A report of A. Skopenkov for the Dynasty Foundation grant, 2015

1. Research results

We work in the smooth category. For a smooth manifold N denote by $E^m(N)$ the set of smooth embeddings $N \to \mathbb{R}^m$ up to smooth isotopy.

In [2] for $p \le q$ and $m \ge 2p+q+3$ I construct a group structure on $E^m(S^p \times S^q)$, and describe this group in terms of homotopy groups of spheres and easier-to-calculate groups of embeddings. Earlier such a description was known only for $2m \ge 3p+3q+4$.

Let N be a closed connected orientable 4-manifold with torsion free integral homology. The main result of [4I, 4II, 4III] is *a complete readily calculable classification of embeddings* $N \to \mathbb{R}^7$. We describe

- the set $E^m(N)/\#$ of smooth embeddings $N \to \mathbb{R}^7$ up to smooth isotopy and connected sums with smooth embeddings $S^4 \to \mathbb{R}^7$ [4I].
- the set $E^m(N)$ up to certain indeterminancy [4II].
- the set of PL embeddings $N \to \mathbb{R}^7$ up to PL isotopy.

Such a classification was earlier known only for simply-connected N: in the PL and 'smooth modulo knots' case by Boéchat-Haefilger-Hudson 1970, in the smooth case by the authors 2008. The case $N = S^1 \times S^3$ allows us to disprove the conjecture on the completeness of the Multiple Haefliger-Wu invariant, and the Melikhov informal conjecture on the existence of a geometrically defined group structure on the set of PL isotopy classes of PL embeddings in codimension 3.

The paper [1] originated from the following problem was suggested by E. Rees in 2002: describe the action of self-diffeomorphisms of $S^p \times S^{n-p}$ on $E^m(S^p \times S^{n-p})$.

Let $g : S^p \times S^{n-p} \to \mathbb{R}^m$ be an embedding such that $g|_{a\times S^{n-p}} : a \times S^{n-p} \to \mathbb{R}^m - g(b \times S^{n-p})$ is null-homotopic for some different points $a, b \in S^p$ and $m \ge n+2+\frac{1}{2}\max\{p, n-p\}$.

Theorem. *For a map* $\varphi : S^p \to SO_{n-p}$ *define an autodiffeomorphism* φ' *of* $S^p \times D^{n-p}$ *by* $\overline{\varphi}(a,b) := (a, \varphi(a)b)$. *Let* φ'' *be the* S^{n-p-1}*-symmetric extension of* φ *to an autodiffeomorphism of* $S^p \times S^{n-p}$. *Then for each map* $\varphi : S^p \to SO_{n-p}$ *embedding* $g \circ \varphi''$ *is isotopic to embedded connected sum* $g\#u$ *for some embedding* $u : S^n \to S^m$.

Let N be an oriented n-manifold and $f : N \to \mathbb{R}^m$, $g : S^p \times S^{n-p} \to \mathbb{R}^m$ are embedding. As a corollary we obtain the following result on S^p-parametric embedded connected sum $f\#_s g$ (defined earlier by the author).

Under certain conditions for orientation-preserving embeddings $s : S^p \times D^{n-p} \to N$ *the 'smooth modulo knots' class in* $E^m(N)/\#$ *of* $f\#_s g$ *depends only on* f, g *and the isotopy (the homotopy or the homology) class of* $s|_{S^p\times 0}$.

In the paper [3] we study conditions under which a finite simplicial complex K can be mapped to $\mathbb{R}^d$ without higher-multiplicity intersections. An *almost* r*-embedding* is a map $f\colon K \to \mathbb{R}^d$ such that the images of any r pairwise disjoint simplices of K do not have a common point. We show that if r is not a prime power and $d \ge 2r$, then there is a counterexample to the topological Tverberg conjecture, i.e., *there is an almost* r*-embedding of the* $(d+1)(r-1)$*-simplex in* $\mathbb{R}^d$. This improves on previous constructions by Frick (for $d \ge 3r+1$) and by the second and the fourth author (for $d \ge 3r$).

The counterexamples are obtained by proving the following algebraic criterion in codimension 2: *If* $r \ge 3$ *and if* K *is a finite* $2(r-1)$*-complex then there exists an almost* r*-embedding* $K \to \mathbb{R}^{2r}$ *if and only if there exists a general position PL map* $f\colon K \to \mathbb{R}^{2r}$ *such that the algebraic intersection number of the* f*-images of any* r *pairwise disjoint simplices of* K *is zero.* This result can be restated in terms of cohomological obstructions or equivariant maps, and extends an analogous codimension 3 criterion by the second and fourth author. It follows from work of Freedman, Krushkal, and Teichner that the analogous criterion for $r = 2$ is false.

As another application of our methods, we classify *ornaments* $f\colon S^3 \sqcup S^3 \sqcup S^3 \to \mathbb{R}^5$ up to *ornament concordance.*

2a. Research papers

[1] A. Skopenkov, How do autodiffeomorphisms act on embeddings, accepted to Proceedings A of The Royal Society of Edinburgh. http://arxiv.org/abs/1402.1853

[2] A. Skopenkov, Classification of knotted tori, http://arxiv.org/abs/1502.04470

[3] S. Avvakumov, I. Mabillard, A. Skopenkov, U. Wagner, Eliminating Higher-Multiplicity Intersections, III. Codimension 2. Submitted to Europ. J. Math. http://arxiv.org/abs/1511.03501

[4I] D. Crowley and A. Skopenkov, Classification of embeddings of non-simply-connected 4-manifolds into R^7, I, classification modulo knots, preprint, 2015

[4II] D. Crowley and A. Skopenkov, Classification of embeddings of non-simply-connected 4-manifolds into R^7, II, classification in the smooth category, preprint, 2015

[4III] D. Crowley and A. Skopenkov, Classification of embeddings of non-simply-connected 4-manifolds into R^7, III, classification in the PL category, preprint, 2015

2b. Expository books.

[6] A. Skopenkov, Algebraic Topology from Geometric Viewpoint, Moscow, MCCME, 2015 http://www.mccme.ru/circles/oim/obstruct.pdf (a new version prepared)

[7] A. Chernov, A. Daynyak, A. Glibichuk, M. Ilyinskiy, A. Kupavskiy, A. Raigorodskiy and A. Skopenkov, Elements of Discrete Mathematics As a Sequence of Problems, Moscow, MCCME, to appear, http://www.mccme.ru/circles/oim/discrbook.pdf

[8] Mathematics via problems, editors: A. Zaslavsky, A. Skopenkov and M. Skopenkov. Moscow, MCCME, to appear (rewritten for 2nd edition).
http://www.mccme.ru/circles/oim/matprob.pdf,
http://www.mccme.ru/circles/oim/materials/sturm.pdf

[9] A. Skopenkov, Basic Differential Geometry As a Sequence of Interesting Problems, MCCME, Moscow, to appear. (rewritten for 3rd edition). http://arxiv.org/abs/0801.1568

2c. Expository papers.

[10] D. Ilyinskiy, A. Raigorodskiy and A. Skopenkov, Existence proofs in combinatorics using independence, Mat. Prosveschenie, 19 (2015), http://arxiv.org/abs/1411.3171

[11] V.V. Prasolov, A.B. Skopenkov, Some reflections on why Lobachevsky geometry was recognized, Mat. Prosveschenie, 19 (2015), http://arxiv.org/abs/1307.4902

[12] A. Skopenkov, Realizability of hypergraphs and Ramsey link theory, Submitted to Arnold Math. J. http://arxiv.org/abs/1402.0658 (rewritten in 2015)

[13] V. Bragin, Ant. Klyachko, A. Skopenkov, When Any Group of N Elements is Cyclic? Submitted to Mat. Prosveschenie. http://arxiv.org/abs/1108.5406 (a new version prepared)

[14] A. Skopenkov, A short elementary proof of the Ruffini-Abel Theorem,
http://arxiv.org/abs/1508.03317

3. Conferences

The 5th German-Russian Week of the Young Researcher 'Discrete Geometry', Dolgoprudniy, September (invited speaker). Talk "Topological Tverberg Conjecture and Eliminating Higher-Multiplicity Intersections".

Conference of winners of Simons and 'Dynasty' grants, Moscow, June (invited speaker). Talk "Classification of knotted tori".

58th Conference of Moscow Institute of Physics and Technology, Dolgoprudniy, November, 2015. Talk "Topological Tverberg Conjecture and Eliminating Higher-Multiplicity Intersections".

4. Work in scientific centers and university groups

I worked in Moscow Mathematical School. In particular, I delivered talks at

- Topology Seminar, Faculty of Mathematics, Higher School of Economics,
- Topology Seminar, Steklov Mathematical Institute,
- Postnikov memorial seminar, Moscow State University,
- Seminar of N.P. Dolbilin and N.M. Moschevitin, Moscow State University,

• Seminar on Geometry, Institute for Transmission of Information Problems,

• Seminar of Department of discrete mathematics, Moscow Institute of Physics and Technology.

I continued collaboration with D. Crowley from Univ. of Edinburgh. I started collaboration with U. Wagner and I. Mabillard from Institute of Science and Technology, Austria.

5. Teaching

[1] Discrete structures and algorithms in topology, III year students, September-December 2015, 4 hours per week. Moscow Institute of Physics and Technology (DIHT)

[2] Modern topological methods in physics, II year students, February-May 2015, 2 hours per week. Moscow Institute of Physics and Technology (DGAP)

[3] Algorithms for recognition of the realizability of hypergraphs, Independent University of Moscow, February-May 2014, 2 hours per week.

[4] Algebraic topology of manifolds, II-V year students, Independent University of Moscow, September-December 2015, 2 hours per week.

[5] Discrete analysis (exercises), II year students, February-December 2015, 2 hours per week. Moscow Institute of Physics and Technology (DIHT)

[6] Realizability of hypergraphs and Ramsey link theory (minicourse), Summer School 'Modern Mathematics', July 2015, 4 hours.

[7] A short elementary proof of the Ruffini-Abel Theorem (minicourse), Summer School 'Modern Mathematics', July 2015, 4 hours.

I was an advisor of papers by D. Akhtyamov (http://arxiv.org/abs/1411.4990) and D. Kolodzey (http://arxiv.org/abs/1509.00370; premaster work).