3. Summary of A. Skopenkov's proposal
Knotting of manifolds below the metastable dimension

This proposal concerns the classical Knotting Problem in topology: *classify embeddings of a given space into another given space up to isotopy.* This problems have played an outstanding role in the development of topology. Various methods for the investigation of the Knotting Problems were created by classical figures. The Knotting Problem is known to be hard. A complete answer for the general case is not to be expected.

The Knotting Problem is most interesting for manifolds of dimension at most 4 because embeddings of such manifolds often appear in other branches of mathematics and its applications.

I work in the smooth category unless PL (piecewise linear) category us explicitly mentioned.

Classical results of Wu, Haefliger, Hirsch (1960-s) on embeddings of n-dimensional manifolds into $\mathbb{R}^m$ have the 'metastable' dimension restriction

$$2m > 3n + 3.$$

In particular, in low dimensions Haefliger and Hirsch classified embeddings of 3-dimensional manifolds into $\mathbb{R}^m$ for $m \geq 7$, and of 4-dimensional manifolds into $\mathbb{R}^m$ for $m \geq 8$.

The main intention of this research proposal is to classify embeddings of closed connected n-dimensional manifolds N into $\mathbb{R}^m$ for

$$2m \leq 3n + 3.$$

For N not a homology sphere until 2005 no classification was known, in spite of the existence of interesting partial results, results in the PL category and interesting approaches of Browder-Wall and Goodwillie-Weiss.

Embeddings $S^n \to S^m$ for $m > n + 2$ were classified by Haefliger in 1960s. There is the 'connected sum' action # of the group of embeddings $S^n \to \mathbb{R}^m$ on the set of embeddings $N \to \mathbb{R}^m$ for a closed connected orientable n-manifold N.

The quotient set of this action was known for some cases including the case $m = 6 = 2n$ and the case N simply-connected, $m = 7 = 2n - 1$; there remained to find the orbits of #. For N not a homology sphere until 2005 no description of the orbits was known. A description of orbits for these cases appeared in papers by Crowley and myself in 2005- 2011. They yielded a classification of embeddings *for 3-dimensional manifolds in* $\mathbb{R}^6$ and for *simply-connected 4-dimensional manifolds in* $\mathbb{R}^7$.

Crowley and I plan to obtain a classification of embeddings of *non-simply-connected 4-manifolds in* $\mathbb{R}^7$. This is a significant new step as it is explained in the proposal. We also plan to obtain a *piecewise linear analogue* of this result. My contribution would be a description of the quotient set of action # and the 'lower estimation' in the description of the orbits of #.

I also plan to study the following *Compression Problem* for $n = 4$ and $m = 6; 7$: characterize embeddings of n-manifolds into $\mathbb{R}^{m+1}$ that are isotopic to embeddings into $\mathbb{R}^m$.

Many interesting examples of embeddings are embeddings $S^p \times S^q \to \mathbb{R}^m$. A classification of such embeddings is a natural next step (after the link theory and the classification of embeddings of highly-connected manifolds) towards classification of embeddings of arbitrary manifolds.

A classification of embeddings $S^p \times S^q \to \mathbb{R}^m$ under certain 'metastable' dimension restrictions was obtained by Haefliger-Hirsch in 1963 and by myself in 2002-2009. I plan to obtain a classification of embeddings $S^p \times S^q \to \mathbb{R}^m$ for $m \geq q + 2p + 3$ and $1 \leq p \leq q$, i.e. below the 'metastable' dimension. For this case new invariants have to be discovered and their images and preimages have to be described. The classification would be in terms of homotopy groups of spheres and embeddings $D^{p+1} \times S^q \to \mathbb{R}^m$ and $S^{p+q} \to S^m$ which are easier to describe.

I plan to study the following *Rees problem*: describe the action of self-diffeomorphisms of $S^p \times S^q$ on isotopy classes of embeddings $S^p \times S^q \to \mathbb{R}^m$. The case $S^1 \times S^3 \to \mathbb{R}^7$ is particularly interesting and difficult.