Frobenius endomorphisms of linear spaces

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My research interests are in ring theory, linear algebra and combinatorics. In particular, I investigate transformations on matrices over different algebraic structures, preserving some matrix properties, sets or invariants. The main goal is to obtain a complete description or characterization of such transformations. This problem is a converse to the question of invariant theory. Namely, in invariant theory it is necessary to characterize orbits and invariants of a given action. Here we reconstruct the action by knowing its invariants. These transformations are sometimes called Frobenius endomorphisms due to Frobenius, who obtained the first result in this area. Namely, in 1896, he characterized all bijective linear transformations on the space of complex matrices $M_n(\mathbb{C})$ that preserve the determinant function. In 1925 Schur, in 1949 Dieudonné, and in 1950 Dynkin continued the work in this direction. Nowdays this area is a field of intensively developing investigations, which are interesting both from theoretical and applied point of view. The detailed and self-contained information on Frobenius endomorphisms can be found in the special volumes of the journal Linear and Multilinear Algebra, volumes 33 and 48, completely devoted to the survey of results in this area.

Main results of my previous work are:

1. The solution of Kaplansky-Watkins problem posed in 1976. We obtained a characterization of non-linear surjective transformations preserving zeros of matrix polynomials. In order to solve this problem the method of elementary operator was developed. This method reduces a non-linear problem to the group of linear problems.

2. I was investigating monotone matrix transformations and developed method of matrix deformations and chain method in order to classify monotone linear and additive transformations on matrices. The following matrix partial orders were considered: Drazin star order, left and right star orders, diamond order, singular value orders. During the last year together with G. Dolinar and J. Marovt we started also to investigate monotone transformations on Banach algebras. We characterized additive monotone with respect to Drazin star order maps on the set of compact operators on Hilbert spaces.

3. For matrices over division rings and some other non-commutative rings Frobenius endomorphisms for Dieudonné determinant are characterized. Analogs of Frobenius and Dieudonne theorems for matrices over semirings are proved.

4. Negative solution of Polya problem over finite fields is given: it is proved that there is no bijective converters of permanent to determinant for the matrix space over a finite field.

The following further investigations are planned:

1. To continue investigations in the frame of Pólya problem: to extend our results to different classes of commutative rings; to find Gibson barriers for the number of ones in sign-convertible $(0,1)$-matrices and symmetric $(0,1)$-matrices, namely, to determine the maximal integer ω_n such that for any $(0,1)$ matrix A with the number of ones $v(A) < \omega_n$ it holds that A is convertible; to determine the minimal integer Ω_n such that for any $(0,1)$ matrix A with $v(A) > \Omega_n$ it holds that A is non-convertible; to characterize convertible matrices with $v(A) = \omega_n$ or $v(A) = \Omega_n$, and to prove that for any r, $\omega_n \leq r \leq \Omega_n$ there exist both a convertible matrix A and a non-convertible matrix B with $v(A) = r = v(B)$. Here convertibility means that permanent of A is equal to the determinant of a certain matrix obtained from A by changing some of its $+1$ entries into -1.

2. To develop interrelations between game theory and tropical linear algebra. In particular, to prove that a tropical polyhedral cone can be reduced to the zero vector if and only if a corresponding mean payoff game has at least one winning initial state.

3. To characterize non-linear and even non-additive monotone matrix transformations and monotone transformations on Banach algebras. In particular, to extend our results for Drazin order to left- and right-star orders and to diamond order. Then it is planned to develop new methods in order to remove the assumption of bijectivity.

4. To investigate equality cases in Marcus-Oliveira conjecture, in particular, to characterize Frobenius endomorphisms for C-determinantal range and radii.