EXPLICIT CONSTRUCTIONS OF INTERESTING PROJECTIVE MANIFOLDS

SERGEY GALKIN

I wish to provide explicit algebro-geometric constructions for various interesting (quasi)-projective varieties, such as Fano fourfolds (with (virtual) Picard number one), Calabi–Yau threefolds (with Picard number one, or instead with one-dimensional moduli space), regular surfaces with vanishing geometric genus.

We know that in each given dimension there are just finitely many deformation families of Fano manifolds, and in dimensions up to three we have explicit descriptions for each family as zero loci of generic sections of a fixed vector bundle on a fixed key space (product of a Grassmannian with a toric variety). Some high-dimensional Fano manifolds (moduli spaces of vector bundles on high-genus curves, in dimensions starting from sixty six) are known to have moduli spaces of general type, so the generic member in the respective deformation family cannot have a uniform description, as it has for Fano threefolds; however we cannot yet exclude the possibility that some very special member of the respective deformation family of Fano manifolds has a good explicit description. The case of Fano fourfolds lies in the shady area (in characteristic zero, otherwise we don't know even threefolds): it is unknown how a typical Fano fourfold shall look like, and how complicated an explicit description of a Fano fourfold could be.

Smooth anti-canonical section of a Fano fourfold (if it exists) is a Calabi–Yau threefold. So far we know only finitely many families of Calabi–Yau threefolds, and many interesting examples arise as such hypersurfaces. For Calabi–Yau threefolds nobody knows even whether there are finitely many or infinitely many deformation families, or worse — whether Betti numbers of Calabi–Yau threefolds are bounded from above: some people secretely believe in finiteness of families of Calabi–Yau threefolds, others produce theories that should show that there are infinitely many diffeomorphism classes of Calabi–Yau threefolds. So far we know just finitely many families of Calabi–Yau threefolds, and if we restrict ourselves to the case, where second Betti number equals to one (or, instead, third Betti number equals to four), then we know just about a hundred examples (and some conjectures, that there should be more examles with some prescribed invariants).

In sixties Mumford formulated a famous (still widely open) question, whether a computer can classify regular surfaces with vanishing geometric genus. Much worse than that, given all the invariants you want, can one explicitly construct a surface with given invariants, that is to find its model, or at least to describe its field of rational functions? It turns out, that in case of Kodaira dimension one there is a description of all the respective surfaces: all surfaces of Kodaira dimension one are elliptic surfaces, and those of them who are regular with geometric genus zero are prinicipal homogeneous spaces over rational elliptic surfaces with a section (thanks to Ogg–Shafarevich theory), however there are no explicit models of the respective surfaces that have Kodaira dimension one.

Finally, using p-adic uniformisation Mumford shown that there exists at least one surface with an ample canonical bundle, that has same Betti numbers as a complex projective plane (these surfaces are now known as fake projective planes). Recently all fake projective planes have been classified: there are just a hundred of them, each one is rigid, each one has no real structure, so there re fifty fundamental groups. We have very explicit descriptions of these groups, given by generators and relations, however we so far don't have even a single example of a fake projective plane given as a projective surface.

It turns out, that very similarly, that we have an implicit description of all regular elliptic surfaces with geometric genus zero (thanks to Ogg, Shafarevich, Halphen, Dolgachev), but we don't know a single example of a regular elliptic surface with geometri genus zero and a finite fundamental group with Kodaira dimension at least one.

The main goal of my project is to provde new and very explicit examples of Fano fourfolds, Calabi–Yau threefolds, and surfaces with $p_g = q = 0$ either explicitly given as projective varieties, or at least some Zariski open subsets of them given as qusi-projective varieties.

In 2014 with collaborators I already made some progress towards this direction: I constructed some Fano fourfolds and Calabi–Yau threefolds that were predicted earlier to exist, and also I've got new understanding for the Dolgachev and Mumford's surfaces, that might finally give an explicit description.

I am going to pursue this further.