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Introduction

The proposed book is devoted to a phenomenon of fractal sets, or simply **fractals**. It has been known for more than a century and was observed in different branches of science. But only recently (approximately, in the last 30 years) it has become a subject of mathematical study.

The pioneer of the theory of fractals was B. Mandelbrot. His book [Man82] first appeared in 1977 and its second enlarged edition was published at 1982. After that the serious articles, surveys, popular papers and books about fractals are counted by dozens (if not hundreds); since 1993 a special periodic journal "Fractals" is published by World Scientific. So, why write one more book?

First, it turns out that in spite of the vast literature, many people, including graduate students and even professional mathematicians, have only a vague idea about fractals.

Second, in many popular books the reader finds a lot of colorful pictures and amazing examples but no accurate definitions and rigorous results. On the contrary, the articles written by professionals are, as a rule, too difficult for beginners and often discuss very special questions without any motivation.

Last and may be the most important reason is my belief that the endeavor of independent study of the Geometry, Analysis and Arithmetic of fractals is one of the best ways for a young mathematician to acquire an active and stable knowledge of the basic mathematical tools.

This subject also seems to me to be an excellent opportunity to test one's ability to produce creative work in mathematics. I mean here not only solving of well-posed problems, but recognizing a hidden pattern and formulating new fruitful problems.

My personal interest in fractals originates from the lecture course I gave in the University of Pennsylvania in 1995 as per the request of our undergraduate students. I repeated this course in 1999, 2003 and 2005. In 2004 I had the opportunity to expose the material in four lectures at the Summer School in Dubna near Moscow organized for high school seniors and first year university students who were winners of the Russian Mathematical Olympiad. I was surprised by the activity of the audience and by the quickness of coomprehending all necessary information.

In this book we deliberately restrict ourselves by only two examples of fractals: Sierpiński and Apollonian gaskets. We describe and rigorously formulate several problems coming from the study of these fractals. Most of them can be formulated and solved independently but only the whole collection gives an understanding of the world of fractals.

Some of these problems are more or less simple exercises, some are relatively new results and a few are unsolved problems of unknown difficulty.

The solution (and even formulating and understanding) of all of the problems requires some preliminary background. It contains, in particular, the following:

- Elements of Analysis: functions of one variable, differential and integral calculus, series.
- Elements of Linear Algebra: real and complex vector spaces, dimension, linear operators, quadratic forms, eigenvalues and eigenvectors. Coordinates and inner products.
- Elements of Geometry: lines, planes, circles, discs and spheres in $\mathbb{R}^3$. Basic trigonometric formulae. Elements of spherical and hyperbolic geometry.
- Elements of Arithmetic: primes, relatively prime numbers, gcd (greatest common divisor), rational numbers, algebraic numbers.
- Elements of Group Theory: subgroups, homogeneous spaces, cosets, matrix groups.

All of this is normally contained in the first two or three years of mathematical curriculum. I consider the diversity of the necessary tools and their interconnection as a great advantage to the whole problem and as a characteristic feature of modern mathematics.

Several words about the style of exposition. I tried to avoid two main dangers: to be dull by explaining too many details in most elementary form and to be incomprehensible by using very effective but sometimes too abstract modern technique. It is to the reader to judge the success of this endeavor.

I also tried to communicate a non-formal knowledge of mathematical tools which distinguishes (almost all) professionals from most beginners. Sometimes one phrase explains more than a long article[1]

So, from time to time, I use intentionally some "high-altitude" notions, explaining each time what they mean in simplest situations.

Additional information is included in the text in the form of short "Info's". The end of an Info is marked by the sign $\diamondsuit$.

I also use "Remarks" as another form of additional information. The end of a Remark is marked by the sign $\heartsuit$.

The end of a proof (or the absence of proof) is marked by the sign $\square$.

[1]In my personal experience it happened when I tried to understand induced representations, spectral sequences, intersection homology, etc...