REPORT ON THE DYNASTY FOUNDATION GRANT 2014

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Description of results

We study surfaces in space $\mathbb{R}^3$ such that through each point of the surface one can draw two circles fully contained in the surface. Hereafter by a *circle* we mean either an ordinary circle in $\mathbb{R}^3$ or a straight line. We reduce finding all such surfaces to the algebraic problem of finding all Pythagorean 6-tuples of polynomials. In a subsequent publication we are going to solve the latter problem.

The problem of finding such surfaces traces back to the works of Darboux from XIXth century. Basic examples — a one-sheeted hyperboid and a nonrotational ellipsoid — are discussed in Hilbert–Cohn-Vossen's "Anschauliche Geometrie". There (and respectively, in a recent paper by Nilov and Skopenkov) it is also proved that a smooth surface containing two lines (respectively, a line and a circle) through each point is a quadric or a plane. A torus contains 4 circles through each point: a meridian, a parallel, and two Villarceau circles.

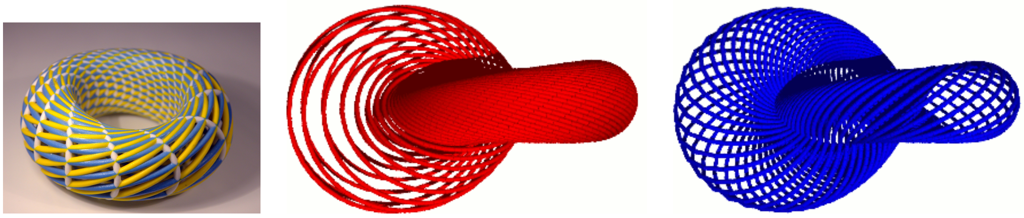

FIGURE 1. A Darboux cyclide and the stereographic projection of a Clifford translational surface.

All these examples are particular cases of a *Darboux cyclide*, i.e., a subset of $\mathbb{R}^3$ given by the equation

$$a(x^2+y^2+z^2)^2+(x^2+y^2+z^2)(bx+cy+dz)+Q(x,y,z)=0,$$

where $a,b,c,d\in\mathbb{R}$ and $Q\in\mathbb{R}[x,y,z]$ of degree ≤ 2 do not vanish simultaneously; see Figure 1 to the left. Almost each Darboux cyclide contains at least 2 circles through each point (and there is an effective algorithm to count their actual number). Conversely, Darboux has shown that the existence of 10 circles through each point guarantees that an analytic surface is a Darboux cyclide. This result has been much improved over the years by Coolodge, Ivey, Lubbes: in fact already 3, or 2 orthogonal, or 2 cospheric circles are sufficient for the same conclusion. Hereafter two circles are called *cospheric*, if they are contained in one 2-dimensional sphere or plane.

Recently there has been a renewed interest to surfaces with 2 circles through each point due to Pottmann who considered their potential applications to architecture. Pottmann noticed that the *Euclidean translational surface* $\Phi(u,v)=\alpha(u)+\beta(v)$, where $\alpha(u),\beta(v)$ are circles in $\mathbb{R}^3$, contains 2 circles through each point but is not a Darboux cyclide (for generic $\alpha(u),\beta(v)$). Another example with similar properties was given by Zubė in 2011: the stereographic projection of a *Clifford translational surface* $\Phi(u,v)=\alpha(u)\cdot\beta(v)$ in S^3, where $\alpha(u),\beta(v)$ are now circles in the sphere S^3 identified with the set of unit quaternions. The latter surfaces may have degree up to 8; see Figure 1 to the right. Euclidean translational surfaces are limiting cases of the images of Clifford translational surfaces under the stereographic projection followed by *Moebius transformations*, i.e., compositions of inversions in $\mathbb{R}^3$.

We conjecture that the above ones are all surfaces containing 2 circles through each point. Let us make this statement precise. We switch to a local problem involving a piece of the surface instead of a closed one and circular arcs instead of circles. By an *analytic surface* in $\mathbb{R}^n$ we mean the image of an injective real analytic map of a planar domain into $\mathbb{R}^n$ with nondegenerate differential at each point. We often use the same notation for the real analytic map and the surface; no confusion arises from this. A circle (or circular arc) *analytically depending* on a point is a real analytic map of a planar domain into the variety of all circles (or circular arcs) in $\mathbb{R}^n$.

Main Conjecture 1. *If through each point of an analytic surface in $\mathbb{R}^3$ one can draw two transversal circular arcs fully contained in the surface (and analytically depending on the point) then the surface is a Möbius transformation of a subset of either a Darboux cyclide, or a Euclidean translational surface, or the stereographic projection of a Clifford translational surface.*

We hope to deduce Main Conjecture 1 from its 4-dimensional counterpart. The 4-dimensional problem seems to be more accessible than the 3-dimensional one because of nice approach using quaternions. In what follows identify $\mathbb{R}^4$ with the skew field $\mathbb{H}$ of quaternions, and $\mathbb{R}^3$ with the set Im$\mathbb{H}$ of purely imaginary quaternions. Möbius transformations in $\mathbb{R}^4$ are precisely nondegenerate maps of the form $q \mapsto (aq+b)(cq+d)^{-1}$ and $q \mapsto (a\overline{q}+b)(c\overline{q}+d)^{-1}$, where $a,b,c,d \in \mathbb{H}$. Circles in $\mathbb{R}^4$ are precisely the curves having a parametrization of the form $\alpha(u) = (au+b)(cu+d)^{-1}$, where $a,b,c,d \in \mathbb{H}$ are fixed and $u \in \mathbb{R}$ runs. Denote by $\mathbb{H}_{mn} \subset \mathbb{H}[u,v]$ the set of polynomials with quaternionic coefficients of degree at most m in the variable u and at most n in the variable v (the variables commute with each other and the coefficients). Denote $\mathbb{H}_{m*} := \bigcup_{n=1}^{\infty} \mathbb{H}_{mn}$. Define $\mathbb{H}_{n*}$, $\mathbb{H}_{**}$, $\mathbb{C}_{mn}$, and $\mathbb{R}_{mn}$ analogously. For each $P \in \mathbb{H}_{mn}$ and real numbers $\hat{u}, \hat{v}$ (but not quaternions) the value $P(\hat{u},\hat{v})$ is well-defined. Let us state the 4-dimensional counterpart of Main Conjecture 1.

Conjecture 2. *Assume that through each point of an analytic surface in $\mathbb{R}^4$ one can draw two noncospheric circular arcs fully contained in the surface (and analytically depending on the point). Assume that for some point of the surface the number of circular arcs passing through the point and fully contained in the surface is finite. Then the surface is a Möbius transformation of either*

1. *a generalized cyclide $\Phi(u,v) = A(u,v)B(u,v)^{-1}$; or*
2. *a generalized translational surface $\Phi(u,v) = C(v)^{-1}D(u,v)E(u)^{-1}$;*

for some $A, B, D \in \mathbb{H}_{11}$, $C \in \mathbb{H}_{01}$, $E \in \mathbb{H}_{10}$.

Conversely, almost each surface (1) and (2) contains two circles $u = \text{const}$ and $v = \text{const}$ through each point. The first result of this paper is the following assertions reducing the 3-dimensional problem to the 4-dimensional one.

Theorem 3. *If the surface $\Phi(u,v) = A(u,v)B(u,v)^{-1}$, where $A, B \in \mathbb{H}_{11}$, is contained in $\mathbb{R}^3$ (respectively, in S^3) then Φ is a subset of a Darboux cyclide (respectively, an intersection of S^3 with another 3-dimensional quadric).*

Theorem 4. *If the surface $\Phi(u,v) = C(v)^{-1}D(u,v)E(u)^{-1}$, where $C \in \mathbb{H}_{01}$, $D \in \mathbb{H}_{11}$, $E \in \mathbb{H}_{10}$, is contained in $\mathbb{R}^3$ (respectively, in S^3) then it is a subset of either Euclidean (respectively, Clifford) translational surface or a Darboux cyclide (respectively, an intersection of S^3 with another 3-dimensional quadric).*

Corollary 5. *If a surface in $\mathbb{R}^3$ is a Möbius transformation of one of the surfaces (1) or (2) in Conjecture 2 then the surface is a Möbius transformation of a subset of either a Darboux cyclide, or a Euclidean translational surface, or the stereographic projection of a Clifford translational surface.*

Surfaces containing two circles through each point are particular cases of surfaces containing two conic sections or lines through each point. The latter surfaces have been classified by Brauner and Schicho. The Schicho classification is up to a week equivalence relation, thus it does not allow automatically to find all surfaces containing two circles through each point. However it provides the first step toward the solution of our problem.

Theorem 6. *Assume that through each point of a complex analytic surface in a domain in n-dimensional complex projective space one can draw two transversal conic sections intersecting each other only at this point (and analytically depending on the point) such that their intersections with the domain are contained in the surface. Assume that through some point of the surface one can draw only finitely many conic sections such that their intersections with the domain are contained in the surface. Then the surface (besides a one-dimensional subset) has a parametrization*

$$\Phi(u,v) = X_0(u,v) : \dots : X_n(u,v)$$

for some $X_0, \dots, X_n \in \mathbb{C}_{22}$ such that the conic sections are the curves $u = \text{const}$ and $v = \text{const}$.

Corollary 7. *Assume that through each point of an analytic surface in S^{n-1} (respectively, in $\mathbb{R}^n$) one can draw two noncospheric circular arcs fully contained in the surface (and analytically depending on the point). Assume that through some point of the surface one can draw only finitely many circular arcs fully contained in the surface. Then the surface (besides a one-dimensional subset) has a parametrization*

$$\Phi(u,v) = X_0(u,v) : \dots : X_n(u,v), \tag{1}$$

where $X_0, \dots, X_n \in \mathbb{R}_{22}$ satisfy the equation

$$X_1^2 + \dots + X_n^2 = X_0^2 \tag{2}$$

(respectively, the equation $X_1^2 + \dots + X_n^2 = X_0 Y$ for some $Y \in \mathbb{R}_{22}$).

Summary of the 3-year grant period

To summarize, during the 3-year grant period all the planned results except Corollary 7 on parametrization of bicircular surfaces in 3-dimensional space have been either published or submitted for publication. The later theorem has been written up and is prepared for publication. Also several unplanned results, e.g., Theorems 3–6 has been prepared for publication.

Papers

[1] A. Bobenko, M. Skopenkov, Discrete Riemann surfaces: linear discretization and its convergence, J. für die reine und angewandte Mathematik, Published Online (2014).
Available at:http://arxiv.org/abs/1210.0561

We develop linear discretization of complex analysis, originally introduced by R. Isaacs, J. Ferrand, R. Duffin, and C. Mercat. We prove convergence of discrete period matrices and discrete Abelian integrals to their continuous counterparts. We also prove a discrete counterpart of the Riemann–Roch theorem. The proofs use energy estimates inspired by electrical networks.

[2] D. Crowley, S. Ferry, M. Skopenkov, The rational classification of links in codimension > 2, Forum Math. 26:1 (2014), 239–269;

Let m and $p_1, \ldots, p_r < m-2$ be positive integers. The set of links of codimension > 2, $E^m(\sqcup_{k=1}^r S^{p_k})$, is the set of smooth isotopy classes of smooth embeddings $\sqcup_{k=1}^r S^{p_k} \to S^m$. Haefliger showed that $E^m(\sqcup_{k=1}^r S^{p_k})$ is a finitely generated abelian group with respect to embedded connected summation and computed its rank in the case of knots, i.e. $r = 1$. For $r > 1$ and for restrictions on $p_1, \ldots, p_r$ the rank of this group can be computed using results of Haefliger or Nezhinsky. Our main result determines the rank of the group $E^m(\sqcup_{k=1}^r S^{p_k})$ in general. In particular we determine precisely when $E^m(\sqcup_{k=1}^r S^{p_k})$ is finite. We also accomplish these tasks for framed links. Our proofs are based on the Haefliger exact sequence for groups of links and the theory of Lie algebras.

[3] A. Pakharev, M. Skopenkov, A. Ustinov, Through the resisting net, Mat. Prosv. 3rd ser. 18 (2014), 33–65;

This is a popular science paper devoted to an elementary proof of the following beautiful folklore result:
Theorem. (a) A man which is randomly walking in a 2-dimensional square lattice will hit the right neighbor of the initial point before returning to the initial point with probability 1/2.
(b) The resistance between neighboring nodes of an infinite 2-dimensional square lattice of unit resistances equals 1/2.

Parts (a) and (b) turn out to be equivalent to each other. The approach to the proof is based on a physical interpretation.

[4] M. Skopenkov, When the set of embeddings is finite?, submitted to Intern. J. Math (2013).
http://arxiv.org/abs/1106.1878

Given a manifold N and a number m, we study the following question: *is the set of isotopy classes of embeddings* $N \to S^m$ *finite*? In case when the manifold N is a sphere the answer was given by A. Haefliger in 1966. In case when the manifold N is a disjoint union of spheres the answer was given by D. Crowley, S. Ferry and the author in 2011.

We consider the next natural case when N is a product of two spheres. In the following theorem, $FCS(i,j) \subset \mathbb{Z}^2$ is a concrete set depending only on the parity of i and j which is defined in the paper.

Theorem. Assume that $m > 2p + q + 2$ and $m < p + 3q/2 + 2$. Then the set of isotopy classes of smooth embeddings $S^p \times S^q \to S^m$ is infinite if and only if either $q + 1$ or $p + q + 1$ is divisible by 4, or there exists a point (x, y) in the set $FCS(m - p - q, m - q)$ such that $(m - p - q - 2)x + (m - q - 2)y = m - 3$.

Our approach is based on a group structure on the set of embeddings and a new exact sequence, which in some sense reduces the classification of embeddings $S^p \times S^q \to S^m$ to the classification of embeddings $S^{p+q} \sqcup S^q \to S^m$ and $D^p \times S^q \to S^m$. The latter classification problems are reduced to homotopy ones, which are solved rationally.

[5] A. Skopenkov, M. Skopenkov, Some short proofs of the unrealizability of hypergraphs, submitted to Arnold Math. J.

We present short elementary proofs of van Kampen–Flores and Ummel's theorems on unrealizability of certain hypergraphs in four-dimensional space. The proofs are based on reduction to *Ramsey linking theory* resuls for graphs in three-dimensional space.

In addition to [1]–[5], several talk abstracts have been published in 2014.

Scientific conferences and seminar talks

[1] International congress of mathematicians, Seoul, Korea, 12-21.08.2014
Talk "Discrete complex analysis: convergence results"

[2] International conference "Embedded graphs" Sankt-Petersburg, Russia 27-31.10.2014
Talk: Discrete complex analysis: convergence results

[3] Talks at several seminars and other conferences in Moscow and Sankt-Petersburg.
The applicant is not involved in the work of any scientific centers or international groups.

Teaching

[1] Topology-1. Independent University of Moscow, I year students, February- May 2014, 4 hours per week + distant excercise class http://dist-math.ru.

Program (short version).

1. Visual topological problems.
2. Graphs and maps on surfaces.
3. Knots and links.
4. Classification of 2-manifolds.
5. Vector fields in the plane and their homotopies.
6. Main theorem of topology.
7. Two-dimensional simplicial complexes.
8. Fundamental group.
9. Vector fields on surfaces and the Euler–Poincare theorem.
10. Homologies of 2-manifolds.

[2] Topology-2. Independent University of Moscow, II year students, September- December 2014, 4 hours per week + distant excercise class http://dist-math.ru.

Program (short version).

1. Immersions. Classification of immersions of a circle into the plane.
2. All maps $S^1 \to S^2$ are homotopic. A square and a cube are not homeomorphic.
3. Homotopy classification of maps $S^n \to S^n$. Degree of a map.
4. Submanifolds in $\mathbb{R}^n$. Vector fields. The Euler–Poincare and Hopf theorems on the existence of a vector field on a manifold.
5. Three-dimensional simplicial complexes and manifolds. Homologies of 2- and 3-manifolds.
6. The Hopf map and the Hopf invariant of maps $S^3 \to S^2$.
7. Fundamental group and coverings.

[3] Visual potential theory, Higher School of Economics, I-III year students, September-December 2014, 2 hours per week.

Program.

1. Definition of an electric network.
2. The existence and uniqueness of a potential in an electric network. Conductance.
3. Physical interpretation of dissections of a rectangle into squares. The Dehn theorem on tiling of a rectangle.
4. Alternating-current networks. The Laszkovich–Freiling–Rinne-Szekeres theorem on dissections into similar rectangles.

[4] Distant courses for mathematical olympiads winners (http://school.dist-math.ru/moodle), 2007–now. Since 2014 supported by Independent University of Moscow. Program in Russian available at the website http://school.dist-math.ru/moodle

[5] Scientific advisor of two undergraduate students, and consulting two more students in writing research papers.