REPORT ON THE DINASTY–IUM FELLOWSHIP 2014

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Results

- Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. The representation theory of $\mathfrak{g}$ is very simple: any finite-dimensional representation is a direct sum of irreducibles and the irreducibles are numbered by highest weights. The category $\mathcal{O}$ of finitely-generated $\mathfrak{g}$-modules with locally finite action of Borel subalgebra (a maximal solvable subalgebra) is no more semisimple, however, admits a nice property called BGG property, first introduced by Bershtein–Gelfand–Gelfand. The BGG property was originally formulated as a duality for certain multiplicities for projective, Verma and irreducible modules. One can reformulate it as a homological property about existence of semi-orthogonal decomposition of the corresponding derived category. We collect different necessary and sufficient conditions to have a BGG property for abelian categories and use them to show that the category of modules over the Lie algebra of currents $\mathfrak{g} \otimes \mathbb{C}[t]$ which are graded, finitely-generated and with finite-dimensional graded components admits this property whenever the Lie algebra $\mathfrak{g}$ is semisimple.

  First, this result gives an approach to work with representations of nonsemisimple Lie algebras and explains the nature of Weyl modules introduced by V. Chari and collaborators. Second, we identify the Macdonald pairing on symmetric functions with a homological pairing on the Grothendieck ring of the latter category of modules. This leads a way to categorify Macdonald polynomials which is not yet known.
- The notion of operad was introduced in 60's by P. May while studding the topology of iterated loop spaces. May realised that the latter topological spaces are in one to one correspondence with algebras over little balls operad. Nowadays, the language of operads became very popular and is well used in algebraic topology, mathematical physics, combinatorics, algebraic geometry and other fields of mathematics. The reason is that this is the good language in order to work with universal formulas (formulas which remains to be true for any object of your category). Any particular homological computation for operads leads many application for the corresponding category of algebras over this operad. Unfortunately, very poor methods are known to produce the homological computations for operads. Let us explain two particular computations we made:
  - We introduce the notion of equivariant homology functor and use it to derive a minimal resolution of the framed little ball operad, which governs the topology of nonbased iterated loop spaces. In particular, we show that the latter operad is not formal for balls of odd dimension.
  - The graph complex is a combinatorial complex with basis given by graphs and differential given by an edge contraction. This complex was introduced by Kontsevich while working with knot invariants. Later on it was realised that homology

of this complex is very complicated and governs different mathematical question. In particular, the zero'th cohomology is a famous Grothendieck-Teichmuller Lie algebra introduced by Drinfeld, the vanishing of the first cohomology will imply the formality theorem and the full homology is a substantial part of the deformation complex of the little discs operad. We use the results on framed operad in order to get an extra differential on the graph complex whose total homology is empty. This allows to find many particular classes in the graph complex.

Papers

[1] with A. Berenstein, M. Bennet, V. Chari, S. Loktev
"Macdonald Polynomials and BGG reciprocity for current algebras"
published in Selecta Mathematica New series, April 2014, Volume 20, Issue 2, pp 585–607

[2] "Characteristic classes of flags of foliations and Lie algebra cohomology."
submitted to Transformation Groups arxiv:1303.1889v2,

[3] "Highest weight categories and Macdonald polynomials."
submitted to Advances in Mathematics arxiv:1312.7053

[4] with T. Willwacher, M. Živković
"Differentials on graph complexes I"
preprint arxiv:1411.2369
[5] "Equivariant homology and Framed little balls operad"
in preparation
[6] with S.Merkulov "Gravity operad: Formality, Deformations and Applications"
in preparation

Scientific conferences and seminar talks

[1] Visit Israel, January
Talk "Highest weight categories and orthogonal polynomials"
at *Algebraic Geometry & Number Theory seminar,* Ben Gurion University, Beer Sheba; and at *Algebra seminar,* Bar Ilan, Ramat Gan;
Talk "Around avoidance problem"
at *Combinatorics seminar,* Bar Ilan, Ramat Gan;
[2] Visit to Japan, August
Talk "Hypercommutative operad as a homotopy quotient of BV"
at *String theory Seminar,* Kavli IPMU, 2014;

Teaching

[1] Homological Algebra. Independent University of Moscow, II–III year students, September-December 2014, 2 hours per week.

Program

The goal of the course is to give the introduction to the language of derived functors and their applications. In particular, we will emphasize on the following subjects:

- Comlexes and homology;
- Basic category theory;
- Projective modules;
- Exact functors and projective resolutions;

- Derived functors;
- Functors Tor and Ext;
- Ext and extensions;
- Homological dimension;
- Spectral sequences;
- Group cohomology;
- Derived and Triangulated categories;

[2] Basic representation theory, III year students, Independent University of Moscow, Math in Moscow, Higher School of Economics, 4 hours per week.

Program.

This is an introduction to representation theory. Upon completion of this course student will know enough to understand classical applications and will be ready to take more advanced courses leading to modern problems and applications of representation theory. In more detail, the aims are to learn

- common notions and problems of representation theory;
- various instruments for dealing with finite-dimensional representations of finite groups: intertwining operators, characters, Maschke's and Burnside's theorems;
- representations of ring and algebra in a particular case of group ring and algebra.
- representations of symmetric group and related algebraic and combinatorial constructions: Young diagrams and tableau, Young symmetrizers;
- basics of Lie algebras and their representations;
- representations of $\mathsf{sl}_2(\mathbb{C})$.