Pierre Deligne contest.

Shkredov Ilya Dmitrievich, report, 2010.

1. In 2010 I wrote several papers :

- (with S. Yekhanin) *Sets with large additive energy and symmetric sets*, accepted to Journal of Combinatorial Theory, Series A, 8 pages.
- (with A. Fish) *A note on non–trivial solutions of Vieta system of equations within any normal set*, Combinatorics, Probability and Computing, 4 pages.
- (with T. Schoen) *Additive properties of multiplicative subgroups of F_p*, submitted for publication, Quartarely Journal of Math., 10 pages.
- (with I. V'ugin) *On additive shifts of multiplicative subgroups*, Mat. Zametki, 10 pages.

2. About our results.

In the *first paper* the set of large values of convolutions that is the set $\mathrm{Sym}_\sigma(A, B) := \{x \ : \ \sum_y A(y)B(x-y) \ge \sigma\}$, A, B are subsets of a group is considered. These sets are dual to so–called sets of large exponential sums. We prove that the set $\mathrm{Sym}(A, B)$ cannot contain a huge set Λ having no any short linear relation (so–called dissociated set i.e. sets having the property : any equation $\sum_j \varepsilon_j \lambda_j = 0$, $\varepsilon_j \in \{0, \pm 1\}$, $\lambda_j \in \Lambda$ implies $\varepsilon_j = 0$). The results of the type were obtained in abealian groups using Fourier method by the applicant and T. Sanders. Unfortunately, the approach cannot give optimal bounds. Our new estimates on the size of Λ are best possible (up to logarithmic factors) and were proved by a new graph–theoretic technique.

In the *next* paper we continue a work of A. Fish on linear equations over normal sets and solve a problem posed by Imre Leader. Recall that an infinite binary sequence $\omega = \{0, 1\}^{\mathbb{N}}$ (or, equivalently, a subset of $\mathbb{N}$) is called normal if every infinite binary word w has frequency $1/2^{|w|}$ within ω, where $|w|$ is the length of w. We prove that for any normal set $A \subseteq \mathbb{N}$ there are infinitely many non–trivial solutions for Vieta system of equations : $x, y \in \mathbb{N} \setminus \{1\}$ such that $\{x + y, xy \in A\}$.

Let R be a multiplicative subgroup of $\mathbb{Z}/p\mathbb{Z}$, p be a prime number, $|R| = O(p^{2/3})$. A well–known result of D.R. Heath–Brown and S.V. Konyagin says that $|R+R| \gg |R|^{3/2}$. Certainly, the result is optimal for subgroups of size $p^{2/3}$ because of $R + R \subseteq \mathbb{Z}/p\mathbb{Z}$. Answering a question of T. Cochrein and C. Pinner we prove in the *third* paper that for any $\epsilon > 0$ and any subgroup R, $p^\epsilon < |R| < p^{2/3-\epsilon}$ the following holds $|R+R| \geq |R|^{3/2+\delta}$, where δ depends on ϵ only. We compute δ for large subgroups, and improve a result of A. Glibichuk showing that $6R$ contains $\mathbb{Z}/p\mathbb{Z}$ provided by $|R| > p^{41/83}$. The method of proving uses previous analytic technique and a new combinatorial description of sumsets due to N. Katz and P. Koester.

The last paper develops the approach from the third article. We prove, in particular, that $|R+R| \gg |R|^{5/3}$ for an arbitrary multiplicative subgroup, $|R| = O(p^{1/2})$. Also, using Stepanov's method we study intersections of R with several additive shifts, and obtain that $|R \cap (R+\mu_1) \cap \dots \cap (R+\mu_k)| \ll_k |R|^{\frac{k+1}{2k+1}}$, $\mu_i \neq \mu_j$, $i \neq j$, $\mu_j \neq 0$, provided by $1 \ll_k |R| \ll_k p^{2/3-\alpha_k}$, $\alpha_k \to 0$, $k \to \infty$.

3. In the year I took part at the conferences :

- "Uniform distribution theory" (Ströbl, Austria, June 4–11, 2010),
- "ICM 2010" (Hyderabad, India, August 19–28, 2010).

4. In our special course "Szemerédi's Theorem and Fourier analysis" we discussed some results of Additive Number Theory. We study a new subexponential bound of T. Schoen concerning so–called Polynomial Freiman–Ruzsa's Conjecture. Also we discussed several results on the structure of sumsets such as N. Alon theorem about huge sumsets in $\mathbb{Z}/p\mathbb{Z}$ and a recent result of N. Alon, A. Granville and A. Ubis about the asymptotic formula for the number of sets having the form $A+B$. Finally, we repeat in the course the simplest application of Stepanov's method : a theorem of D.R. Heath–Brown and S.V. Konyagin on sumsets of multiplicative subgroups. I read a course "Ordinary Differential Equations" for second year students at Moscow State University.

2007–2010, summary.

In 2007–2010 I wrote 12 papers which can be divided onto four groups and a survey *Fourier analysis in combinatorial number theory*, UMN, v.5, 393:3 (2010), 88–144 which is concern some applications of Fourier method to combinatorial number theory.

At the first group of articles

- *On sumsets of dissociated sets*,
- (with S.V. Konyagin) *On a result of J. Bourgain*,
- *Some applications of W. Rudins inequality to problems of combinatorial number theory*
- (with S. Yekhanin) *Sets with large additive energy and symmetric sets*,

we study sets of large exponential sums, i.e. the subsets of an abealian group **G**

$$\mathcal{R}_\epsilon(A) = \{\xi \in \widehat{\mathbf{G}} \ : \ |\sum_{x\in A} e^{2\pi i \xi \cdot x}| \ge \epsilon |A|\} \,.$$

Here $\epsilon \in (0,1]$ is a parameter and A is an arbitrary subset of **G**. These sets appear in any problem where Fourier approach is used. In our papers we study the structure of the sets (and also the structure of the dual sets in the third and the fourth article). Any knowledge about the structure potentially allows us to obtain improvements of plenty previous results in additive combinatorics. Relatively simple result on the dimension of $\mathcal{R}_\epsilon(A)$ (so–called Chang's theorem) already gave a series of useful applications. We try to obtain a full description of the sets. Unfortunately, simple hypothesis are not true (see the first and the second paper). Nevertheless, some clever modifications of our positive results and methods (see the third and the forth article) allow T. Sanders (see papers "On a theorem of Shkredov", "Structure in sets with logarithmic doubling", and also "On certain other sets of integers" at the arXiv) to obtain a principally new result for the old problem about sets having no solutions of any fixed linear equation (with restriction that number of variables tend to infinity). Thus, I think that our programme of studying the sets of large exponential sums is successful, although we still not have full description of the sets.

The question of having full description of sets of large exponential sums can be solved if so–called Polynomial Freiman–Rusza's Conjecture was true. The hypothesis has plenty applications, e.g. asymptotic formula for the number of arithmetic progressions in the primes number. Recently, N. Katz and P. Koester discover a new combinatorial property of sumsets which allows to obtain subexponential bounds in Polynomial Freiman–Rusza's Conjecture. Nevertheless, we have feeling that the method should give stronger results. The simplest problem where the technique can be applied is the question about sumsets of multiplicative subgroups. We beat the previous analytic approach here in the papers

- (with T. Schoen) *Additive properties of multiplicative subgroups of* F_p,
- (with I. V'ugin) *On additive shifts of multiplicative subgroups.*

In articles

- *On some two–dimensional configurations in dense sets*,
- *On an inverse theorem for* $\mathbf{U}^3(\Box)$*-norm*,
- *On Gowers norms of some functions*,

we try to extend Szemerèdi's theorem on arithmetic progressions. Although, some difficult result were obtained (the first article is about a hundred pages long) we realized that, unfortunately, our method is purely two–dimensional. If one can prove the existence of three–dimensional corners using our approach (that is the configuration of the form $\{(x,y,z),(x+d,y,z),(x,y+d,z),(x,y,z+d)\}$ in a sufficiently dense set A) then it means that he/she automatically obtain a variant of so–called Szemerédi's Regularity Lemma which is known gives just very bad (tower–type) bounds.

The last group of our papers

- *On monochromatic solutions of some nonlinear equations in* $\mathbf{Z}/p\mathbf{Z}$,
- (with A. Fish) *A note on non–trivial solutions of Vieta system of equations within any normal set*,

stand apart from the previous. Here we try to use different approaches and, in particular, Fourier method to solve some problems concerning coloring (e.g. Schurs result on existence of monochromatic solutions of the equation $x+y=z$, $x,y,z\in\mathbb{N}$). Roughly speaking, it was a purely combinatorial area but now T. Sanders and the applicant with co–authors suggest some new analytic approaches to obtain quantitative results in the direction. For example, we prove that for any finite coloring there are $x,y\in\mathbb{Z}/p\mathbb{Z}$ such that xy and $x+y$ have the same color and give even more stronger so–called dense result.