Feynman checkerboard:
pointwise continuum limit

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The main result of this note is contained in second part and don't requires for understanding first. So, the first part is motivation coming from applications.

1 Introduction (by M. Skopenkov and A. Ustinov)

We start with an informal definition in the spirit of [1] and then give a precise one.

On an infinite checkerboard, a checker moves to the diagonal-neighboring squares, either upwards-right or upwards-left. To each path s of the checker, assign a vector $a(s)$ as follows. Take a clock, which ticks when the checker moves. Initially the clock arrow looks upwards. While the checker moves straightly, the arrow does not move, but each time when the checker changes the direction, the arrow is rotated through $90°$ clockwise (independently of the direction the checker turns). The final position of the arrow is the direction of the required vector $a(s)$. The length of the vector is set to be $1/2^{(y-1)/2}$, where y is the total number of moves (this is just a normalization). For instance, for the path in Figure 1 to the left, the vector $a(s) = (1/8,0)$ is directed to the right and has length $1/8$.

Denote by $a(x,y) := \sum_s a(s)$ the sum over all the checker paths from the square $(0,0)$ to the square (x,y), *starting with the upwards-right move*. For instance, $a(1,3) = (0,-1/2) + (1/2,0) = (1/2,-1/2)$; see Figure 1 to the right. The length square of the vector $a(x,y)$ is called the *probability to find an electron in the square* (x,y)*, if it was emitted from the square* $(0,0)$.

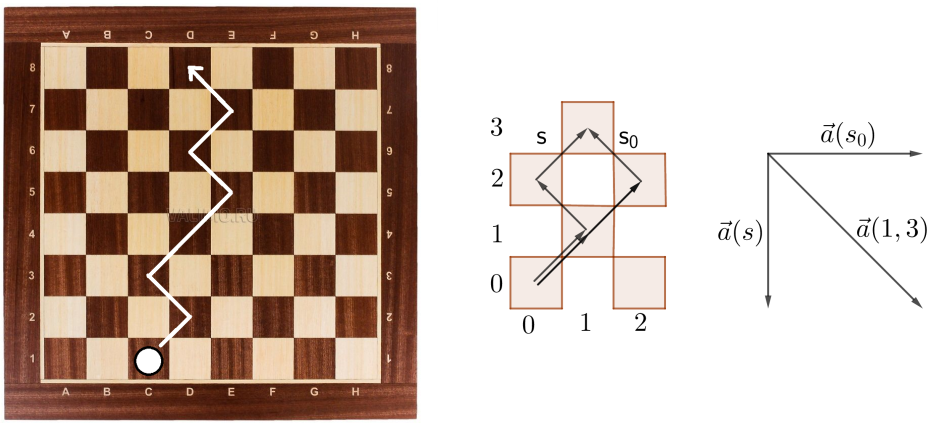

Figure 1: Checker paths and $a(x,y)$ for small x,y

Let us summarize this construction rigorously.

Definition 1. A *checker path* is a finite sequence of integer points in the plane such that the vector from each point (except the last one) to the next one equals either $(1,1)$ or $(-1,1)$. A *turn* is a point of the path (not the first and not the last one) such that the vectors to the next and to the previous points are orthogonal. Denote by

$$a(x,y) := 2^{(1-y)/2}\, i \sum_s (-i)^{t(s)}$$

the sum over all checker paths s from (0,0) to (x,y) with the first step to (1,1), where $t(s)$ is the number of turns in s. Hereafter an empty sum is 0 by definition. Denote

$$P(x,y) := |a(x,y)|^2.$$

Remark. The model is a rather rough description of electron motion. Physically, it means approximation of continuum by a lattice of huge step.

A natural generalization of this model is to determine the mass and lattice step.

Definition 2. Fix $\varepsilon,m > 0$ called *lattice step* and *particle mass* respectively. Denote by

$$a(x,y,m\varepsilon) := (1+m^2\varepsilon^2)^{(1-y)/2}\, i \sum_s (-im\varepsilon)^{t(s)}$$

the sum over all checker paths s from (0,0) to (x,y) with the first step to (1,1). Denote

$$P(x,y,m\varepsilon) := |a(x,y,m\varepsilon)|^2.$$

For any vector $a \in \mathbb{C}$ denote by a_1,a_2 the real and the imaginary part of a respectively. In particular, $a(x,y,m\varepsilon) = a_1(x,y,m\varepsilon) + i\, a_2(x,y,m\varepsilon)$.

For instance, $P(x,y,1) = P(x,y)$.

In an experiment, we measure the probabilities to find the electron in *intervals* $x_0 \le x \le x_0 + \Delta x$, $y = y_0$ rather than at particular *points*. Here $x_0,y_0,\Delta x$ are not integers but actual lengths measured in meters. If all squares have small size $1/n \times 1/n$, then the interval is approximated by the collection of squares $\left(2\left\lfloor \frac{nx}{2}\right\rfloor, 2\left\lfloor \frac{ny}{2}\right\rfloor\right)$ with x,y satisfying the above (in)equalities.

Remark. The limit in main theorem(see Section 2) coincides with the retarded fundamental solution of Dirac's equation, describing motion of an electron on a line. The normalization factor of n before "a" comes naturally from the proof, but is hard to explain a priori. A naive passing from probability to probability density suggests a factor of $\sqrt{n}$ rather than n.

2 Pointwise continuum limit (by A. Lvov)

Denote *Bessel function of the first kind*:

$$J_0(z) := \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k}}{(k!)^2}, \qquad J_1(z) := \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k+1}}{k!(k+1)!},$$

Definition. For each $y > |x|$ such that $x + y$ is even we denote

$$a_1(x,y,m\varepsilon) := (1+m^2\varepsilon^2)^{(1-y)/2} \sum_{r=0}^{\lfloor y/2 \rfloor} (-1)^r \binom{(x+y-2)/2}{r} \binom{(y-x-2)/2}{r} (m\varepsilon)^{2r+1}, \tag{1}$$

$$a_2(x,y,m\varepsilon) := (1+m^2\varepsilon^2)^{(1-y)/2} \sum_{r=1}^{\lfloor y/2 \rfloor} (-1)^r \binom{(x+y-2)/2}{r} \binom{(y-x-2)/2}{r-1} (m\varepsilon)^{2r}; \tag{2}$$

Theorem. *Pointwise continuum limit.* *For $|x| < y$ we have*

$$\lim_{n\to\infty} n\, a_1\left(2\left\lfloor \frac{nx}{2}\right\rfloor, 2\left\lfloor \frac{ny}{2}\right\rfloor, \frac{m}{n}\right) = mJ_0(m\sqrt{y^2-x^2});$$

$$\lim_{n\to\infty} n\, a_2\left(2\left\lfloor \frac{nx}{2}\right\rfloor, 2\left\lfloor \frac{ny}{2}\right\rfloor, \frac{m}{n}\right) = -m\frac{x+y}{\sqrt{y^2-x^2}} J_1(m\sqrt{y^2-x^2}).$$

Remark. Pointwise continuum limit was derived in 1972 by Narlikar[3], in 1984 by Jacobson, Schulman[2] and mathematically proven by M. Skopenkov and A. Ustinov in 2019 with technically difficult bound of rate of convergence. So, given proof is much simpler than previously known.

Remark. Let us present direct restatement of the theorem: For $|p| < q$ we have

$$\lim_{n\to\infty}(n^2+m^2)^{1/2-\lfloor nq\rfloor}\sum_{r=0}^{\lfloor nq\rfloor}(-1)^r\binom{\lfloor nq\rfloor+\lfloor np\rfloor-1}{r}\binom{\lfloor nq\rfloor-\lfloor np\rfloor-1}{r}\frac{m^{2r}}{n^{2r+1-2\lfloor nq\rfloor}}=$$
$$=\sum_{k=0}^{\infty}(-1)^k\frac{m^{2k}(q^2-p^2)^k}{(k!)^2};$$
$$\lim_{n\to\infty}(n^2+m^2)^{1/2-\lfloor nq\rfloor}\sum_{r=0}^{\lfloor nq\rfloor}(-1)^r\binom{\lfloor nq\rfloor+\lfloor np\rfloor-1}{r}\binom{\lfloor nq\rfloor-\lfloor np\rfloor-1}{r-1}\frac{m^{2r-1}}{n^{2r-2\lfloor nq\rfloor}}=$$
$$=-(p+q)\sum_{k=0}^{\infty}(-1)^k\frac{m^{2k+1}(q^2-p^2)^k}{k!(k+1)!}.$$

Denote $x_n := 2\lfloor \frac{nx}{2}\rfloor$, $y_n := 2\lfloor \frac{ny}{2}\rfloor$, $A := \frac{x_n+y_n}{2}$, $B := \frac{y_n-x_n}{2}$.

Proposition 1. $\lim\limits_{n\to\infty}(1+(\frac{m}{n})^2)^{\frac{y_n-1}{2}} = 1.$

Proof. We have $1 < (1+(\frac{m}{n})^2)^{\frac{y_n-1}{2}} < (1+(\frac{m}{n})^2)^{ny} = \sqrt[n]{(1+(\frac{m}{n})^2)^{n^2y}} \sim \sqrt[n]{e^{m^2y}} \to 1$ as $n\to\infty$. Hence by the squeeze theorem we have $\lim\limits_{n\to\infty}(1+(\frac{m}{n})^2)^{\frac{y_n-1}{2}} = 1$. □

Proposition 2. $\lim\limits_{n\to\infty}\binom{A-1}{l}\binom{B-1}{l}\frac{1}{n^{2l}} = \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l}{(l!)^2}$.

Proof. We have

$$\lim_{n\to\infty}\binom{A-1}{l}\binom{B-1}{l}\frac{1}{n^{2l}} = \lim_{n\to\infty}\frac{(A-1)\dots(A-l)\cdot(B-1)\dots(B-l)}{(l!)^2}\cdot\frac{1}{n^{2l}} = \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l}{(l!)^2}$$

because

$$\lim_{n\to\infty}\frac{A-i}{n} = \lim_{n\to\infty}\frac{A}{n} = \lim_{n\to\infty}\frac{\lfloor\frac{nx}{2}\rfloor+\lfloor\frac{ny}{2}\rfloor}{n} = \lim_{n\to\infty}\frac{\frac{nx}{2}+\frac{ny}{2}+o(n)}{n} = \frac{x+y}{2}$$

and analogously, $\lim\limits_{n\to\infty}\frac{B-i}{n} = \frac{y-x}{2}$. □

Proposition 3. $\binom{A-1}{l}\binom{B-1}{l}\frac{1}{n^{2l}} \le \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l}{(l!)^2}$

Proof.

$$(A-i)(B-i) \le (A-1)(B-1) = \left(\left\lfloor\frac{nx}{2}\right\rfloor+\left\lfloor\frac{ny}{2}\right\rfloor-1\right)\left(\left\lfloor\frac{ny}{2}\right\rfloor-\left\lfloor\frac{nx}{2}\right\rfloor-1\right) \le \left(\frac{nx}{2}+\frac{ny}{2}\right)\left(\frac{ny}{2}-\frac{nx}{2}\right) =$$
$$= n^2\left(\frac{x+y}{2}\right)\left(\frac{y-x}{2}\right).$$

So,

$$\binom{A-1}{l}\binom{B-1}{l}\frac{1}{n^{2l}} = \frac{(A-1)\dots(A-l)\cdot(B-1)\dots(B-l)}{(l!)^2}\cdot\frac{1}{n^{2l}} \le \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l}{(l!)^2},$$

□

Lemma. *Suppose $(\{a_0(n)\},\{a_1(n)\}\dots)$ is a sequence of nonnegative sequences such that $\lim\limits_{n\to\infty}a_i(n) = b_i$, $\sum\limits_{i=0}^{\infty}b_i$ is finite and $a_i(n)\le b_i$ for each i,n. Then $\lim\limits_{n\to\infty}\sum\limits_{i=0}^{\infty}a_i(n) = \sum\limits_{i=0}^{\infty}b_i$.*

Proof. Denote $b := \sum_{i=0}^{\infty} b_i$. Then for each n we have $\sum_{i=0}^{\infty} a_i(n) \le b$. Take any $\varepsilon > 0$. Take such N that $\sum_{i=0}^{N} b_i > b - \varepsilon$. For each $0 \le i \le N$ take M_i such that $\forall t \ge M_i : a_i(t) > b_i - \frac{\varepsilon}{2^{i+1}}$. Then for all $n > \max(M_0, M_1, \ldots, M_N)$ we have $\sum_{i=0}^{\infty} a_i(n) > b - 2\varepsilon$. Therefore, $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_i(n) = b$. □

Proof of Theorem. From Propositions 2, 3 and Lemma it follows that

$$\lim_{n\to\infty} \sum_{l=0;2|l}^{\infty} \binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} = \sum_{l=0;2|l}^{\infty} \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l \cdot m^{2l+1}}{(l!)^2},$$

and analogously for $2 \nmid l$.

Also for every n we have

$$\sum_{l=0}^{\infty}(-1)^l\binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} = \sum_{l=0;2|l}^{\infty} \binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} - \sum_{l=0;2\nmid l}^{\infty} \binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}},$$

because $\binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} = 0$ for $l > \max(A, B)$, so this sums are finite.

Therefore, by "Explicit" formula and Proposition 1 we have

$$\lim_{n\to\infty} n\, a_1(x_n, y_n, \frac{m}{n}) = \lim_{n\to\infty} n \cdot \sum_{l=0}^{\infty}(-1)^l\binom{A-1}{l}\binom{B-1}{l}\left(\frac{m}{n}\right)^{2l+1} =$$

$$= \lim_{n\to\infty} \sum_{l=0;2|l}^{\infty} \binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} - \lim_{n\to\infty} \sum_{l=0;2\nmid l}^{\infty} \binom{A-1}{l}\binom{B-1}{l}\frac{m^{2l+1}}{n^{2l}} =$$

$$= \sum_{l=0;2|l}^{\infty} \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l \cdot m^{2l+1}}{(l!)^2} - \sum_{l=0;2\nmid l}^{\infty} \frac{(\frac{x+y}{2})^l(\frac{y-x}{2})^l \cdot m^{2l+1}}{(l!)^2} = m \cdot J_0(m\sqrt{y^2 - x^2}).$$

It can be proven analogously that $\lim_{n\to\infty} n a_2(x_n, y_n, \frac{m}{n}) = -m\frac{x+y}{\sqrt{y^2-x^2}} \cdot J_1(m\sqrt{y^2 - x^2})$. □

References

[1] R. Feynman, QED: The strange theory of light and matter. Princeton University Press, 2006.

[2] T Jacobson and L S Schulman, Quantum stochastics: the passage from a relativistic to a non-relativistic path integral, J Physics A 17:2 (1984), 375–383.

[3] J. Narlikar, Path Amplitudes for Dirac particles, J. Indian Math. Society, 36, (1972) 9–32.