Report on the Dynasty–IUM fellowship 2016

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New results

[1] Большое кольцо многообразий

Было введено большое кольцо многообразий, снабжённое двумя эпиморфизмами — в кольцо Гротендика многообразий и в кольцо Левина–Мореля алгебраических кобордизмов (следовательно, и в кольцо комплексных кобордизмов). Образующие большого кольца это символы $[X]$ для всех связных гладких проективных многообразий X, а каждому гладкому подмногообразию $Z \subset X$ соответствует соотношение "вырождения к нормальному конусу":

$$[Bl_Z X] - [X] = [\mathbb{P}_{\mathbb{P}_Z(N_{X/Z})}(\mathcal{O} \oplus \mathcal{O}(1))] - [\mathbb{P}_Z(N_{X/Z} \oplus \mathcal{O})]$$

Таким образом большое кольцо является "гибридом" кольца Гротендика многообразий с кольцом бордизмов, и какие-то неприятные свойства кольца Гротендика в нём заменяются более приятными свойствами кольца Тома: например, оно естественным образом градуировано размерностью (а на кольце Гротендика есть лишь натуральная фильтрация). Это кольцо помнит и числа Ходжа и числа Черна. Кроме бинарной операции умножения на нём можно ввести две унарные операции, соответствующие взятию схём Гильберта двух и трёх точек, а также паре гнездящихся схем Гильберта. В настоящее время я пытаюсь понять, можно ли поднять из бордизмов в большое кольцо действие алгебры операций Ландвебера–Новикова, а также доказать интересные тождества в этом кольце, соответствующие интересным бирациональным разложениям.

Papers

[1] With V. Golyshev and H. Iritani
Gamma classes and quantum cohomology of Fano manifolds: Gamma conjectures
Duke Math. J. 165 (2016), no. 11, 2005–2077.

We propose Gamma Conjectures for Fano manifolds which can be thought of as a square root of the index theorem. Studying the exponential asymptotics of solutions to the quantum differential equation, we associate a principal asymptotic class A_F to a Fano manifold F. We say that F satisfies Gamma Conjecture I if A_F equals the Gamma class $\hat{\Gamma}_F$. When the quantum cohomology of F is semisimple, we say that F satisfies Gamma Conjecture II if the columns of the central connection matrix of the quantum cohomology are formed by $\hat{\Gamma}_F Ch(E_i)$ for an exceptional collection E_i in the derived category of coherent sheaves $D^b_{coh}(F)$. Gamma Conjecture II refines part (3) of Dubrovin's conjecture. We prove Gamma Conjectures for projective spaces and Grassmannians.

[2] With T. Coates, A. Corti, A. Kasprzyk
Quantum periods for 3-dimensional Fano manifolds
Geometry & Topology 20 (2016) 103–256.

The quantum period of a variety X is a generating function for certain Gromov–Witten invariants of X which plays an important role in mirror symmetry. In this paper we compute the quantum periods of all 3-dimensional Fano manifolds. In particular we show that 3-dimensional Fano manifolds with very ample anticanonical bundle have mirrors given by a collection of Laurent polynomials called Minkowski polynomials. This was conjectured in joint work with Golyshev. It suggests a new approach to the classification of Fano manifolds: by proving an appropriate mirror theorem and then classifying Fano mirrors. Our methods are likely to be of independent interest. We rework the Mori–Mukai classification of 3-dimensional Fano manifolds, showing that each of them can be expressed as the zero locus of a section of a homogeneous vector bundle over a GIT quotient V/G, where G is a product of groups of the form $GL_n(\mathbb{C})$ and V is a representation of G. When $G = GL_1(\mathbb{C})^r$, this expresses the Fano 3-fold as a toric complete intersection; in the remaining cases, it expresses the Fano 3-fold as a tautological subvariety of a Grassmannian, partial flag manifold, or projective bundle thereon. We then compute the quantum periods using

the Quantum Lefschetz Hyperplane Theorem of Coates–Givental and the Abelian/non-Abelian correspondence of Bertram–Ciocan-Fontanine–Kim–Sabbah.

[3] With I. Karzhemanov and E. Shinder
Acyclicity of non-linearizable line bundles on fake projective planes
arXiv:1602.06107 and IPMU 15-0202.

On the projective plane there is a unique cubic root of the canonical bundle and this root is acyclic. On fake projective planes such root exists and is unique if there are no 3-torsion divisors (and usually exists but not unique otherwise). Earlier we conjectured that any such cubic root (assuming it exists) must be acyclic. In the present note we give a new short proof of this statement and show acyclicity of some other line bundles on those fake projective planes with at least 9 automorphisms. Similarly to our earlier work we employ simple representation theory for non-abelian finite groups. The novelty stems from the idea that if some line bundle is non-linearizable with respect to a finite abelian group, then it should be linearized by a finite (non-abelian) Heisenberg group. Our argument also exploits J. Rogawski's vanishing theorem and the linearization of an auxiliary line bundle.

Scientific conferences and seminar talks

[1] Twenty-third Gökova Geometry/Topology Conference, Gökova, May 30-June 3
Talk "Hyperkähler manifolds and mirror symmetry"
[2] miniPAGES: Polish Algebraic Geometry mini-Semester, Banach Center, Warsaw, May 1–13
Talk "Hyperkähler manifolds and modular forms"
[3] Third SwissMAP Geometry&Topology conference, Engelberg, Jan 24–27
Talk "Curve counting on varieties with special holonomy"
[4] Talk "Algebras of geometries" at Moscow Mathematical Society, April 12 (Moscow State University)
[5] Talk "Gamma conjectures and mirror symmetry" at University of Geneva, March 18
[6] Talk "Yau–Zaslow formula" at "Automorphic forms and their applications", Feb 9, Moscow (HSE)
[7] Talk "Minuscule varieties and mirror symmetry" at University of South Carolina, Feb 1, Columbia (USC)
[8] Talk "Quantum indices of real curves and non-commutative Ginzburg-Landau potential" at "Riemann surfaces, Lie groups and Math physics seminar", Jan 22, Moscow (IUM)
[9] Talk "Arnold–Maxwell theorem and Oguiso–Shröer manifolds" at "Geometric Structures on Manifolds", Dec 31 2015, Moscow (HSE)

Teaching

[1] With N. Kurnosov
Mini-course "Cubic forms" (4 lectures) at summer School Modern Math, Dubna, July 24–29 2016.
[2] With V. Gritsenko
Seminar "Automorphic forms and their applications", National Research University Higher School of Economics (joint with Poncelet lab), students from 3 year, October 2016 - June 2017, 2 hours per week.
[3] With M. Verbitsky and V. Zhgoon
Seminar "Geometric structures on complex manifolds", National Research University Higher School of Economics, students from 3 year, September 2016 - June 2017, 3 hours per week.

Научное руководство.

[1] Павел Попов, аспирант 2 года в НМУ и на матфаке ВШЭ.
[2] Артём Приходько, аспирант 2 года в НМУ и на матфаке ВШЭ.
[3] Артём Калмыков, студент 1 курса магистратуры матфака ВШЭ.
[4] Андрей Давыдов, выпускник бакалавриата матфака ВШЭ в 2016 году.
В 2016 году Андрей написал дипломную работу "Производная категория многообразия прямых на кубической гиперповерхности" ("Derived category of the variety of lines on a cubic hypersurface").
[5] Александра Мазурова, выпускница бакалавриата матфака ВШЭ в 2016 году.
В 2016 году Саша написала дипломную работу "Исключительные наборы на исключительных микровесовых многообразиях" ("Exceptional collections on exceptional minuscule varieties").
[6] Евгений Маршаков, выпускник бакалавриата матфака ВШЭ в 2016 году.
В 2016 году Женя написал дипломную работу про поверхности Энриквеса степени 16 и пучки плоских кубических кривых.