Report of Dmitry Kaledin on the work done in 2013.

Results

Let me recall that the general goal of my research project was to try to prove a weaker version of Tate Conjecture for crystalline cohomology of algebraic varieties -- roughly speaking, one wanted to prove that for a smooth algebraic variety over a finite field of characteristic p, the pro-p completion of its algebraic K-theory can be expressed in terms of the Frobenius action on cristalline cohomology. The plan was to first generalize the conjecture to non-commutative algebraic varieties, that is, homologically smooth and proper DG algebras. The project splits naturally into several steps:

1. Find an algebraic generalization of cristalline cohomology to non-commutative associative algebras, in the spirit of TR theory known in algebraic topology.

2. Generalize it further to DG algebras and small DG categories.

3. Reprove the comparison result for K-theory and TR known in algebraic topology in the case of finite-dimensional associative algebras over a finite field.

4. Extend the comparison result from algebras to DG algebras.

Of these, part 4 was the one I was not certain about at the time of writing the proposal: I did have a sketch of an argument, but not more than that.

I now know that the sketch I had was wrong -- there was an unfixable mistake. Moreover, the conjecture itself needs a correction: to express algebraic K-theory of a smooth and proper DG algebra, one needs to restrict oneself to the canonical truncation at 0 of the cristalline version of periodic cyclic homology. What one removes in this truncation is certainly not trivial -- conversely, it is canonically dual to what remains. Thus this part of the story is significantly more complicated than I hoped. I spent some time studying it, but eventually couldn't get anywhere; so, for now, this part of the project is suspended.

Other part of the project are less problematic. Part 2 above is essentially finished and published as arxiv:1308.3743 ("Trace theories and localization"). There I develop a general formalism of "trace functors" that allows one to twist the notion of Hochschild homology of an algebra and/or DG algebra by plugging in a possibly non-additive functor from vector spaces to vector spaces with some additional structures. In good cases, the resulting homology theory has all the propeties of Hochschild homology such as Morita invariance and Keller's localization theorem. The functors needed for the cristalline cohomology are good, so that this part of the story works.

Part 1 of the project is what I started to work on next. In fact, I was trying to finish it before writing this report, but it takes slightly more time to do everything properly, and having spent three years on this stuff already, it makes no sense to rush. I still hope to finish by the end of year. The results will be two papers of 70-80 pages each, with the construction of a Hochschild-Witt" complex generalizing the H ochschild homology complex in exactly the same way as de Rham-Witt complex of Deligne and Illusie generalizes the usual de Rham complex.

In addition, I wrote a short paper where I sketched a non-commutative generalization of Beilinson's conjecture on the regulator map for varieties over Q ("Beilinson conjecture for finite-dimensional associative algebras", posted to arxiv today). It doesn't contain much to brag about, the only new result is an observation that the generalized conjecture is true for finite-dimensional algebras -- and this is almost trivial, unlike the situation is char p. Still, I like the general picture that emerges, and I think it should be fundamentally correct. Let me mention that to formulate the conjecture, I need to use an idea of K. Kato and state it not as an isomorphism of a certain map, but as an exactness of a certain triangle. The triangle involves K-theory, a version of Deligne cohomology, ad the complex dual to K-theory tensored with R. As I understand now, the correct statement in char p should be very similar and also involve a third term; for varieties over Z, the two conjectures match quite nicely.

The plans for the next year are quite clear: after finishing part 1 -- hopefull still in this month -- I am going to move to part 3 and write up the comparison result for finite-dimensional algebras. This needs two preliminary papers on category theory extending my earlier work on "derived Mackey functors"; both are semi-ready, and finishing them would not take much time. Also, I plan to return to my earlier work on non-commutative Hodge-to-de Rham degeneration, rewrite that paper and remove the unneccesary technical assumptions (technology developed in arxiv:1308.3743 allows to do it quite nicely). As for the main goal of the project, the K-theory comparison for DG algebras, I will keep it running in the background; whether some helpful new ideas come up, only the future will tell.

2. Publications

D. Kaledin, {\em Cyclotomic complexes}, Izv. RAN ser. mat. 77:5 УЕТ. НБФ., 77:5 (2013), 3-70 (in Russian). A version of arXiv:1003.2810 badly mutilated by the compulsory editing at Izvestiya (this is the Shafarevich anniversary volume, otherwise I would never publish in this journal).

D. Kaledin, {\em Trace theories and localization},
arXiv:1308.3743, submitted.

D. Kaledin, {\em Beilinson conjecture for finite-dimensional associative algebras}, to appear on arxiv on 17.12.2013, accepted to Contemp. Math.

3. Conferences

Homological Mirror Symmetry, Miami, USA, 28.01 -- 01.02

Third Latin Congress on Symmetries in Geometry and Physics,
Maranhao, Brazil, 01.02 -- 09.02

Birational Geometry and Geometric Invariant Theory, Vienna, Austria,
21.05 -- 24.05

Colloque Solstice, Paris, France, 20.06 -- 21.06

Group actions and homotopy theory, Copenhagen, Denmark, 19.08 --
23.08

Quantum and motivic cohomology, Fano varieties and mirror symmetry,
Saint Petersburg, Russia, 26.09 -- 28.09

Symplectic Algebraic Geometry, Kyoto, Japan, 30.09 -- 04.10

4. Work in "scientific centers and international teams"

I don't quite understand what this means.

5. Teaching

None. Well, I do supervise a couple of students jointly with colleagues, but I'm too bad at it to try supervising people on my own, so it shouldn't count.

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