REPORT

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1. New results

In 2012, the following results were obtained.

1.1. **Cohomological Hall algebras.** First result [E1] is the proof of homotopy finiteness of derived categories of coherent sheaves (conjectured by Kontsevich) and coherent matrix factorizations.

Namely, a $\mathbb{Z}$-graded DG algebra A over a field k is called homotopically finitely presented if it is a homotopy retract of a finite cellular DG algebra B. By definition, a finite cellular DG algebra is a DG algebra B which as a graded algebra is isomorphic to a free finitely generated algebra with homogeneous generators $x_1, \dots, x_n$ with the assumption

$$dx_i \in k\langle x_1, \dots, x_{i-1}\rangle, \quad 1 \leq i \leq n.$$

Here we say that A is a homotopy retract of B if it is a retract in the homotopy category of DG algebras (obtained by inverting quasi-isomorphisms): there exist morphisms

$$f : A \to B,\ g : B \to A \text{ in } \mathrm{Ho}(\mathrm{dgcat}_k),$$

such that

$$g \circ f = \mathrm{id}_A .$$

Another way to define homotopically finitely presented DG algebras is to say that they are precisely homotopically compact objects in the model category of DG algebras (with weak equivalences being quasi-isomorphisms).

Homotopy finiteness for (essentially) small DG categories is defined in the similar way. Again, these are homotopically compact objects in the model category of small DG categories, with weak equivalences being Morita equivalences.

We show that for any separated scheme of finite type X (e.g. quasi-projective) over a field of characteristic zero k (e.g. complex numbers), the DG category $D^b_{coh}(X)$ (we mean any DG enhancement, e.g. by complexes of injectives bounded from below) is homotopically finitely presented. Characteristic zero is important here, since our proof uses Kuznetsov-Lunts categorical resolutions of singularities [KL], which is based on Hironaka theorem.

We have an analogous result for $\mathbb{Z}/2$-graded DG categories coherent matrix factorizations (see [Pos]), associated with any regular function W on X. Actually, the proof is a generalization of the proof for coherent sheaves. In particular, we generalize the categorical resolution of singularities in this setting.

It is expected that these results should be valid over any perfect field (not necessarily characteristic zero), although we don't have a resolution of singularities in finite characteristic

1.2. **Homotopically finite associative algebras.** In [E2], we show that an associative algebra A over any commutative ring k (considered as a DG algebra) is homotopically finite if and only if it is homologically smooth and finitely presented (finite number of generators and relations). This result can be generalized for a large class of operads. It can be considered (as well as the previous result) as a special case of the following principle: a DG algebra should be homotopically finite if it is homologically smooth, and is "defined by a finite amount of data".

By definition, an h-flat DG algebra B over k is said to be homologically smooth if the diagonal bimodule A is perfect over $A \otimes_k A^{op}$. For non-flat DG algebras one can take any h-flat quasi-isomorphic replacement, and then apply the same definition.

One can show that homotopically finite DG algebras are homologically smooth but not vice versa. For example, the field of rational functions in one variable $\mathbb{C}(t)$ is smooth but not homotopically finite over $\mathbb{C}$. The requirement of finite representation for an algebra is necessary for homotopy finiteness, and in [E2] we show that it is actually sufficient, if we require smoothness.

One can also consider smoothness as a condition that the bimodule of differentials $\Omega_A := \ker(A \otimes_k A \to A)$ is perfect. Note that the bimodule Ω_A governs infinitesimal deformation theory. It has an analogue, called the cotangent complex for any sufficiently nice operad. For such operads, the notion of homotopy finiteness makes sense, and smoothness is defined by the requirement that the cotangent complex is perfect. We show the same results for algebras over such operads.

By saying "sufficiently nice", we mean that the category of DG algebras over such operad admits a model structure with weak equivalences being quasi-isomorphisms and fibrations being surjective morphisms.

We believe that our description of homotopicallu finite associative algebras will allow to prove coherence for such algebras, which is known to be a very difficult problem.

1.3. **Derived categories of Grassmannians over integers.** In [E3] (in preparation), we describe the derived categories of Grassmannians over integers. Namely, we prove that they have a natural semi-orthogonal decomposition into derived categories of polynomial representations of $GL_k(\mathbb{Z})$, with suitable assumptions on weights with respect to the maximal torus. This in particular gives an exceptional collection, which by an extension of scalars to any field of characteristic zero gives Kapranov's collection [Ka]. We also show that such description naturally gives a Koszul duality functor for strict polynomial functors of Friedlander-Suslin [FS].

For any $0 \leq k \leq n$, and $0 \leq d \leq k(n-k)$, define the triangulated category $\mathcal{T}_{k,n;d}$ to be the derived category of polynomial representations U of $GL_k(\mathbb{Z})$ of degree d, such that all the weights of U with respect to the maximal torus of diagonal matrices have the form $(d_1, \dots, d_k)$ with $0 \leq d_i \leq n-k$.

There is a natural functor $\Phi_d : \mathcal{T}_{k,n;d} \to \mathrm{Gr}(k,n)$, which is obtained by "applying" a representation to the tautological bundle. We shoiw that these functors are fully faithful and provide a semi-orthogonal decomposition

$$D^b(\mathrm{Gr}(k,n)) = \langle \mathcal{T}_{k,n;0}, \mathcal{T}_{k,n;1}, \dots, \mathcal{T}_{k,n;k(n-k)} \rangle.$$

We show that the categories $\mathcal{T}_{k,n;d}$ are generated by exceptional collections consisting of appropriate Schur functors applied to the tautological representations. This gives an exceptional collection on $\mathrm{Gr}(k,n)$. Over a field of cahracteristic zero it is just Kapranov's collection [Ka]. Our collection is not strong and higher Exts are torsion abelian groups, which in particular implies that the Kapranov's collection is strong (which is of course well known).

We also show that if we consider left dual semi-orthogonal decomposition, it would be

$$D^b(\mathrm{Gr}(k,n)) = \langle \mathcal{T}_{n-,n;k(n-k)}, \mathcal{T}_{n-k,n;k(n-k)-1}, \dots, \mathcal{T}_{n-k,n;0} \rangle,$$

and the natural equivalence is provided by the Koszul duality [Kr] for strict polynomial functors of Friedlander-Suslin [FS].

We expect that our results for Grassmannians should generalize to moduli of stable quiver representations.

2. Papers and preprints.

Papers in 2012:

Efimov, Alexander I. Homological mirror symmetry for curves of higher genus. Advances in Mathematics vol. 230 issue 2 June 1, 2012. p. 493-530.

Efimov, Alexander I. Cohomological Hall algebra of a symmetric quiver. Efimov, Alexander I. Compositio Mathematica vol. 148 issue 4 July 2012. p. 1133-1146

Preprints in 2012:

A. Efimov, Cyclic homology of categories of matrix factorizations, arXiv:1212.2859.

3. Conferences and Schools

1) "Conference on Homological Mirror Symmetry", January 2012, Miami, USA Talks: Quantum cluster variables via Donaldson-Thomas theory, I, II

2) "Conference dedicated to F. Bogomolov's 65th birthday", January 2012, Miami, USA Talk: "Cyclic homology of matrix factroizations"

3) TQFT, Langlands and Mirror Symmetry, 2012, Huatulco, Mexico Talk: "Cyclic homology of categories of matrix factorizations"

4) "Birational Geometry and Derived Categories", August 2012, Vienna, Austria Talk: "Homotopy finiteness for derived categories of coherent sheaves and categories of matrix factorizations"

5) "Homological Projective Duality and Noncommutative Geometry", October 2012, Warwick, UK Lecture course: "Homotopy finiteness of DG categories from algebraic geometry"

6) "Homological Projective Duality and Quantum Gauge Theory", November 2012, Kashiwa, Japan Talk: "Derived categories of Grassmannians over integers and modular representation theory"

References

[FS] E. M. Friedlander and A. Suslin, Cohomology of finite group schemes over a field, Invent. Math. 127 (1997), no. 2, 209270.

[Ka] M. Kapranov, The derived category of coherent sheaves on Grassmann varieties, Functional Analysis and Applications 17 (1983), 145-146.

[KL] A. Kuznetsov, V. Lunts, Categorical resolutions of irrational singularities, in preparation.

[Kr] H. Krause, Koszul, Ringel, and Serre duality for strict polynomial functors, arXiv:1203.0311.

[Pos] L. Positselski, Coherent analogues of matrix factorizations and relative singularity categories, arXiv:1102.0261v6 (preprint).

[E1] A. Efimov, Homotopy finiteness of DG categories from algebraic geometry, in preparation.

[E2] A. Efimov, Homotopy finiteness of algebras over operads, in preparation.

[E3] A. Efimov, Derived categories of Grassmannians over integers and modular representation theory, in preparation.

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