Summary

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Mirror Symmetry came from physics as a so called $N = 2$ superconformal field theory. I am interested in Fano varieties — those whose some multiple of sheaf of higher holomorphic differentials embeds them to a projective space. Mirror conjectures state an existence of a so called dual Landau–Ginzburg model, that is an open smooth Calabi–Yau variety with non-trivial complex-valued function called potential. Mirror Symmetry conjectures relate symplectic properties of a Fano variety with algebro-geometric ones for its dual Landau–Ginzburg model and vice-versa, relate algebro-geometric properties of the variety with symplectic ones of the Landau–Ginzburg model.

The main Mirror Symmetry conjecture is Homological Mirror Symmetry going back to Kontsevich. It formulates algebraic-symplectic duality in terms of derived categories. Homological Mirror Symmetry conjecture is very powerful but unfortunately it is very hard to prove it for particular mirror pairs. A natural invariant of a category is its Hochschild cohomology. The isomorphism of Hochschild cohomologies, or, the same, non-commutative Hodge structures given by them, is called Mirror Symmetry conjecture of variations of Hodge structures. This conjecture and its applications to birational geometry is the main subject of the project.

The conjecture briefly can be described as follows. Given a Fano variety one can, via its Gromov–Witten invariants, associate a particular differential equation. They are computed by Beauville, Givental, Golyshev, Kuznetsov, and the applicant for all Picard rank 1 Fano threefolds (Golyshev's conjecture). The main condition on a dual object to n-dimensional Fano variety — a Laurent polynomial f in n variables called toric Landau–Ginzburg model — is that a generating series of f^i, $i \geq 0$, is a solution of a differential equation associated to the Fano variety. These objects are found by applicant for Picard rank 1 Fano threefolds, by the applicant with Ilten and Lewis for complete intersections. Some other partial results in this direction are obtained by the applicant, Ilten, Lewis, Coates, Corti, Galkin, Golyshev, and Kasprzyk.

The project has 3 main directions. The first one use Mirror Symmetry to get a new, simplified view on a classification of Fano threefolds. In the second one we study arithmetic and variational properties of threefold Landau–Ginzburg models. In the last one we study Mirror Symmetry of variations of Hodge structures. All of these project have a deep categorical and birational outcome.

1. Originally Fano varieties were defined by del Pezzo and Fano as anticanonically embedded ones. Having canonically embedded varieties we can classify them using projective geometry. For instance, all anticanonically embedded del Pezzo surfaces — Fano varieties of dimension 2 — are projections of $\mathbb{P}^2$ from up to 6 points, from $\mathbb{P}^2$ itself to cubic surface, and quadric surface whose projection is a projection of $\mathbb{P}^2$ from two points. After projection we can get a singular variety — one with canonical Gorenstein singularities (du Val singularities, ADE singularities, etc.). Nevertheless they determine smooth del Pezzo surface as they can be smoothed to them.

In the project with Cheltsov and Katzarkov we develop the similar picture for threefolds. Of course this can't be done for smooth varieties straightforward as for del Pezzo case: say, because there are non-rational Fano threefolds. However the situation becomes more flexible if we consider singular varieties. It turned out that all of them can be included in a tree with several roots like $\mathbb{P}^3$. The tree is connected by toric basic links — projections from smooth or cDV points, lines or conics.

This approach has several other outcomes. First it gives a method to construct new four-dimensional Fano varieties. Then we can hope to complete a classification of canonical Gorenstein Fano threefolds. Partial results in this direction are obtained by Mori, Mukai, Karzhemanov, Prokhorov, Jahnke–Radloff, and Cheltsov, Shramov, and the applicant. Finally we get a new view on variations of Landau–Ginzburg models, Kontsevich–Soibelman wall crossing, Simpson's theory of nonabelian Hodge structures.

2. The next project, with Doran, Lewis, and Katzarkov, is related to modular properties of periods of Landau–Ginzburg models — those series that establish Mirror Symmetry of variations of Hodge structures. Mirror Symmetry predicts duality between anticanonical sections of Fano variety and fibers of its dual Landau–Ginzburg model. For threefold case this duality is a classical Dolgachev–Nikulin duality of K3 surfaces, Calabi–Yau varieties of dimension 2. This means that the periods give modular forms, Hilbert modular forms, Borcherds forms for Picard ranks 1, 2, 3 respectively. We study modular structure of Landau–Ginzburg models and answer on a series of questions concerned with them.

3. The last project is to prove Mirror Symmetry of variations of Hodge structures in non-proved cases. In particular, by Katzarkov's approach the structure of central fibers of toric Landau–Ginzburg models gives Hodge numbers of an initial Fano variety. This fact was proven by the applicant for Picard rank 1 Fano threefolds. We are going to extend it for higher ranks and dimensions.