REPORT

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1. New results

In 2013, the following results were obtained.

1.1. **MacLane (co)homology of the second kind and Wieferich primes.** We show that for any number field K and an element $w \in \mathcal{O}_k$, there is a direct connection between Maclane (co)homology of the second kind of $\mathcal{O}_K$ with curvature w and the "critical" points of w in $\operatorname{Spec} \mathcal{O}_K$. Our basic reference for MacLane (co)homology is [L], chapter 13.

MacLane cohomology of an associative ring R with coefficients in $R-R$-bimodule M is defined as Hoschschild cohomology of the cubical Maclane construction of R:

$$HML^{\bullet}(R,M) := HH^{\bullet}_{\mathbb{Z}}(Q(R),M).$$

Here, for any abelian group A, $Q(A)$ is a functorial nonnegative chain complex of abelian groups which computes stable homology of Eilenberg-MacLane spaces:

$$H_n(Q(A)) \cong H_{n+k}(K(A,k)), \quad k > n.$$

In particular, $H_0(Q(A)) = A$, $H_1(Q(A)) = 0$, and $H_{\geq 2}(Q(A))$ is torsion for any abelian group A. Also, we have that $H_{>0}(A) = 0$ if A is a $\mathbb{Q}$-vector space.

The functor $Q(-)$ is (nonunital) monoidal (but not symmetric monoidal!), so that we have natural maps of complexes

$$Q(A) \otimes Q(B) \to Q(A \otimes B),$$

with natural associativity isomorphisms, satisfying cocycle condition. This makes $Q(R)$ into a DG ring whenever R is a ring.

Similarly, Maclane homology is defined as Hochschild homology:

$$HML_{\bullet}(R,M) := HH_{\bullet}(Q(R),M).$$

For simplicity, we will restrict ourselves to Maclane *cohomology.* We write $HML(R)$ instead of $HML(R,R)$.

The classical computations shows that for a finite field $\mathbb{F}_q$ we have

$$HML^{2n}(\mathbb{F}_q) = \mathbb{F}_q,\, n \geq 0, \quad HML^{odd}(\mathbb{F}_q) = 0.$$

For the ring of integers, one has

$$HML^n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0; \\ \mathbb{Z}/l\mathbb{Z} & \text{for } n = 2l > 0; \\ 0 & \text{otherwise.} \end{cases}$$

We would like to illustrate our results for the ring $\mathbb{Z}$. Using the notion of Hochschild (co)homology of the second kind, introduced by A. Polishchuk and L. Positselski [PP], one can define MacLane (co)homology of the second kind. Namely, let $w \in R$ be a central element. Then define R_w to be a $\mathbb{Z}/2$-graded curved DG ring R with curvature w. We define the Maclane cohomology of the second kind by the formula

$$HML^{\bullet,II}(R_w) := HH^{\bullet,II}(Q(R)_{[w]}, R),$$

where $[w] \in Q(R)_0$ is the natural cycle associated to w (by definition, the $\mathbb{Z}$-basis of $Q(R)_0$ is formed by non-zero elements of R). Our main result for the ring of integers states that for any $w \in \mathbb{Z}$ we have

$$HML^{\bullet,II}(\mathbb{Z}_w) = \mathbb{Z}[\{p^{-1}\}_{p\notin S}] \oplus \bigoplus_{p\in S, n>0} \mathbb{Z}/p^{\nu_p(n)}\mathbb{Z},$$

where $S \subset \text{Spec}\,_m\mathbb{Z}$ is the set of those primes p for which $w^p \equiv w \bmod p^2$. If moreover p does not divide w, then such p is called a base w Wieferich prime. It is natural to consider S as the set of critical points of w on $\text{Spec}\,\mathbb{Z}$.

Open problem. *For any $w \in \mathbb{Z}$ the set of base w Wieferich primes is infinite.*

Our result (and its obvious grneralization for localizations of $\mathbb{Z}$) shows that this open problem is equivalent to the following statement.

Conjecture 1.1. *For any positive integer n, and any $w \in \mathbb{Z}$, we have that*

$$HML^{\bullet}\left(\mathbb{Z}\left[\frac{1}{n!}\right]_w\right) \ncong \mathbb{Q}.$$

Our result may be considered as an arithmetic analogue of the following geometric statement. Let A be a smooth finitely generated commutative algebra over a field k of characteristic 0 or greater than $\dim A$, and put $X := \text{Spec}\,A$. Let $w \in A$ be an element. Then we have

$$HH^{\bullet,II}(A_w) \cong H^{\bullet}(\Lambda^{\bullet}T_X, [w, -]),$$

where $[-,-]$ is the Schouten-Nijenhuis bracket. So this cohomology is a coherent sheaf on X, and its support is precisely the critical locus of w.

1.2. **DG categorical dinamics.** Another result is about DG categorical dynamics. Let $\mathcal{T}$ be a triangulated DG category generated by one object G, and $F:\mathcal{T}\to\mathcal{T}$ an endoufunctor. We consider $(\mathcal{T},F)$ as a DG-categorical discrete dynamical system. In [DHKK], the authors defined an entropy for the endofunctor F, which is a function $h(F):\mathbb{R}\to\mathbb{R}$, $t\mapsto h_t(F)$. In the case when $\mathcal{T}$ is smooth and proper over $\mathbb{C}$, one of the equivalent definitions is the following:

$$h_t(F)=\lim_{n\to\infty}\frac{1}{n}\log(\sum_{k\in\mathbb{Z}}\dim\operatorname{Ext}^k(G,F^n(G))e^{-kt}).$$

One can show that this limit exists and does not depend on the choice of generator G.

In all examples the exponent of the entropy $e^{h(F)}$ is integral element (in the sense of commutative algebra) over $\mathbb{Z}[e^{\pm t}]$. So the natural question arises: is the (exponential of) entropy always algebraic?

We consider the generating function

$$Q_{F;E_1,E_2}(q,x):=\sum_{n\geq 0}(\sum_{k\in\mathbb{Z}}\dim\operatorname{Ext}^k(E_1,F^n(E_2))q^k)x^n\in\mathbb{Z}[q^{\pm 1}][[x]].$$

Clearly, in the case when $E_1=E_2$ is a generator, after substitution $q=e^{-t}$, we get that the radius of convergence of $Q_{F;E_1,E_2}(e^{-t},x)$ is just $e^{-h_t(F)}$.

Unfortunately, in general the generating function $Q_{F;E_1,E_2}$ is not rational, and not even differentially finite on x. However, we show that its "complexity" reduces to the very special case of a category generated by exceptional collection of length 2.

Theorem 1.2. *Let $\mathcal{T}$ be smooth and proper, $F:\mathcal{T}\to\mathcal{T}$ a DG functor, $E_1,E_2\in\mathcal{T}$ two objects. Then there exists another smooth and proper DG category $\mathcal{T}'$, a DG functor $F':\mathcal{T}'\to\mathcal{T}'$ and two objects E_1', E_2' such that the following holds:*

(i) the category $\mathcal{T}'$ is generated by exceptional collection $\langle A_1,A_2\rangle$;

(ii) We have $F'(A_i)\cong W_i\otimes A_i$ for some $W_i\in\operatorname{Perf}(k)$;

(iii) the difference

$$Q_{F;E_1,E_2}(q,(1+q)x)-Q_{F';E_1',E_2'}(q,x)$$

is a rational function.

2. Comparison with the application

The following conjecture of Kontsevich was proved, as well as its generalization for coherent matrix factorizations.

Theorem 2.1. *Let Y be a separated scheme of finite type over a field k of characteristic zero. Then the DG category $D^b_{coh}(Y)$ is hfp.*

It was mentioned as a plan for future work.

On the other hand, I did not prove HMS conjecture for punctured spheres of genus $g \geq 1$, and the conjecture about existence of full strong exceptional collections on projective toric DM stacks, which were also mentioned in the plan of future work. In fact I was mostly working on other topics.

3. Papers and preprints.

Papers in 2013:

Abouzaid, Mohammed; Auroux, Denis; Efimov, Alexander I.; Katzarkov, Ludmil; Orlov, Dmitri, Homological mirror symmetry for punctured spheres. J. Amer. Math. Soc. 26 (2013), no. 4, 10511083.

Preprints in 2013:

A. Efimov, Homotopy finiteness of some DG categories from algebraic geometry, arXiv:1308.0135.

4. Conferences and Schools

1) "Conference on Homological Mirror Symmetry", January 2013, Miami, USA.

Talk: "Homotopy finiteness of DG categories from algebraic geometry"

2) "Third Latin Congress on Symmetries in Geometry and Physics", February, Sao Luis, Brazil.

Talks: "Homotopy finiteness of DG categories from algebraic geometry";
"Derived categories of Grassmannians over integers and modular representation theory"

3) "DT-invariants in Paris", June 2013, Paris, France.

Talk: "Topological Hochschild homology of the second kind and Wieferich primes"

4) "Quantum and motivic cohomology, Fano varieties and mirror symmetry", September 2013, Saint-Petersburg, Russia.

Talk: "Topological Hochschild (co)homology of the second kind and Wieferich primes"

5) "Categories and Complexity", November 2013, Vienna, Austria.

Talk: "Categories and Complexity"

References

[DHKK] G. Dimitrov, F. Haiden, L. Katzarkov, M. Kontsevich, Dynamical systems and categories, arXiv:1307.8418 (preprint).

[L] J.-L. Loday, Cyclic homology. Appendix E by María O. Ronco. Second edition. Chapter 13 by the author in collaboration with Teimuraz Pirashvili. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 301. Springer-Verlag, Berlin, 1998. xx+513 pp.

[PP] A. Polishchuk, L. Positselski, Hochschild (co)homology of the second kind I. Trans. Amer. Math. Soc. 364 (2012), no. 10, 53115368.

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