}$ such that $\pi B \cap \operatorname{Lin} \pi A' = \{0\}$. Then $A := A' \cup \{e_1\}$ is the set we need. Indeed, if

$$v = v' + ke_1 \in B \cap \operatorname{Lin} A \quad \text{where} \quad v' \in \operatorname{Lin}\{e_2, \dots, e_n\}, \quad \text{then} \quad \pi v' \in \pi B \cap \operatorname{Lin} \pi A'.$$

Hence $v' = 0$, which implies $k = 0$ and so $v = 0$.

Hint to 6b for $n = m = 2$. For each $j \in \{1, 2\}$ there exists $i \in \{1, 2\}$ such that $e_i - f_j \notin B$. (Because otherwise $(e_1 - f_j) - (e_2 - f_j) \in B$.) Fix such a pair i, j. Suppose to the contrary that each of the 8 subspaces has non-zero intersection with B. Then B contains

$$\alpha_1(e_i - f_j) + \beta_1(e_{3-i} - f_{3-j}) \quad \text{and} \quad \alpha_2(e_i - f_j) + \beta_2(e_{3-i} - e_i - f_{3-j})$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2$. Since $e_i - f_j \notin B$, we have $\beta_1\beta_2 \neq 0$. Therefore for

$$\delta := \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0 \quad \text{we have} \quad b := \delta(e_i - f_j) + \beta_1\beta_2 e_i \in B.$$

Since $e_i \notin B$, we have $\delta \neq 0$. Since $b \in B$ and $B \cap \operatorname{Lin}\{e_1, e_2\} = \{0\}$, we have $e_{3-i} - f_j \notin B$. (I.e. the first sentence of the proof is true even if the form '*for each i, j...*'.) Then analogously to $b \in B$ we obtain $b' := \delta'(e_{3-i} - f_j) + \beta_1'\beta_2' e_{3-i} \in B$ for some non-zero $\delta', \beta_1', \beta_2'$.

Hence $\delta' b - \delta b' = \delta'(\delta + \beta_1\beta_2)e_i + \delta(\delta' + \beta_1'\beta_2')e_{3-i} \in B$.

Since $\delta\delta' \neq 0$, we obtain $\delta + \beta_1\beta_2 = \delta' + \beta_1'\beta_2' = 0$. This and $B \cap \operatorname{Lin}\{e_1, e_2\} = \{0\}$ imply that $\delta + \beta_1\beta_2 = \delta' + \beta_1'\beta_2' = 0$. Hence $f_j \in B$.

Analogously $f_{3-j} \in B$. Since $\dim B \leq 2$, we have $B = \operatorname{Lin}\{f_1, f_2\}$. Then $B \cap \operatorname{Lin}\{e_1 - f_1, e_2 - f_2\} = \{0\}$.

Hint to 6b. *The case* $(m, 2)$ follows by applying the case $(2, 2)$ to the subspace $\operatorname{Lin}\{e_1, e_2, f_1, f_2\}$. ∎

Hint to 6b. *Deduction of the case* (m, n) *from the case* $(m, 2)$. Denote by V our vector space, by $V' := \operatorname{Lin}\{e_3, \dots, e_n\}$ and by $\pi : V \to V/V'$ the projection. By the case $(m, 2)$ there exists a subspace

$$X = \operatorname{Lin}\{e_i - f_j, e_{3-i} - ke_i - f_{3-j}\} \quad \text{of} \quad V/V' \quad \text{such that} \quad X \cap \pi B = \{0\}.$$

Then $\pi^{-1}X$ is the required subspace among the given 8 subspaces of V. Indeed, each element of $B \cap \pi^{-1}X$ can be represented as

$$a + b, \quad \text{where} \quad a \in \operatorname{Lin}\{e_1, e_2, f_1, \dots, f_m\} \quad \text{and} \quad b \in V'.$$

Then $\pi(a + b) = \pi a \in X \cap \pi B$. Therefore $\pi a = 0$, so $a = 0$. Since $B \cap V' = \{0\}$, we have $v = b = 0$.