REPORT ON THE SIMONS–IUM FELLOWSHIP 2013 AND DYNASTY FOUNDATION

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Results of the 2013 year

On solvability of linear differential systems. Consider the system

$$\frac{dy}{dz} = B(z)y, \qquad y(z) \in \mathbb{C}^n, \quad B(z) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$

with rational coefficient matrix $B(z)$, and irregular singular points $a_1, \dots, a_m$. Suppose that the formal exponents λ_i^j are pairwise distinct, satisfy the condition

$$\mathrm{Re}\,\lambda_i^j > -\frac{1}{m(n-1)},$$

and for every pair λ_i^j, λ_i^l one of the following two conditions

$$1)\mathrm{Re}\,\lambda_i^j - \mathrm{Re}\,\lambda_i^l \notin \mathbb{Q} \qquad \mathrm{Im}\,\lambda_i^j \neq \mathrm{Im}\,\lambda_i^l$$

holds. Then this system is solvable by quadratures, if and only if there exists a constant matrix $C \in \mathrm{GL}(n, \mathbb{C})$ such that the matrix $CB(z)C^{-1}$ is upper-triangular (see [1]).

On the linear independence of the system of functions. Consider the scalar Fuchsian linear diferential equation

$$u^{(n)} + b_1(z)u^{(n-1)} + \ldots + b_n(z) = 0$$

with Fuchsian singularities $a_1, \dots, a_m$, $u_1(z), \dots, u_n(z)$ — are fundamental solutions. Let suppose that all exponents β_i^j satisfy $\mathrm{Re}\beta_i^j > D$. Consider the polynomial form

$$\Phi(z) = \sum_{\alpha_1+\ldots+\alpha_n<M} C_\alpha u_1^{\alpha_1} \ldots u_n^{\alpha_n}(z).$$

If the form $\Phi(z)$ is not identity to zero then the order of zero in any nonsingular point $z_0 \notin \{a_1, \dots, a_m\}$ satisfy

$$\mathrm{ord}_{z_0}\Phi(z) \leq \frac{(C_{M+n-1}^{n-1})^2(m-2)}{2} - MmnD.$$

Let us consider the linear combination

$$\tilde{\Phi}(z) = \sum_{\alpha_1+\ldots+\alpha_n<M,\alpha_0} C_\alpha z^{t\alpha_0} u_1^{\alpha_1} \ldots u_n^{\alpha_n}(z).$$

Suppose that $t > \frac{(C_{M+n-1}^{n-1})(m-2)}{2} - MmD$ and $u_1(z), \dots, u_n(z)$ are linearly indpendent. Then the linear form $\tilde{\Phi}(z)$ is not identity to zero.

Papers

[1] (With R.R. Gontsov) Solvability of linear differential systems in the Liouvillian sense
arXiv:1312.2518 , 2013, (submited to journal).

The paper concerns the solvability by quadratures of linear differential systems, which is one of the questions of differential Galois theory. We consider systems with regular singular points as well as those with (non-resonant) irregular ones and propose some criteria of solvability for systems whose (formal) exponents are sufficiently small.

[2] On the linear independence of some system of functions
Proceedings of the young mathematician conference, 2013, p. 14-18 (an extended version will be submited to journal soon).

The question of linear independece of some system of functions has been studied. The system consists of products of the functions $x^{\alpha t}$ and powers of fundumental solutions $y_i^{\alpha_i}(x)$ of some linear differential equation. Such system is linearly independent if t is sufficiently large. The estimate of t has been obtained.

Scientific conferences and seminar talks

[1] Workshop "Formal and Analytic Solutions of Differential, Difference and Discrete Equations", Warsaw, 25.08.2013-31.08.2013,
Talk "On solvability of linear differential systems by quadratures" (joined with R.R. Gontsov).
[2] International conference dedicated to the centenary of Israel Gelfand, Moscow, 22.07.2013-15.07.2013,
Talk "Linear independence of functions and intersections of subsets of $\mathbb{Z}_p$".
[3] Conference "Geometric Days in Novosibirsk-2013", Novosibirsk, 28.08.1013-1.09.1013,
Talk "Local form of solutions of some Painlevé equations".
[4] Seminar "Analitic theory of differential equations" (D.V Anosov, V.P. Lexin), MIAN RAS
Talk "Linear independence of functions and the orders of zeros".

Teaching

[1] Calculus (lectures and seminars). Independent University of Moscow, I year students, Feburary-May 2013, 2+2 hours per week.
Program
1. Functions of several variables.
2. Implicit function theorem and its corollaries. Morse lemma.
3. Jordan measure and Lebesgue measure.
4. Measurable functions.
5. Lebesgue integral.
6. Fubini's theorem and the theorem of Radon-Nikodym.
7. Space L_2.
8. Orthogonal system of functions. Fourier series.

[2] Analitic theory of differential equations (lectures) Special course in HSE and IUM.
Program

Linear differential equations

1. Linear differential equations: monodromy, singular points. Levelt decomposition.
2. Elements of global theory.
3. Hypergeometric equation and hypergeometric functions.

Painlevé property for first order nonlinear equations.

4. Fuchs conditions.
5. Riccati equation.
6. Equations of the genius zero are equivalent to Riccati equation.
7. Some of second order nonlinear differential equations. Painlevé equations.

Theory of the local normal forms.

8. Formal normal forms. Poincaré-Dulac theorem.
9. Poincaré and Siegel domains. Theorem of Poincaré.

[3] Calculus (seminars-exercises) Higher School Economics, II year students, January-June 2013, 2 hours per week.

Program

1. Jordan measure and Lebesgue measure.
4. Measurable functions.
5. Lebesgue integral.
6. Absolutely continuous functions.
7. Space L_2.
8. Orthogonal system of functions. Fourier series.
9. Spatial integrals.

[4] PDE (seminars-exercises). Higher School Economics, II year students, January-June 2013, 2 hours per week.

1. Canonical form of a linear second order PDE.
2. The wave equation. The Dalamber formula.
3. The heat equation.
4. The Poisson equation.
5. Fourier method.
6. The wave equation in the second and third dimensional spaces.
7. Harmonic functions.