Report on the Dynasty–IUM fellowship 2017

SERGEY GALKIN

New results

[1] Property $\mathcal{O}$ for odd cohomology

This new result has been written in the new appendix of paper [1] and reported in talk [21]. We show that one can deduce the validity of property $\mathcal{O}$ (about eigenvalues of on the operator of quantum multiplication by first Chern class) for the whole cohomology space from a priori weaker validity of property $\mathcal{O}$ on the subspace of even cohomology, or on the subspace of (p,p)-cohomology. This result and its proof is similar to Hertling-Manin-Teleman's result about semi-simplicity of quantum cohomology.

[2] Polar dg-rings

For a given base field k (e.g. complex numbers) we give a couple of definitions of *polar dg-ring* P — an abelian group with grading, binary operation of product $\cdot$ and unary operation of differential d (or boundary δ), whose generators are geometric entities — log-CY varieties over k. Polar dg-ring $P = \bigoplus_d P_d$ is a $\mathbb{Z}$-graded abelian group generated in degree d by isomorphism classes of d-dimensional log-Calabi-Yau pairs (X, ω) — smooth projective variety X equipped with meromorphic volume form ω with anti-effective divisor $(\omega) = -D = -\sum_{i=1}^{n} D_i$ with only simple poles along D_i and D being divisor with simple normal crossings. For technical reasons one has to decorate pair by numbering of divisors D_i. Cartesian product induces structure of product on P, however the product is less significan than differential (boundary operator). Boundary of a pair (X, D) is a formal sum $\sum_i (-1)^i (D_i, (D - D_i)_{|D_i})$ or in terms of volume forms it is a sum of residual forms $(D_i, res_{D_i}\omega)$. Numbering of divisors and sign $(-1)^i$ (or just tensoring everything with $\mathbb{Z}/2$) guarantees that $\delta^2 = 0$, so it makes sense to consider closed elements $Z_d(P) = Ker(\delta)$, exact elements (boundaries) $B_d(P) = Im(\delta)$ and homology groups $H_d(P) = Z_d(P)/B_d(P)$. And it is a very interesting question whether these homology groups are non-zero and how to prove that some particular element does not vanish. Every smooth projective Calabi-Yau d-fold gives a closed element in Z_d, however many CYs considered in physics literature (such as quintic threefold) lie in subspace of exact elements (boundaries) B_d, since by construction they are anti-canonical divisors somewhere. So in some sense homology groups H_d measure a gap between (small) world of Calabi-Yau related to geometry close to Fano manifolds and (supposedly much larger) world of Calabi-Yaus on their own. One can use classification of surfaces to show that over non-closed fields k already group $H_1(P)$ is non-zero. Regarding surfaces I expect that abelian surfaces $K3$ surfaces of Picard number one and large enough degree of polarization also produce non-exact elements. This is related to more classical problem of smoothability of cones over projective manifolds, but does not directly follow from it. For Calabi–Yau threefolds I expect that majority of them are non-boundary, however I do not have any explicit simply-connected candidate. In terms of "Oxford dictionary" between real topology and holomorphic geometry the problem above can be phrased as construction of characteristic numbers (that give obstruction to manifold to be a boundary).

I reported on this construction and range of related topics in talks [2] and [15].

[3] On derived involutions of very general polarized hyperkähler manifolds

Derived McKay correspondence establishes an equivalence between derived category of coherent sheaves on Hilbert scheme of n points on a surface X with category of S_n-equivariant coherent sheaves on Cartesian power X^n. Since symmetric group S_n has normal subgroup A_n with quotient-group S_2 any S_n-equivariant category $A = B^{S_n}$ can be represented as S_2-equivariant category of A_n-equivariant category: $A = C^{S_2}$ where $C := B^{A_n}$. Elagin's reciprocity law says that if A is equivariant for C with respect to an action of any finite abelian group G, then there is a natural action of the dual group of characters $G^\vee = Hom(G, U(1))$ on A and moreover C itself is a category of $G^\vee$-equivariant objects in A. In the example above this says that if some perfect dg-category C is Ganter-Kapranov's symmetric power of any other category, then category C has a non-trivial involution σ (in particular all Hilbert schemes of surfaces have non-trivial derived involutions).

In principle it is possible that category $C = D(X)$ of coherent sheaves on a surface X has non-commutative deformation C' such that for some number n its Ganter-Kapranov's symmetric power $S^n_{GK}C'$ is equivalent to $D(Y)$ for commutative space Y ("Hilbert schemes of non-commutative deformations could nevertheless be commutative").

In particular five and three years ago I conjectured that this is the case for the non-commutative K3 category inside derived category of cubic fourfold with representatives for $Hilb_2$ and $Hilb_4$ being Beauville–Donagi's variety of lines F and Lehn–Lehn–Sorger–van Straten's variety Z related to twisted cubics. Application of Elagin's reciprocity explained in the last paragraph then says that if my conjectures are true, then bounded derived categories $D(F)$ and $D(Z)$ should have some non-trivial involutions. It is interesting that for generic enough cubic fourfold its variety of lines F does not have any geometric involution (so the conjectural involution should really be derived), however LLSvS's 8-fold Z actually has a geometric involution related to the structure of double-sixes on a cubic surface.

Kontsevich's picture of homological mirror symmetry conjectures a relation between autoequivalences of derived category of coherent sheaves on a Calabi-Yau variety with monodromy of the mirror-dual family.

Two years ago I reported that with van Straten we figured out that Picard-Fuchs equation for mirror-dual family of Beauville-Donagi fourfold is likely to be the ODE that kills modular form of weight 4 and level 3 and its monodromy group is a modular group of level 3, which has a natural Fricke involution $\tau \to -\frac{1}{3\tau}$.

I reported this in talk [8] and mentioned in some talks related to the next topic.

[4] Hyperkähler fourfolds and Hilbert squares of non-commutative K3 surfaces in Renormalization Group flow for Gauged Linear Sigma Models

I produce three gauged linear sigma models with peculiar property that one of their phases is geometric and gives a holomorphic symplectic fourfold, whereas in another phase we see Ganter-Kapranov's symmetric square of a category which is (usually) non-commutative deformation of K3 surface.

Fix a polynomial functor F (in discussion below either S^3 of Λ^3), (flavour) vector space V of rank N and an element $f \in FV^*$ (e.g. for $F = S^3$ form f is a cubic form in N variables). Let U be a vector space of rank k. Let matter fields (ϕ, p) be elements of a vector space $Hom(U, V) \oplus FU =: M$. Equip M with natural action of (gauge) group $G = GL(U)$ and a G-invariant function (superpotential) $W(\phi, p) := Tr(f(F\phi)p)$.

Then for $(F = S^3, k = 2, N = 6)$ in geometric phase we have Beauville–Donagi's fourfold (variety of lines on cubic fourfold), and for $(F = \Lambda^3, k = 6, N = 10)$ in geometric phase we have Debarre–Voisin's hyperkähler fourfold.

On the other hand, in another phase we naturally can see a category which is Ganter-Kapranov's symmetric square of a category with k twice smaller than original, and is a non-commutative K3. This is related to the facts that symmetric cubic form of two variables (and skew-symmetric cubic form of 6 variables) have standard forms with stabilizers of generic elements looking as $H^2 \to Stab(p) \to S_2$. The respective S_2 in the quotient should be related to conjectural derived involution from the previous discussion.

I reported this in talks [1], [3], [4], [5], [6], [18].

Papers

[1] With H. Iritani

Gamma conjecture via mirror symmetry

arXiv:1508.00719v2 (26 Jun 2017), *to appear in the proceedings of the conference "Primitive Forms and Related Subjects" (IPMU, Feb 2014), to be published in a volume of Advanced Studies in Pure Mathematics*

The asymptotic behaviour of solutions to the quantum differential equation of a Fano manifold F defines a characteristic class A_F of F, called the principal asymptotic class. Gamma conjecture of Vasily Golyshev and the present authors claims that the principal asymptotic class A_F equals the Gamma class $\widehat{\Gamma}_F$ associated to Euler's Γ-function. We illustrate in the case of toric varieties, toric complete intersections and Grassmannians how this conjecture follows from mirror symmetry. We also prove that Gamma conjecture is compatible with taking hyperplane sections, and give a heuristic argument how the mirror oscillatory integral and the Gamma class for the projective space arise from the polynomial loop space.

Scientific conferences and seminar talks

[1] Conference "Algebraic Geometry in Mexico", Nov 27 - Dec 3, Oaxaca and Puerto Escondido

Talk "Cubic hypersurfaces, their moduli spaces and beautiful formulae"

[2] Conference "Arithmetic, geometry, automorphic forms and physics", Oct 30-Nov 3, Laboratory of Mirror Symmetry, HSE, Moscow

Talk "Polar dg-algebra and another view on Calabi-Yau geography"

[3] Annual memorial conference dedicated to the memory of Andrei Nikolaevich Tyurin, Oct 26, Steklov Math Institute, Moscow

Talk “Beautiful formulae and gauged linear sigma models”

[4] Conference “Instruments of Algebraic Geometry”, Sep 18-22, Bucharest

Talk “Beautiful formulae, gauged linear sigma models and problems of semi-classical algebraic geometry”

[5] Workshop “Geometry of String and Gauge Theories”, July 10-21, CERN

Talk “Two Gauged Linear Sigma Models for non-spherical Calabi-Yau manifolds”

[6] Inaguration of Gökova Geometry/Topology Institute, June 5-7 , Gökova

Lecture series “Cubic forms and related geometries”

[7] Thematic Program “Advances in Birational Geometry”, workshop “closing workshop - future directions”, May 15-19, ESI Vienna

Talk “Degenerations to Normal Cone as Relations for a Grothendieckuesque Group”

[8] Thematic Program “Advances in Birational Geometry”, workshop “Categorical approach to rationality”, May 2-5, ESI Vienna

Talk “On a derived involution on variety of lines on a cubic fourfold”

[9] Thematic Program “Advances in Birational Geometry”, workshop “Recent developments in rationality questions”, Apr 24-28, ESI Vienna

Talk “Relations between cycles on cubics”

[10-11] School-conference “Algebra and Number Theory in Kaliningrad”, Apr 17-21, Immanuel Kant Baltic Federal University, Kaliningrad

Talk “Representations of Kronecker quiver and determinantal representations of cubic forms”

Talk “Points, lines, squares, and cubics”

[12] Conference on Gamma-conjectures, Mar 10-11, Schloss Hünigen

Talk “Gamma conjectures”

[13] Informal Geometry/Topology workshop, Jan 16-19, Belalp

Talk “On tubular neighbourhoods and degenerations to normal cones”

[14] Talk “On gamma-class and hemisphere” at “Riemann surfaces, Lie algebras and math physics” seminar, Dec 8, Independent University of Moscow

[15] Talk “On geography of log Calabi-Yau” at Algebraic Geometry seminar, Dec 6, University of Georgia, Athens

[16] Talk “27” at “Algebro-geometric methods in integrable systems and quantum physics” seminar, Nov 16 , MIPT, Dolgoprudny

[17] Talk “Beauville surfaces and dessins d’enfant” at “Graphs on surfaces and number fields” seminar, Nov 8, Moscow State University

[18] Talk “Extension of moduli and gauged linear sigma models” at “Automorphic forms and their applications” seminar, Oct 23, LMS HSE, Moscow

[19] Talk “Virtual Symmetries” at “Automorphic forms and their applications” seminar, Sep 11, LMS HSE, Moscow

[20] Talk “Fano threefolds, K3 surfaces, Mathieu group, and Brav-Dyckerhoff relative Calabi-Yau structures” at the weekly seminar of Lab of Algebraic Geometry, August 4, HSE, Moscow

[21] Talk “Property O for odd cohomology” at “Automorphic forms and their applications” seminar, June 20, LMS HSE, Moscow

[22] Talk “On Kashaev’s simple model of 4d TQFT and realizations of Pachner moves”, May 18, Vienna

[23] Talk “The conifold point” at weekly seminar of Lab of Algebraic Geometry, Apr 14, HSE, Moscow

[24] Talk “Relations between zero-cycles and curves on cubic (hyper)surfaces”, Mar 23, Universität Zürich

[25] Talk “Relations between zero-cycles and curves on cubic (hyper)surfaces”, Feb 23, Jussieu, Paris

[26] Talk “Octonions and projective planes” at “Geometric structures on manifolds” seminar, Jan 12, HSE, Moscow

[27] Talk “Mirror symmetry and automorphic forms for some hyperkähler manifolds” at “Automorphic forms and their applications” seminar, Jan 10, LMS HSE, Moscow

Teaching

[1] Course “Algebraic Geometry”, National Research University Higher School of Economics, students from 3 year, Fall 2017, 3 hours per week.

Programme: Схемы и их морфизмы. Расслоенные произведения и их применения. Относительная точка зрения. Функтор точек. Отделимость, собственность и проективность. Плоскость и многочлен Гильберта.

Семейства схем и пределы. Дифференциалы, гладкость. Когерентные пучки, их когомологии и высшие прямые образы, теорема полунепрерывности. Proj, раздутие. Геометрические применения — кривые, поверхности.

[2] Seminars on Geometry, National Research University Higher School of Economics, 1st year students, Fall 2017, 1.5 hours per week.

[3] With V. Gritsenko and V. Spiridonov

Seminar "Automorphic forms and their applications", National Research University Higher School of Economics), students from 3 year, 1.5 hours per week.

[4] With C. Brav

Seminar "Variety of varieties", National Research University Higher School of Economics, students from 3 year, Fall 2017, 1.5 hours per week.

Научное руководство.

[1] Павел Попов, аспирант 3 года в НМУ и на матфаке ВШЭ.

[2] Артём Приходько, аспирант 3 года в НМУ и на матфаке ВШЭ.

[3] Артём Калмыков, студент 2 курса магистратуры матфака ВШЭ.

[4] Антон Шляпугин, студент 1 курса магистратуры матфака ВШЭ.

[5] Александр Шлимович, студент 1 курса бакалавриата матфака ВШЭ.