QUANTUM CLUSTER VARIABLES VIA VANISHING CYCLES

ALEXANDER I. EFIMOV

1. New results

In 2011, the following results were obtained.

1.1. **Cohomological Hall algebras.** First result [E1] is the proof of a conjecture of Kontsevich and Soibelman on freeness of Cohomological Hall algebra of a symmetric quiver. In the paper [KS], Kontsevich and Soibelman in particular associate to each finite quiver Q with a set of vertices I the so-called Cohomological Hall algebra $\mathcal{H}$, which is $\mathbb{Z}_{\geq 0}^I$-graded. Its graded component $\mathcal{H}_\gamma$ is defined as cohomology of Artin moduli stack of representations with dimension vector γ. The product comes from natural correspondences which parameterize extensions of representations.

In the case of symmetric quiver, one can refine the grading to $\mathbb{Z}_{\geq 0}^I \times \mathbb{Z}$, and modify the product by a sign to get a super-commutative algebra $(\mathcal{H}, \star)$ (with parity induced by $\mathbb{Z}$-grading). It is conjectured in [KS] that in this case the algebra $(\mathcal{H} \otimes \mathbb{Q}, \star)$ is free super-commutative generated by a $\mathbb{Z}_{\geq 0}^I \times \mathbb{Z}$-graded vector space of the form $V = V^{prim} \otimes \mathbb{Q}[x]$, where x is a variable of bidegree $(0,2) \in \mathbb{Z}_{\geq 0}^I \times \mathbb{Z}$, and all the spaces $\bigoplus_{k\in\mathbb{Z}} V_{\gamma,k}^{prim}$, $\gamma \in \mathbb{Z}_{\geq 0}^I$. are finite-dimensional. We prove this conjecture in [E1].

Passing to generating functions, we obtain the positivity result for quantum Donaldson-Thomas invariants, which was used by S. Mozgovoy to prove Kac's conjecture (on the number of absolutely indecomposable representations of quivers over finite fields [Kac]) for quivers with sufficiently many loops [M].

1.2. **Quantum cluster algebras.** In [E2], we provide a Hodge-theoretic interpretation of Laurent phenomenon for general skew-symmetric quantum cluster algebras, using Donaldson-Thomas theory for a quiver with potential [KS].

As an application, we show that the positivity conjecture (and actually a stronger result on Lefschetz property) holds if either initial or mutated quantum seed is acyclic. For acyclic initial seed the positivity has been already shown by F. Qin [Q] in the quantum case, and also by Nakajima [Nak] in the commutative case.

Cluster algebras were introduced in [FZ02]. They form a certain class of commutative algebras with a distinguished set of generators, which are called *cluster variables.* If the cluster algebra has rank n, then the set of generators is a union of distinguished n-element subsets called *clusters.* There is a rule of mutation of such clusters, when one cluster variable is replaced by some very simple rational function in the variables of the same cluster:

$$xx' = M_1 + M_2,$$

where x is the cluster variable, x' is its replacement, and M_1 and M_2 are monomials in the other cluster variables in the same cluster. Moreover, all clusters are obtained by such mutations from any given cluster.

The most surprising property of cluster algebras is *Laurent phenomenon:* any cluster variable is actually a Laurent polynomial in the variables of any given cluster. It leads to the well-known *positivity conjecture:* all such Laurent polynomials have non-negative integer coefficients.

In [P1], Plamondon obtains a general formula for cluster monomials for skew-symmetric cluster algebras. The same formulas are actually obtained in [DWZ2], the coincidence is shown in [P2]. The resulting coefficients are Euler characteristics of some quiver Grassmanians. However, this does not imply the positivity conjecture, since a priori Euler characteristic can be negative.

We obtain somehow related formulas for quantum cluster variables. It can be viewed as a generalization of quantum cluster character [Q] to arbitrary skew-symmetric quantum cluster algebras. In our case positivity does not follow automotically (as it does in [Q]), but it reduces to a certain conjecture on purity of monodromic mixed Hodge structures on the cohomology with the coefficients in the sheaf of vanishing cycles on the moduli of stable framed representations..

It turns that purity is very simple when either initial or the mutated quantum seed is acyclic, in this case we get just cohomology of smooth projective varieties. In general the purity is not clear, but in [E2] we conjecture that it holds.

1.3. **Cyclic homology of categories of matrix factorizations.** In [E3] (in preparation), we prove that for any smooth quasi-projective complex algebraic variety X with a regular function $W; X \to \mathbb{C},$ the periodic cyclic homology of the category $MF(X,W)$ of matrix factorizations (viewed as alocal system on the formal punctured disk) is identified (under Riemann-Hilbert correspondence) with the cohomology of the (shifted perverse) sheaf of vanishing cycles $\phi_W\mathbb{C}_X,$ with natural monodromy:

$$HP_\bullet(MF(X,W)) \cong \widehat{RH}^{-1}(H^\bullet_{an}(X^0,\phi_W\mathbb{C}_X), T\cdot(-1)^{parity}). \tag{1.1}$$

Let us describe the RHS and LHS. First, if (V,A) is a finite-dimensional $\mathbb{C}$-vector space with an automorphism $A; V \to V,$ then one can choose a logarithm $M,$

$$A = \exp(-2\pi i M),$$

and put

$$\widehat{RH}^{-1}(V,A) = (V \otimes_{\mathbb{C}} \mathbb{C}((u)), \nabla_u = \frac{d}{du} + \frac{M}{u}).$$

For any complex-analytic variety Y with holomorphic function f, the functor

$$\phi_f : D_c^b(Y) \to D_c^b(Y^0)$$

is the usual functor of vanishing cycles between derived categories of sheafs of vector spaces with constructible cohomology, preserving the perverse t-structure, and the monodromy T is actually an automorphism of the functor ϕ_W, $T : \phi_W \to \phi_W$.

Now we describe the LHS. First, locally, if X is affine, $MF(X,W)$ is a $\mathbb{Z}/2$-graded DG category of 2-periodic "complexes" of locally free sheaves (E,δ), such that

$$\delta^2 = W \cdot id.$$

Actually, if $W \neq 0$, then these are not complexes, but nevertheless the morphism δ is called a differential. Morphisms in $MF(X,W)$ are usual Hom-complexes, with differential being super-commutator with δ's. In the non-affine case, one should quotient by locally contractible matrix factorizations (analogs of acyclic complexes), see [Or], [PV], [LP], [Pos].

Periodic cyclic homology is a non-commutative analog of de Rham cohomology, see [FT]. First, it is a vector bundle (in general, of infinite rank) over the formal punctured disk. The origin of connection on HP is explained in [KKP].

The isomorphism (1.1) in particular implies a Chern character

$$ch : K_0(MF(X,W)^{\kappa}) \to H_{an}^{even}(X^0, \phi_W \mathbb{C}_X))^T,$$

which gives the usual Chern character for $W = 0$.

The image can be shown to be rational. Moreover, at least in the case when the set

$$Crit(W) \cap W^{-1}(0)$$

one can formulate the generalization of the Hodge conjecture.

2. Papers and preprints.

Papers in 2011:

Efimov, Alexander I.; Lunts, Valery A.; Orlov, Dmitri O. Deformation theory of objects in homotopy and derived categories III: Abelian categories. Adv. Math. 226 (2011), no. 5, 38573911.

A. Efimov, A proof of the Kontsevich-Soibelman conjecture, Mat. Sb. 202 (2011), no. 4, 65–84; translation in Sb. Math. 202 (2011), no. 3-4, 527546

Preprints in 2011:

A. Efimov, Cohomological Hall algebra of a symmetric quiver, arXiv:1103.2736v2, to appear in Compositio Mathematica.

Mohammed Abouzaid, Denis Auroux, Alexander I. Efimov, Ludmil Katzarkov, Dmitri Orlov, Homological mirror symmetry for punctured spheres, arXiv:1103.4322v1.

A. Efimov, Quantum cluster variables via vanishing cycles, to appear on arXiv on 16 Dec.

3. Conferences and Schools

1) Bielefeld, Conference "Workshop on Matrix Factorizations"

Talk: "Reconstruction of hypersurface singularity from its triangulated category of singularities"

2) Vienna, Conference "Memorial conference for Maximilian Kreuzer"

Talk: "Quantum cluster algebras and motivic Hall algebras"

3) Cetraro, Conference "Mirror Symmetry and Tropical Geometry"

Talk: "Cohomological Hall algebra and Kac conjecture"

4) Split, Conference "Homological Mirror Symmetry and Category Theory"

Talk: "Quantum cluster algebras and motivic Hall algebras"

5) Tianjin, Conference "Geometry and Quantization"

Talk: "Quantum cluster variables via vanishing cycles"

6) Kashiwa, Conference "Curves and categories in geometry and physics"

Talk: "Quantum cluster variables via vanishing cycles"

References

[DWZ2] Quivers with potentials and their representations II: Applications to cluster algebras, J. Amer. Math. Soc. 23 (2010), 749-790.

[FZ02] S. Fomin, A. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497-529 (electronic).

[FT] B.L. Feigin, B.L. Tsygan Additive K-theory. K-theory, arithmetic and geometry (Moscow, 1984-1986), 67-209, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[Kac] V.G. Kac, Root systems, representations of quivers and invariant theory, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 74-108.

[KKP] l. Katzarkov, M. Kontsevich, T. Pantev, Hodge theoretic aspects of mirror symmetry. From Hodge theory to integrability and TQFT tt*-geometry, 87-174, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

[KS] M. Kontsevich, Y. Soibelman, Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, arXiv:1006.2706 (preprint).

[LP] K. Lin, D. Pomerleano, Global matrix factorizations, arXiv:1101.5847 (preprint).

[M] S. Mozgovoy, Motivic Donaldson-Thomas invariants and Kac conjecture, arXiv:1103.2100 (preprint).

[Nak] H. Nakajima, Quiver varieties and cluster algebras, Kyoto J. Math. 51 (2011), no. 1, 71-126.

[Or] D. Orlov, Matrix factorizations for nonaffine LG-models, arXiv:1101.4051v2 (preprint).

[P1] P.-G. Plamondon, Cluster characters for cluster categories with infinite-dimensional morphism spaces, Adv. Math. 227 (2011), no. 1, 1-39.

[P2] P.-G. Plamondon, Cluster algebras via cluster categories with infinite-dimensional morphism spaces, Compositio Mathematica November 2011 147 : pp 1921-1954.

[PV] A. Polishchuk, A. Vaintrob, Matrix factorizations and singularity categories for stacks, arXiv:1011.4544v1 (preprint).

[Pos] L. Positselski, Coherent analogues of matrix factorizations and relative singularity categories, arXiv:1102.0261v6 (preprint).

[Q] F. Qin, Quantum Cluster Variables via Serre Polynomials, arXiv:1004.4171v2.

[E1] A. Efimov, Cohomological Hall algebra of a symmetric quiver, arXiv:1103.2736v2 (preprint), to appear in Comp. Math.

[E2] A. Efimov, Quantum cluster variables via vanishing cycles, to appear on arXiv on 16 Dec.

[E3] A. Efimov, Cyclic homology of categories of matrix factorizations, in preparation.

Steklov Mathematical Institute of RAS, Gubkin str. 8, GSP-1, Moscow 119991, Russia
E-mail address: efimov@mccme.ru