REPORT ON THE DYNASTY FOUNDATION 2015

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Results of the 2015 year

Results in additive combinatorics.

We study an algebraic equation $P(x,y)=0$ over a field $\mathbb{F}_p$, where p is a prime. Let $P\in\mathbb{F}_p[x,y]$ be a polynomial of two variables x and y, G be a subgroup of $\mathbb{F}_p^*$. We study the upper bound of the number solutions of the polynomial equation, such that $x\in g_1G$, $y\in g_2G$. The estimate

$$\#\{(x,y)\mid P(x,y)=0,\ x\in g_1G,\ y\in g_2G\}\le 16mn^2(m+n)|G|^{2/3}.$$

is obtained using Stepanov method. This estimate was obtained by a different method in the paper [1]. We improve this estimate in average.

Let us consider a homogeneous polynomial $P(x,y)$ of degree n such that $\deg P(x,0)\ge 1$, $P(0,0)\ne 0$ and $l_1,\dots,l_h$ belong to different cosets g_iG. We estimate the sum N_h of numbers of solutions of the set of equations:

$$P(x,y)=l_i,\qquad i=1,\dots,h,\quad x\in g_1G_1,\ y\in g_2G.$$

Then the sum N_h does not exceed $32h^{3/4}n^5|G|^{2/3}$.

Now let us consider some generalization of the additive energy which we call *polynomial energy*. Polynomial energy is the following

$$E_P^q(A)=\#\{(x_1,y_2,x_2,y_2)\mid P(x_1,y_1)=P(x_2,y_2),\ x_1,y_1,x_2,y_2\in A\},$$

where $P(x,y)\in\mathbb{F}_p[x,y]$ is a polynomial.

Theorem *Let us suppose that $100n^3<|G|<\left(\frac{p}{3}\right)^{\frac{12}{17}}$, $P\in\mathbb{F}_p[x,y]$ is a homogeneous polynomial. Then the following holds: if $q\le 3$ then $E_P^q(G)\le C(n,q)|G|^{\frac{7q+16}{12}}$; if $q=4$ then $E_P^4\le C(n,q)|G|^{1+\frac{2q}{3}}\ln|G|$; if $q\ge 5$ then $E_P^q(G)\le C(n,q)|G|^{1+\frac{2q}{3}}$, where $C(n,q)$ depends only on n and q.*

The results of the talk can be found in the paper [2].

Results in analytic theory

Consider the system

$$Y(z+1)=A(z)Y(z) \tag{1}$$

of linear difference equations, with a rational coefficient $n\times n$-matrix function $A(z)=A_rz^r+\ldots+A_0$. Birkhoff proved that the class of meromorphic equivalence of such systems is described by a set of characteristic constants $\{\rho_i\},\{c_{kl}^{(s)}\}$. Birkhoff also studied the inverse problem (Generalized Riemann–Hilbert problem for difference equations).

We improve the Birkhoff's result of inverse problem. We prove that for any set of eigenvalues $\rho_1,\dots,\rho_n$ of the matrix A_r, such that $\rho_i/\rho_j\notin\mathbb{R}$ for $i\ne j$, there are matrices $A_0,\dots,A_{r-1}$, such that the system (**??**) with $A_r=\mathrm{diag}(\rho_1,\dots,\rho_n)$ has the given characteristic constants $\{d_k\},\{c_{kl}^{(s)}\}$. Birkhoff has proved that there are the system with constants $\{d_k+l_k\},\{c_{kl}^{(s)}\}$, where l_k are some integers.

Общий итог за 3 года.

Тематика, в которой я работал последние три года состоит из двух направлений: аналитической теории дифференциальных уравнений и аддитивной теории чисел.

В первой были достигнуты два основных результата:

1. Обобщен на случай систем с иррегулярными особыми точками критерий Ильяшенко-Хованского разрешимости в квадратурах системы линейных дифференциальных уравнений с малыми коэффициентами.

2. Усилена теорема Дж. Биркгофа о разрешимости обобщенной проблемы Римана-Гильберта для линейных разностных уравнений. Этот результат готовится к публикации. Возможно, он является лучшим за данный период.

В области аддитивной теории чисел имеются два обобщения результата,

полученного ранее мной и И.Д. Шкредовым об оценки мощности пересечения аддитивных сдвигов мультипликативной подгруппы простого конечного поля.

А также имеется новый результат о числе решений уравнения P(x,y)=0, принадлежащих заданной подгруппе F_p^*.

В целом считаю план выполненным. Получилось так, что за указанный период у меня вышло мало серьезных публикаций. Это связано в основном с периодом 2011-2012 годов, когда я болел. Сейчас, только что, вышла статья в Arnold Mathematical Journal, кроме этого поданы в печать еще 3 работы, одна из которых уже принята.

План преподавательской деятельности считаю выполненным полностью

Papers

[1] (With R.R. Gontsov) Solvability of linear differential systems in the Liouvillian sense // *Arnold mathematical journal* , Volume 1, Issue 4, P. 445-471, 2015,

The paper concerns the solvability by quadratures of linear differential systems, which is one of the questions of differential Galois theory. We consider systems with regular singular points as well as those with (non-resonant) irregular ones and propose some criteria of solvability for systems whose (formal) exponents are sufficiently small.

[2] (With I.D. Shkredov, E.V. Solodkova) Intersections of multiplicative subgroups // *Mathematical notes*, 2016, (to be appear).

The paper is devoted to some applications of Stepanov method. In the first part of the paper we obtain the estimate of the cardinality of the set, which is obtained as an intersection of additive shifts of some different subgroups of F_p^*.

[3] (With S. Makarychev) The number of solutions of polynomial equation over the field $\mathbb{F}_p$ and new bounds of additive energy // *Proceedings of the Conference Diferential Equations and Related Topics*, 2015, p. 24-26.

An algebraic equation $P(x,y)=0$ over a field $\mathbb{F}_p$, where p is a prime is studied. Let G be a subgroup of $\mathbb{F}_p^*$. We study the upper bound of the number solutions of the polynomial equation, such that $x \in g_1G$, $y \in g_2G$. The estimate

$$\#\{(x,y) \mid P(x,y)=0,\ x \in g_1G,\ y \in g_2G\} \le 16mn^2(m+n)|G|^{2/3}.$$

is obtained using Stepanov method. We improve this estimate in average. (The extended version of this paper has been submited to journal.)

Scientific conferences and seminar talks

Talks:

1. Conference Diferential Equations and Related Topics in Zaraisk, talk: "The number of solutions of polynomial equation over the field $\mathbb{F}_p$ and new bounds of additive energy June 11-12, 2015.

Seminar talks:

2. Analitic theory of differential equations in Steklov Mathematical Institute - several talks.

3. Seminar on Complex Analysis (Gonchar Seminar) in Steklov Mathematical Institute, talk: "On the generalized RiemannвЂ“Hilbert problem for linear difference equations".

4. Seminar Contemporary Problems in Number Theory, talk: "On the number of solutions of polynomial equation $P(x, y) = 0$ over $\mathbb{F}_p$, such that x and y belong to some subgroup of $\mathbb{F}_p^*$".

Teaching

[1] Analitic theory of differential equations (lectures) Special course in HSE and IUM (joint with V.A. Poberezhny).

Program

Linear differential equations: Riemann–Hilbert problem, isomonodromic deformations and Painlevé equations

1. Riemann–Hilbert problem
2. Vector bundles with connection
3. Isomonodromy deformations
4. Schlesinger equation
5. Painleveé equations

[2] Diferential equations and isomonodronic deformations (scientific seminar for students in HSE, joint with V.A. Poberezhny)

[3] Analitic theory of differential equations (scientific seminar in Steklov Mathematical Institute, joint with V.P. Lexin, R.R. Gontsov, A.V. Klimenko)

I have teached the exercises of Calculus, ODE and PDE.

I am a supervisor of 5 students. Student Victoria Malyasova has passed your graduate thesis under my supervision.