Report on the work done in 2014
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1. Results

Roughly speaking, my original research proposal came in two parts – a realistic one, and an optimistic one. Last year, I reported that the optimistic part is suspended for being too optimistic (the sketch of an argument I had was totally wrong). This year, the situation is the same but after one more iteration. Late last year, I discovered a new approach that looked very promising; by about September, I had to acknowledge that it is still not enough.

More concretely, the subject of my research was a new homological invariant of associative algebras that I call "Hochschild-Witt homology a non-commutative generalization of de Rham-Witt complex of Deligne and Illusie. The realistic part of the project was to define it for DG algebras, and then to prove that for a finite-dimensional algebra over a field, there is a comparison theorem that expresses pro-p completion of K-theory in terms of Hochschild-Witt homology. The optimistic part was to generalize this comparison to smooth and proper DG algebras, and since this includes smooth projective algebraic varieties, it would lead to magnificent applications. The problem is, generalizing things from algebras to DG algebras is never automatic, since K-theory has notoriously bad descent properties; this is where my original sketch of an argument broke down.

As things stand now, the situation looks even more intriguing, since there appears to be a third ingredient in the story.

Namely, Hochschild-Witt homology is defined first for pairs of an algebra R over a finite field, and a projective R-bimodule M; essentially this is a completed version of K_1 of the tensor algebra of M over R. Higher K-groups are not in the picture since they actually vanish (this will eventually come out of my project, but it is also known independently from the work of L. Hesselholt). Then this theory is extended first to arbitrary bimodules, then to DG algebras; the extension uses the classic machinery of Dold derived functors. The resulting theory is pretty computable, it has reasonable descent properties, but it is not immediately comparable to K-theory.

The new third ingredient is the following: right away, take a DG algebra R and a DG bimodule M over M, and compute the completed K-theory of the tensor DG algebra (let us provisionally call it "Witt K-theory"). This is something that one can compare to Hochschild-Witt homology, on one hand,

and to pro-p completion of K-theory, on the other hand.

For Hochschild-Witt homology, one can conjecture that Witt K-theory coincides with the truncation of Hochschild-Witt homology at 0. This would be presumptious, but it happens to be true for smooth and proper DG algebras that correspond to smooth and proper algebraic varieties – it turns out that Thomason localization is just strong enough to establish this fact. At present, I have no idea how to approach the conjecture in the general case.

As far as K-theory is concerned, one can prove that the pro-p completion of K-theory coincides with the Frobenius-invariant of the p-typical part of "Witt K-theory"for finite-dimensional algebras. However, at present, I do not know whether the same is true for DG algebras. The conjectural comparison theorem here would relate two things of the same nature, two different versions of K-theory. The proof for finite-dimensional algebras is essentially by induction, similar to Goodwillie Theorem on comparison between relative K-theory and cyclic homology. Perhaps if one really understands why the result holds and obtains a direct proof, one could figure out how to extend to the general case. For now, I do not know how to do it.

Let me now return to reality and describe the parts of the project where I did accomplish something tangible. Last year, I planned to finally finish writing up the definition of Hochschild-Witt homology and its basic properties (the plan was to have two papers of about 80 pages each). Well, maybe I was wrong, but the desire to do things properly took the better of me – so, instead of finishing those two papers, I decided first to write another two setting up the linear algebra. Namely, Hochschild-Witt homology groups are more than just groups – on one hand, they form a cyclic object in the sense of A. Connes, and on the other hand, they have the structure of a so-called "Mackey functor"for the infinite cyclic group Z (it is this second structure that allows one to speak of the "p-typical part"). However, both the standard theory of Mackey functors and its derived version that I developed earlier in arXiv:0812.2519 only work well for finite groups. It turns out that for arbitrary groups, there is an interesting version of this theory that I call the theory of "Mackey profunctors". This year, I developed with theory. The resulting preprint arXiv:1412.3248 contains both the underived and derived version of the theory. I work with a general finitely generated group, since it is not much different from the case of the group Z, and then also prove additional results only valid for Z.

The next goal is to combine Z-Mackey functors and cyclic objects into a single structure provisionally called "cyclotomic Mackey functors". The preprint where I do this is half-finished and should appear early next year.

Unfortunately, both arXiv:0812.2519 and arXiv:1412.3248 and over 100 pages long. To make the theory slightly more accessible, I wrote a short overview with all the constructions but no proofs; this is arXiv:1412.3584.

There were also two small things that I did this year. One is a simple direct construction of the Eilenberg-Maclane spectrum corresponding to algebraic K-theory when it exists – namely, then the algebraic K-theory spectum becomes an Eilenberg-Maclane spectrum after localization. Specifically, this is known to happen for algebras over a finite field k of characteristic p, after localization at p. The reason is, K-theory spectrum is then a module-spectrum over the K-theory spectrum of k, and the latter is Eilenberg-Maclane after localization by the standard theorem of Quillen. However, my construction is much more elementary – it only uses homological algebra, and one does not even need to know the definition of a module-spectrum. The preprint is arXiv:1412.2537.

Another preprint is arXiv:1410.7121, joint with A. Kuznetsov. Here we observe that two main steps in the categorical resolution procedure of Kuznetsov and Lunts are actually parts of a single construction, a certain very natural refinement of the usual blowup.

The plans for the next year are clear: finish the cyclotomic Mackey functors paper, and then finally finalize and publish the two papers on Hochschild-Witt homology (with all the technicalities done, this should be a breeze). I will also continue thinking about the general conjectures sketched in the beginning of this report, but at present, I do not expect to come up with anything worthwhile in the next year or two.

2. Publications

1. D. Kaledin, *Derived Mackey functors and profunctors: an overview of results*, arXiv:1412.3584.

2. D. Kaledin, *Mackey profunctors*, arXiv:1412.3248.

3. D. Kaledin, *K-theory as an Eilenberg-Maclane spectrum*, arXiv:1412.2537.

4. D. Kaledin, A. Kuznetsov, *Refined blowups*, arXiv:1410.7121.

5. V. Baranovsky, V. Ginzburg, D. Kaledin, J. Pecharich, *Quantization of line bundles on Lagrangian subvarieties*, arXiv:1403.3493.

I also have three papers in the proof stage, but since they are not published yet, I'd better report them next year.

3. Conferences

Homological Mirror Symmetry, Miami, USA, January

Langlands, Mirror Symmetry and TWFT, Playa del Carmen, Mexico, March

Non-commutative Geometry, Oberwolfach, Germany, May

Kontsevich 50, Paris, France, June

Frontiers of Rationality, Longyearbyen, Norway, July

Causian Mathematical Conference, Tbilisi/Batumi, Georgia, September

4. Work in "scientific centers and international teams"

I suppose I should mention here a visiting position at the IBS Center for Geometry and Physics, POSTECH, Pohang, Rep. of Korea, November-December 2014

5. Teaching

I'm supervising one graduate student, E. Balzin, jointly with Carlos Simpson. I'm very bad at it. Fortunately, the student is good. By now, he has one preprint out, so it's not completely hopeless.

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