COMBINATORIAL GAME THEORY

PIERRE DEHORNOY
DUBNA
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In this course, we will see how one can study simple combinatorial games by associating to them rather strange numbers. These correspond to an synthetic but exact evaluation of both player positions. This procedure reduces the complexity and the variety of the possible situations, with a minimal loss of information. Every material in this course comes from the wonderful books by Berlekamp, Conway and Guy [1], and Conway [2]. I would recommend to read [1] at first, and then [2] for a theoretically complete exposition.

Which games? The games on which we shall focus verify the following properties:

- there are two players;
- there are positions, and the rules define which positions can be reached from a given one;
- there is full information, *i.e.* no hidden cards nor secret missions, the rules are known by both players;
- there is no luck, *i.e.* no rolling dice nor card shuffle;
- the game always ends after a finite number of steps;
- there is always a winner, *i.e.* no draw is possible.

For convenience we transform the last condition into the following convention (which is stronger, but does not exclude all possible games)

- a player loses at some position if and only if there is no allowed move for him from this position.

In the whole text, the two players will be denoted Left and Right (L and R sometimes).

Formally speaking, here is a definition

Definition 0.1. *A* game *is a (possibly infinite) set of positions $\mathcal{G}$, and a set of rules defining for each position g in $\mathcal{G}$ two sets $\mathcal{G}^L = \{g_1^L, g_2^L, \dots\} \subset \mathcal{G}$ (resp. $\mathcal{G}^R = \{g_1^R, g_2^R, \dots\} \subset \mathcal{G}$) of positions that can be reached by Left (resp. Right) from g. If $\mathcal{G}^L$ (resp. $\mathcal{G}^R$) is empty, we say that g is a losing position for Left (resp. Right).*

Notation: In order to remember the options offered to the two players from the position g, we write $g = \left\{g_1^L, g_2^L, \dots \middle| g_1^R, g_2^R, \dots\right\}$.

Remark that the game is not the main thing. The only important things are the positions and the rules describing their relations. Note also the we have no basic blocks for this recursive construction. The only building block we will use here is the empty game $\{|\}$. The following remark is crucial, although not difficult.

Proposition 0.2. *Let a position of game and a beginning player be given. Then one of the two player always has a winning strategy, i. e. whatever his opponent plays, he can win the game.*

Proof. Exercise. □

Given a position of a game, there are therefore four possible distinct outcomes:

- the first player has a winning strategy;
- the second player has a winning strategy;
- Left has a winning strategy, whoever begins;
- Right has a winning strategy, whoever begins.

Since we suppose our players smart, we will say that a player *wins* if he has a winning strategy.

1. Partizan games and numbers

Partizan games are games such that, for some positions g, the set g_L and g_R are different. This is the general situation, which contrasts with non-partizan games for which $g_L = g_R$. First, we turn to a partizan game – Hackenbush – verifying a very special property, namely that the first player never has a winning strategy (lemma 1.3).

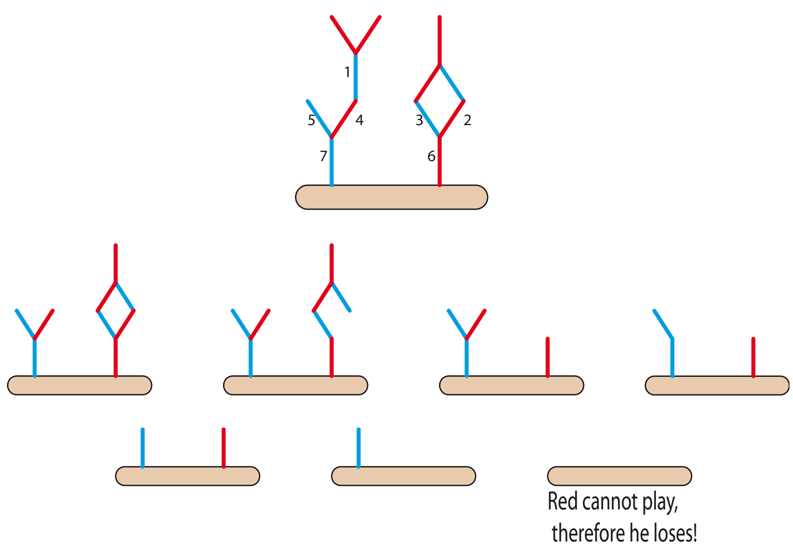
1.1. **The game of Hackenbush.**

Rules 1.1 (Partizan Hackenbush)**.** Positions: *a finite numbers of bLue and Red edges, each of which is connected to the ground by a path;*

Moves: *Left cuts a bLue edge, each edge which is not any more connected to the ground disappears; Right cuts a Red edge, each edge which is not any more connected to the ground disappears;*

Loser: *If there is no more edge of its color when he has to play, a player loses.*

Example 1.2. *Here is a starting position, and a game. After the seventh move, Right has no more Red edge to cut, so he loses.*



What makes partizan Hackenbush nice to begin with is the following:

Lemma 1.3. *Given a position of Hackenbush, the first player never has a winning strategy, i.e. either Left wins, either Right wins, either the second player wins.*

Proof. Assume that the first player has a winning strategy. Then, if Left begins, his winning strategy dictates him to cut a given edge E (and therefore removing some other edges $E_1, E_2, \dots$ which were supported by E. Now, suppose that Right begins. Then, Left can adopt the same strategy as before, except if Right cuts one of the E_i, in which case Left answers by cutting E. Left is back to a situation to which its original strategy applies. □

1.2. **Sum of games. Zero, positive and negative games.** The sum of two games corresponds to playing both games simultaneously: at each turn, each player play on one and only one of the two board. He loses if and only if he is stuck on both boards. Formally,

Definition 1.4. *The* sum *of two games $\mathcal{G}$ and $\mathcal{H}$ is defined as follows:*

Positions: *If g is a position for $\mathcal{G}$ and h is a position for $\mathcal{H}$, then $g \cup h$ is a position for $\mathcal{G} + \mathcal{H}$;*

Moves: $g \cup h = \left\{g_1^L \cup h, g_2^L \cup h, \dots, g \cup h_1^L, g \cup h_2^L, \dots \middle| g_1^R \cup h, g_2^R \cup h, \dots, g \cup h_1^R, g \cup h_2^R, \dots \right\}$;

Loser: *$g \cup h$ is a losing position for Left if and only if g_L and h_L are empty, i.e. g and h are losing positions for Left.*

For example, playing Hackenbush on two separate diagrams is the same as playing the sum of the Hackenbushs associated to the single diagrams. The following remark is crucial for simplifying positions, and allows to introduce some terminology.

Lemma 1.5. *Suppose that $\mathcal{H}$ is a game in which the second player has a winning strategy. Then, for any game $\mathcal{G}$ the winner of $\mathcal{G} + \mathcal{H}$ is the same as the winner of $\mathcal{G}$.*

Proof. The winning strategy for the winner of $\mathcal{G}$ consists in never playing in $\mathcal{H}$, unless the other player does so, and if so, to answer following the winning strategy for the second player in $\mathcal{H}$. □

We now fix notations and vocabulary.

Definition 1.6. *The number 0 is the game* $\{|\}$.

Definition 1.7. *A game is said to be* a zero game *if the second player has a winning strategy,* positive *if Left has a winning strategy,* negative *if Right has a winning strategy,* fuzzy *if the first player has a winning strategy.*

For example, 0 is a zero game! Note that this is not the only one: for example, the position with one bLue edge and one Red edge, each related to the ground is also a zero game. The ambiguity will be soon removed, by defining the equality between games. The lemma 1.5 claims that adding a zero game never affects the outcome.

We now want to define equality. Since $A = B$ can be rephrased $A - B = 0$, we first define the opposite of a game. The definition will be transparent after the following remark.

Lemma 1.8. *Let $\mathcal{G}$ be a position of Hackenbush. Consider the position $\bar{\mathcal{G}}$ where all the colors are changed (bLue becomes Red, and Red becomes bLue), then the game $\mathcal{G} + \bar{\mathcal{G}}$ is a zero game.*

Proof. The strategy for the second player is to copy its opponent's moves in order to win! We call it the *mirror strategy*. □

The mirror strategy is not particular to Hackenbush, it works in any game. This is actually a key stone of the theory.

Definition 1.9. *We define recursively the inverse $-g$ of a position $g = \left\{g_1^L, g_2^L, \dots \middle| g_1^R, g_2^R, \dots\right\}$ as the position $\left\{-g_1^R, -g_2^R, \dots \middle| -g_1^L, -g_2^L, \dots\right\}$.*

We say that two games $\mathcal{G}$ and $\mathcal{H}$ are equal *if the sum $\mathcal{G} - \mathcal{H}$ is a zero game. One writes $\mathcal{G} = \mathcal{H}$.*

First, one checks that a zero game is a game which is equal to zero, we are safe! Then, one checks that the opposite of a positive game is a negative game, and rephrasing lemma 1.8 gives the equation $\mathcal{G} - \mathcal{G} = 0$, or equivalently $-(-\mathcal{G}) = \mathcal{G}$. One can even check that the standard additive arithmetic inequalities hold:

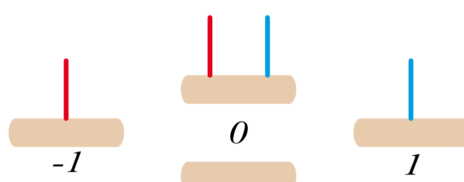
Lemma 1.10. *The sum of two positive games is a positive, the sum of two negative games is negative.*

As for numbers, the sum of positive and negative can be positive, negative, or zero.

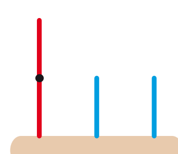
1.3. **Integer values.** Consider the Hackenbush position with only one bLue edge. Then, whoever begins, Left wins. Therefore, this position is positive. How much? Since, it gives exactly one free move to Left, we declare this position to be 1. As well as we defines 0, we even DEFINE THE NUMBER 1 as this game.

Definition 1.11. *The number 1 is the game* $\{0|\} = \left\{\{|\}\middle|\right\}$.

Immediately, by taking the opposite we deduce that the number -1 is the position with one Red edge.



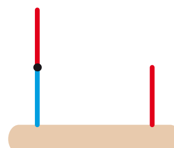
Then, if we want to be coherent with the definition of sum of games, we immediately deduce that $p - q$ is a position with p bLue edges and q Red edges, each touching the ground. This is coherent with the fact that Left wins if $p > q$, Right if $p < q$, and the second player wins if $p = q$. Note that other different situations may be equal to these new defined integers, provided their sum with an integer is zero.



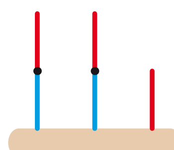
In the preceding position, the second player wins. Since the situation with two bLue edges is 2, we deduce that the bamboo tree formed by the two Red edges is also 2. Not surprisingly, it also gives two free moves to Right.

1.4. **A $\frac{1}{2}$-position.** It is now time to try computing non trivial positions. Denote by g the following position.

In this position, it is easy to check that Left always wins. But unfortunately, if we give one free move to Right,



it turns out that Right always wins. Since g is positive, but not as good as 1, all we can then say is $0 < g < 1$. Let us try to evaluate then $g + g$. It is still positive, and the interesting thing is that, if we give a free move to Right, it becomes a zero game!

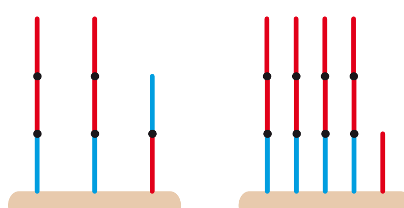


Then we have the equation $g+g-1 = 0$, which we simplify by DEFINING THE NUMBER $\frac{1}{2}$ as g. Note that with our previous convention, g is the game $\{0|1\}$. We then have the equation $\{0|1\}+\{0|1\} = 1$, which we simplified by declaring

$$\frac{1}{2} = \{0|1\}.$$

1.5. **Rational dyadic positions.** Following the preceding idea, it is now possible to compute any bamboo stick, by taking several copies of it, and compare them with smaller stick. By this process we inductively define new numbers.

Example 1.12. *The following figure shows to distinct ways for evaluating the position g formed by that two Red edges above a bLue one. The first picture shows a zero-game yielding the equation* $g + g - \frac{1}{2} = 0$*, while the second yields the equation* $g + g + g + g - 1 = 0$.



We then DECLARE $\frac{1}{4} = g$*, or equivalently*

$$\frac{1}{4} = \left\{0\middle|\frac{1}{2}, 1\right\} = \left\{0\middle|\{0|1\}, 1\right\}.$$

We encode a bamboo stick by the sequences of the colors of the edge from bottm to top. For example, the bamboo stick whose value is $\frac{1}{2}$ is encoded by LR.

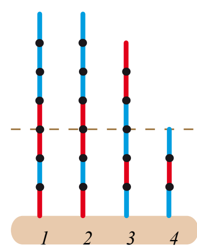
Lemma 1.13. *A bamboo stick is larger than any of its substicks obtained by erasing a bLue edge, and lower than any of its substicks obtained by erasing a Red edge.*

Proof. Exercise □

Lemma 1.14. *i) Let $R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots R^{i_k}L^{j_k}$ be a bamboo stick, $k \geq 1$. Then its double is equal to the sum of sticks $R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots R^{i_{k-1}}L^{j_{k-1}}R^{i_k-1} + R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots R^{i_k}L^{j_k-1}$.*

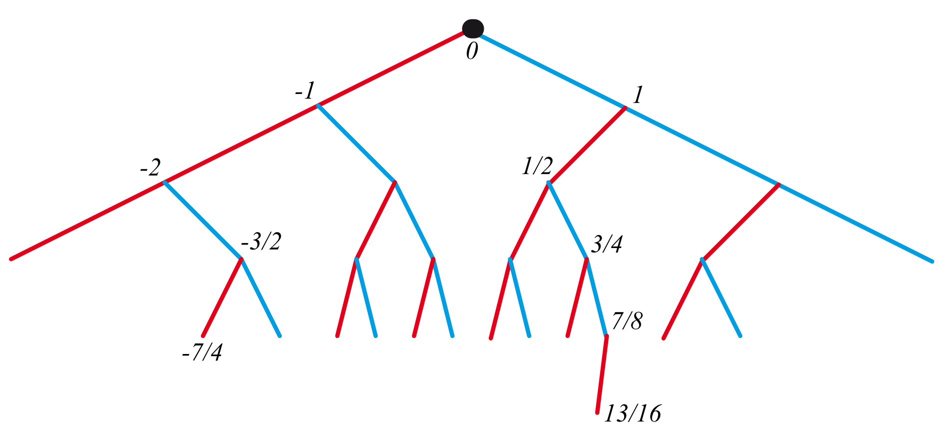
ii) Let $R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots L^{j_{k-1}}R^{i_k}$ be a bamboo stick, $k \geq 2$. Then its double is equal to the sum of sticks $R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots L^{j_{k-1}}R^{i_k-1} + R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots R^{i_{k-1}}L^{j_{k-1}-1}$.

Proof. We prove that the following position is zero, the other case is a variant.



Note that the upper part of column 1, 2, and 3 correspond two the situation describing the equality $\frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0$. Here is the strategy for the second player: while the first player plays in he upper part, the second answers as if the ground was three floors higher (doted line). If the first player never stops playing in the upper part, the latter becomes empty after a move of the second player, since this is a zero-game. Then, the first player loses since was remains (the lower part) is a symmetric game. If at some point, before the upper part be empty, the first player decides to play in the lower part, lemma 1.13 tells us that the result is worse than if it remains is the upper part. □

All of these lemma together gives the following tree T of values for bamboo trees:



Each external node of T is labelled by a relative integer, and each internal node v is labelled by the mean between values of the rightmost node above and on the left of v and of the leftmost node above and on the right of v. Lemma 1.14 claims that the value of a bamboo stick is the label of the node reached when starting from the origin, and descending T according to the colors of the bamboo stick.

1.6. **A new construction of numbers.**

Example 1.15. *Consider the game $\mathcal{G} = \left\{-1\middle|5\right\}$. What is its value? A first guess would be 2, since 2 is the mean value between -1 and 5. Unfortunately, if one give 2 move to Right, considering the game $\mathcal{G} - 2$, one see that Right always win: even if Right starts, he first chooses the -1, and then Left is left with -1 - 2, which is negative, then Left loses.*

The right answer, surprisingly, is 0! Whoever starts loses: if Left starts, he goes to -1, a win for Right, and if Right starts, he goes to 5, a win for Left.

Example 1.16. *Consider now the game $\left\{\frac{1}{4}\middle|1\right\}$. What is its value? Once again, a first guess would be $\frac{5}{8}$, the mean value between $\frac{1}{4}$ and 1. But this is wrong. We already know that $\frac{5}{8}$ is equal to $\left\{\frac{1}{2}\middle|\frac{3}{4}\right\}$, so if we consider the game $\left\{\frac{1}{4}\middle|1\right\} + \left\{-\frac{3}{4}\middle|-\frac{1}{2}\right\}$, it should be zero. But this is not the case: one can check that Right always wins this game since if he starts, he can choose to play into the second game, leading to $\left\{\frac{1}{4}\middle|1\right\} - \frac{1}{2} = \left\{\frac{1}{4}\middle|1\right\} + \left\{-1\middle|0\right\}$, then Left has to play into the first game, leading to $\left\{0\middle|\frac{1}{2}\right\} + \left\{-1\middle|0\right\}$, which is a winning position for Right. We can check that the right value here is $\frac{1}{2}$!*

Generalizing the last observations, the following result justify all our previous work.

Theorem 1.17. *Let $\mathcal{G} = \left\{g_1^L, g_2^L, \dots \middle| g_1^R, g_2^R, \dots\right\}$ be a game such that all options are numbers, and $g_i^L < g_j^R$ hold for all i, j. Then it is equal to a number, which is the simplest number x such that $g_i^L < x < g_j^R$ for all i, j. Here, simplest means "with the highest position in the bamboo sticks value tree T", or, equivalently, "equal to the value of the shortest possible bamboo stick greater that all Left options and smaller that all Right options".*

Proof. Denote by $x = \{x^L|x^R\}$ the highest number in the tree T which is larger than the g_i^L's and smaller than the g_j^R's. Let us show that the game $\mathcal{G} - x$ is zero, *i.e.* the second player wins. We have

$$\begin{aligned} \mathcal{G} - x &= \left\{g_1^L, g_2^L, \dots \middle| g_1^R, g_2^R, \dots\right\} + \left\{-x^R | - x^L\right\} \\ &= \left\{g_1^L - x, g_2^L - x, \dots, -x^R + \mathcal{G} \middle| g_1^R - x, g_2^R - x, \dots, -x^L + \mathcal{G}\right\}. \end{aligned}$$

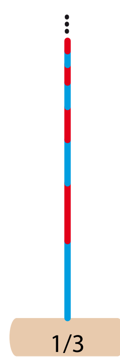
Since $g_i^L < x$, all Left's options, except the last ones are negative numbers. Since x in the highest number is T with the desired property, x^R, which is higher than x, must be larger than one of the g_j^R's! Otherwise we would have chosen x^R instead of x. Therefore $\mathcal{G} - x^R$ is also negative, hence all Left's options are negative. In the same way, all Right's options are positive. Thus $\mathcal{G} - x$ is zero, and $\mathcal{G} = x$. □

Corollary 1.18. *Any position of Hackenbush is a number.*

With the help of theorem 1.17, one can also compute recursively the value of any game of Hackenbush. This is done by induction on the total number of moves from a position. The point is that in general, this procedure is not much easier nor faster than computing the whole graph of the game's positions. In order to make this computation tractable, we have to find on every game simplification rules, as we did for bamboo sticks with lemma 1.14, or sums of games with lemma 1.5.

1.7. **Real and sur-real numbers.** We have seen some games whose values are dyadic numbers by considering finite games. One can then wonder if it is possible to construct all the real numbers by this way? The answer is yes, provided we leave the world of finite games. We do not go too far from this world: we keep the hypothesis that the game ends in finite time. The point is that this finite time will not be bounded before the game is played. How does all of this works?

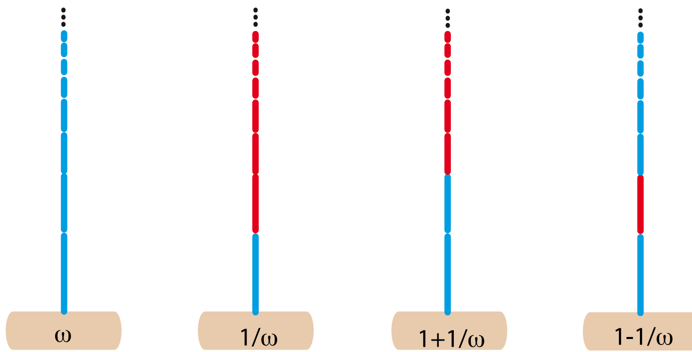
Remember – or learn – how real numbers are constructed by Dedekind: a real number is defined as the set of rationals which are lower than him. Likewise, we defined numbers associated to games inductively, by defining a number of depth d in the bamboo tree T as a number lying between some numbers of depth at most $d-1$ which are smaller – the Left options – and some others which are larger – the Right options. Therefore this is natural to construct a game equal to $\frac{1}{3}$ by giving to Left options which are smaller than $\frac{1}{3}$, giving to Right options which are larger than $\frac{1}{3}$ in such a way that no number simpler than $\frac{1}{3}$ will fit. Consider the following infinite stick.



Two consecutive edges are of different colors. Starting from this position, the game ends after a finite time, since after the first cut, there will remain only finitely many edges. Hence the apparent infinity of the beginning position is not a problem for us. Comparing this game with previously constructed numbers, one checks that this game is indeed larger than any other smaller than $\frac{1}{3}$, and smaller than any other larger than $\frac{1}{3}$. Therefore, we can safely DEFINE $\frac{1}{3}$ as this game.

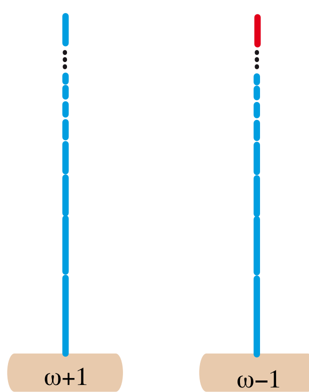
Continuing on this path, we can then define any real number. We leave it as an exercise two compute the stick associated to any number from its dyadic expansion.

The surprising thing is that many other numbers arise is the same way. Consider the following sticks:

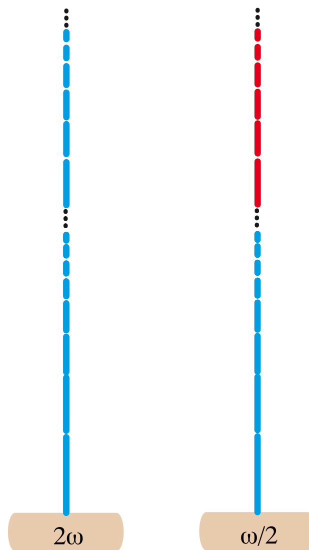


The first stick is a strictly positive game, larger than any integer. Therefore, we call it ω, and declare this number greater than all the real numbers. Then, trying to invert it, we obtain the second game, which we call $\frac{1}{\omega}$. It is positive, but smaller than any real game. And so on, with the third and the fourth games.

We do not stop there: now that we have infinite sticks, why not continuing on top of them?



These games still ends after finite time, are still larger than any real number, but are different from ω. And now, we are not afraid to combine these sticks.



All the numbers constructed this way are called *surreal numbers*, because there are the next step after the real with this so-called Conway-construction. We leave the construction of other surreals like $\omega^2, \omega^\omega, \frac{\omega}{3}, \sqrt{\omega}$ and many others as exercises.

We close this section by pointing out that all the numbers constructed using only bLue edges form the set of *ordinal numbers*, which are very useful in all mathematics, when recursion on the set of the integers is not enough. Their important property is that there is no infinite decreasing sequence, *i.e.* every game ends in finite time.

2. Non-partizan games and nimbers: Sprague-Grundy theory

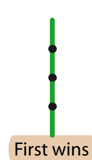
So far we made a very strong hypothesis on our games: First player never wins. This hypothesis was very fruitful, since it allowed us to construct all real numbers, and many more. But it is now time to drop it. In order to keep a tractable analysis, we add another strong – and in a sense orthogonal – hypothesis in this section: both players always have the same options. The immediate corollary is that nor Left nor Right never wins: the winner is either the First player, either the Second. Let us see an example.

Rules 2.1 (Nim/Green hackenbush)**.** Positions: *A finite number of green bamboo sticks of finite height;*

Moves: *A player chooses one bamboo stick and cuts as many edges as he wants;*

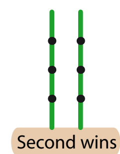
Loser: *As usual, who cannot play loses.*

Of course, if there is no stick, the First player loses. If there is one stick, a new situation is encountered:

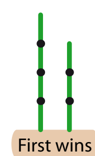


the First player finally wins! He just has to cut all the sticks, and then second player loses.

With two sticks, remember the mirror strategy that worked for Hackenbush (lemma 1.8): if we consider two copies of any game, reversing options for Left and Right, the result of this sum is always zero. Since Left and Right have the same options in Nim, reversing does not change a position, then we deduce that two copies of a position always yield a zero position: if you are the second player, you just have to mimic your opponent's moves for winning the game. Therefore, in the Nim game, if the position consists of two single sticks of the same height, the second player wins. His strategy is two cut exactly as many edges as the first did, but in the other stick.



Then one easily deduce who wins if the two sticks are of different heights: First begins by equalizing the sticks, then Second loses. So First wins if and only if the sticks are of different heights.



2.1. **Nimbers.** Remember that we defined a zero game to be a game in which the second players wins, and lemma 1.5 tells us that we can simply subtract such a game when we meet him. Let us now introduce new numbers, called *nimbers*: write $*n$ for the Nim stick with n edges. Since one can cut as many edges as we want from a stick, we deduce the following inductive definition for nimbers:

$$*n = \left\{0, *1, *2, \ldots, *(n-1) \middle| 0, *1, *2, \ldots, *(n-1)\right\}$$

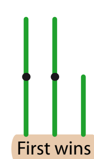
. We can now rephrase our conclusion by the equations

$$*n \neq 0, \text{ for } n > 0,$$

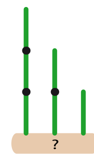
$$*n + *n = 0,$$

$$*m \neq *n, \text{ for } m \neq n,$$

Now we can ask about more complicated values like $*2 + *2 + *1$? Since $*2 + *2 = 0$, this game is equivalent to $*1$, and one check that First wins in this situation.



What about $*3 + *2 + *1$?



It is not hard to show that in this game, Second wins, giving us the equation $*3 + *2 + *1 = 0$, which can be also reformulated by adding $*1$ on both sides $*3 + *2 = *1$, or even $*2 + *1 = *3$, or $*1 + *3 = *2$.

So we see that nimbers, like numbers, seem to be summable. But how to determine the rules? In order to answer, let us turn to another game.

2.2. **The poker Nim and the mex-rule.** In order to understand the nimber addition rules, we will introduce a variant of Nim in which player can postpone a little bit the end of the game.

Rules 2.2 (Poker Nim)**.** Positions: *A finite number of green bamboo sticks of finite height, a finite number of edges in Left's pocket, and a finite number of edges in Right's pocket (not necessarily the same number as in Left's);*

Moves: *A player chooses one bamboo stick. Then either he cuts as many edges as he wants, either he adds from his pocket as many edges as he wants (but no more that what remains in his pocket);*

Loser: *As usual, who cannot play loses.*

Although the game is not any more symmetric, the choice we gave to the players does not affect the result.

Proposition 2.3. *Given a position of poker Nim, the winner is the same as the winner of the corresponding Nim position.*

Proof. The key is that the move consisting in adding edges to a bamboo stick is reversible: if your opponent adds k edges, you can cut them just after. Since his reserves are finite, he can only add edges a finite number of times. If you can win in the corresponding Nim position, this strategy makes you win in the poker Nim position. □

In terms of nimbers, we obtain

$$\left\{0, *1, *2, \ldots, *(n-1), *(n+1), \ldots \middle| 0, *1, *2, \ldots, *(n-1), *(n+1), \ldots\right\} = *n,$$

where the second dots in each side represent some nimbers greater than $*(n+1)$. In other words,

Theorem 2.4. *Any position of Nim game $\mathcal{G}$ is equal to a nimber, which is the* MINIMAL EXCLUDED NUMBER *from the different positions that can be reached from $\mathcal{G}$.*

This rule determining the value of a sum is called the *mex-rule*, "mex" for m(inimal) ex(cluded). The proof of proposition 2.3 in actually more general than poker Nim. It is one of the oldest result of combinatorial game theory

Theorem 2.5 (Sprague-Grundy)**.** *Any finite non partizan game is equal to a Nim game.*

Example 2.6. *Let us come back to the equation $*2 + *1 = *3$. By definition of the sum, the game $*2 + *1$ is equal to*

$$\left\{0 + *1, *1 + *1, *2 + 0 \middle| 0 + *1, *1 + *1, *2 + 0\right\}.$$

We already know $*1 + *1 = 0$*, therefore we have*

$$*2 + *1 = \left\{ *1, 0, *2 \middle| *1, 0, *2 \right\} = *3$$

by the mex-rule.

Example 2.7. *Now let us try to compute* $*3 + *1$*. By definition, the game* $*3 + *1$ *is equal to*

$$\left\{ 0 + *1, *1 + *1, *2 + *1, *3 + 0 \middle| 0 + *1, *1 + *1, *2 + *1, *3 + 0 \right\},$$

as before, we can symplify $*1 + *1$ *by* 0*,* $*2 + *1$ *by* $*3$*, and we thus obtain*

$$*3 + *1 = \left\{ *1, 0, *3, *3 \middle| *1, 0, *3, *3 \right\} = *2,$$

by the mex-rule.

In the same way we can check that $*3 + *2 = *1$, but we already know it from the to preceding example and the mirror strategy.

Now we can compute an addition table for nimbers by successive applications of the mex-rule. This table is constructed as follows: in the first row and first column we write all nimbers in increasing order (corresponding to the trivial equality $*n + 0 = *n$. Then, starting from the top left corner, we put in a cell the smallest nimber that is nor in top of the cell, neither on the left.

+	0	*1	*2	*3	*4	*5	*6	*7	*8	*9
0	0	*1	*2	*3	*4	*5	*6	*7	*8	*9
*1	*1	0	*3	*2	*5	*4	*7	*6	*9	*8
*2	*2	*3	0	*1	*6	*7	*4	*5	*10	*11
*3	*3	*2	*1	0	*7	*6	*5	*4	*11	*10
*4	*4	*5	*6	*7	0	*1	*2	*3	*12	*13
*5	*5	*4	*7	*6	*1	0	*3	*2	*13	*12
*6	*6	*7	*4	*5	*2	*3	*0	*1	*14	*15
*7	*7	*6	*5	*4	*3	*2	*1	0	*15	*14
*8	*8	*9	*10	*11	*12	*13	*14	*15	0	*1
*9	*9	*8	*11	*10	*13	*12	*15	*14	*1	0

The first remark one can do are the following: as expected, 0's are on the diagonal. This means that the second player wins in a two-sticks game if and only if the sticks have the same height. The second remark is that we can see square blocks of size 2^n for each n: when going 2^n cells on the right, one sees the same nimber plus or minus 2^n. For example the block $\{*4, \cdots *7\} \times \{0, \cdots *3\}$ is the same as the block $\{0, \cdots *3\} \times \{0, \cdots *3\}$ plus 4. This observation can be generalized in order to get a general formula for computing the sum of two nimbers. We define the addition $\mod 2$ for digits by $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$.

Theorem 2.8. *Let* $*m$ *and* $*n$ *be two nimbers. Then the digits of their Nim-sum are the sum* $\mod 2$ *of their digits.*

Proof. We leave this induction as an exercise □

For example, the sum of $*3 = *11^{(2)}$ and $*5 = *101^{(2)}$ is $*110^{(2)} = *6$. This simple operation gives an easy way to determine the winning move in the Nim game.

Example 2.9. *The position* $*1 + *3 + *5 + *7$ *(also called Marienbad in reference to the movie* Last year in Marienbad*) is equal to the nimber* $*1^{(2)} + *11^{(2)} + *101^{(2)} + *111^{(2)} = 0$*. Therefore the first player loses. From the position* $*1 + *3 + *5 + *6 = *1^{(2)} + *11^{(2)} + *101^{(2)} + *110^{(2)} = *1$*, a winning move is a move modifying only the last digit of one of the nimbers, then it is* $*1 \mapsto 0$ *or* $*5 \mapsto *4$*, since the other moves modify more than the last digit.*

2.3. **The hungry knight.** As an illustration of the Sprague-Grundy theorem, consider the following game.

Rules 2.10 (Hungry knight)**.** Positions: *A chess knight an a finite chessboards, with a finite numbers of breads in his pockets.*

Moves: *A player can move the knight as a chess knight, but only in the four NNE, NNW, NWW and SWW directions, or he can order him to eat as many breads from his pockets as he wants.*

Loser: *If the knight cannot move nor eat, the player loses.*

Our theory applies perfectly here, since our game is the sum of a Nim game with one stick (the bread game) and another non partizan game (the knight's move). So, any position is a nimber, the sum of the nimber associated to bread, *i.e.* the number of breads, and the nimber associated to the knight. Starting from the 4 cells in the *NW* corner (which are the terminal positions for the knight), we inductively compute nimbers associated to each cell.

0	0	*1	*1	0	0	*1	*1	0	0
0	0	*2	*1	0	0	*1	*1	0	0
*1	*2	*2	*2	*3	*2	*2	*2	*3	*2
*1	*1	*2	*1	*4	*3	*2	*3	*3	*3
0	0	*3	*4	0	0	*1	*1	0	0
0	0	*2	*3	0	0	*2	*1	0	0
*1	*1	*2	*2	*1	*2	*2	*2	*3	*2
*1	*1	*2	*3	*1	*1	*2	*1	*4	*3
0	0	*3	*3	0	0	*3	*4	0	0

It turns out that this table is ultimately periodic, we leave this as an exercise. The main thing is that we can now evaluate the position of a hungry knight: it is the Nim-sum of his position and his nimber of breads! For example, if the knight begin at coordinates $(3, 6)$, with 4 breads in his pocket, then First has a winning strategy consisting in reducing the number of breads to 2, since the position is $*2$.

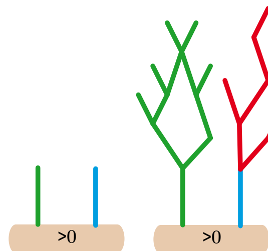
3. Mixing nimbers and numbers

Let us now mix the two theories we have seen so far. We already know that some games are numbers, and some others are nimbers. What if we consider them together?

Rules 3.1 (Tricolor hackenbusch)**.** *Now there are bLue, Red and green edges. Left can cut bLue and green, Right can cut Red and green. The victory conditions remain the same: who cannot play loses.*

With bLue/Red hackenbush we have seen that Left, Right or Second wins, while with green hackenbush, First or Second wins. Thus in tricolor hackenbush all four positions outcomes arise. Remember that a game $\mathcal{G}$ is *fuzzy* if First wins. We write it $\mathcal{G}||0$. In green hackenbush, the sum of two fuzzy games was either fuzzy, or zero. Is it still true in our tricolor hackenbush?

3.1. **Nimbers and numbers.** Let us first compare nimbers and numbers.



Proposition 3.2. *The sum of a positive number and a nimber is still a positive game*

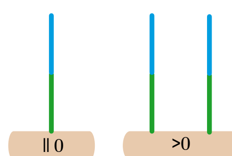
Proof. Since playing in a number is losing some advantage, the players play the nimber at first. Then they turn to the number, which is won by Left, whoever won the nimber. □

Therefore sums of numbers and nimbers are very simple: although they may not be numbers, the winner is decided by the number.

From now on we will simplify our notation, writing $*$ instead of $*1$. We already now the equations $* + * = 0$ and $* + 0 = *$. What about $x + *$, where x is a real number? Let us apply our definition of sum, we get $1 + * = \{0|\} + \{0|0\} = \left\{\{0|0\}, \{0|\}\middle|\{0|\}\right\} = \left\{*, 1\middle|1\right\} = \{1|1\}$, since $1 > *$. The same proof shows for any x real the equality

$$x + * = \{x|x\}.$$

3.2. **Strange behavior around zero.** Consider the following positions.



The first game is $\left\{0, *\middle|0\right\} = \left\{0, \{0|0\}\middle|0\right\}$, which First always wins. But if we double it, we can easily check that Left always wins!

From this example we can deduce that none can be said about the sum of a general fuzzy game with any other. In particular if stars appear as options, we need a finer study.

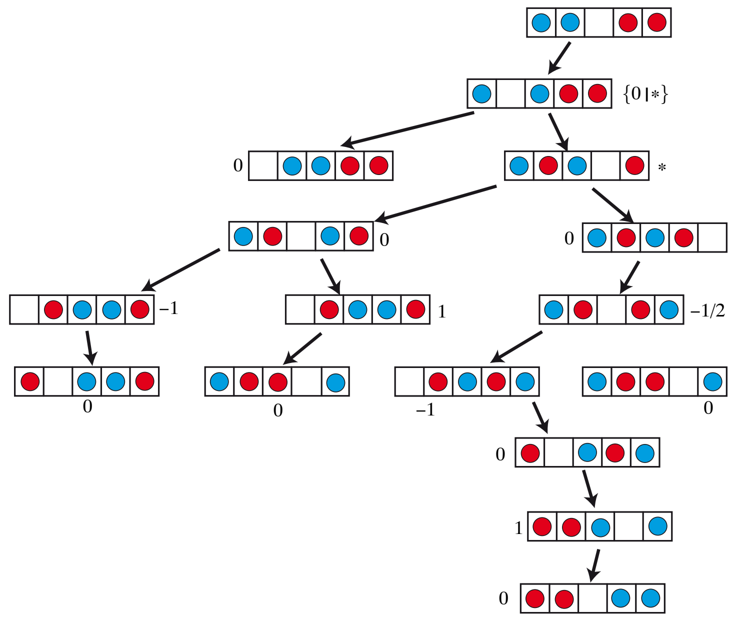
Rules 3.3 (Tods and frogs)**.** Positions: *A finite collection of bands, which consists of a finite number of squares, some of which contains bLue tods, some of which contains Red frogs;*

Moves: *Left chooses a tod and make him move one square eastwards if possible, or jump to the next square if the first east square is occupied by a frog and the next one is free. Similarly, Right chooses a frog and make him move westwards if possible, or jump to the next west square if the first one is occupied and the next one is free;*

Loser: *As usual, who cannot move loses.*

If there is at most one frog and at most one tod per row, it is easy to see that the associated game is the sum of a number and a nimber. This situation is not new for us.

Now consider a position with five squares, two frogs and two tods. Let us construct the whole tree asuming that Left starts.



We then encounter a new game corresponding the the second row: $\{0|*\}$. Let us denote it $\uparrow$, say "Up". Since Left always wins, this is a positive game. Is it a number? No, since its Left's option – namely the game 0 – is not smaller than its Right's – the game $*$. Is it a nimber? No, since it is positive.

Proposition 3.4. *For any positive surreal number x, we have the inequality $\uparrow < x$.*

Proof. Play the game $\uparrow - x$. Since $\uparrow = \left\{0\middle|\{0|0\}\right\}$, Right can always plays in this game, until it is zero. Then, there remain only $-x$ or a smaller surreal number if Left has already played in x, a win for Right. □

3.3. **All small games.** Is $\uparrow$ an isolated phenomenon or are there many such small games? At least, one can add $\uparrow$'s, yielding to the hierarchy $\cdots < -2\uparrow < -\uparrow < 0 < \uparrow < 2\uparrow < \ldots$. The small Hackenbush position $\left\{0, *\middle|0\right\}$ is also in the same family:

Lemma 3.5. *We have $\left\{0, *\middle|0\right\} = \uparrow + *$.*

Proof. Let us play the difference $\left\{0, *\middle|0\right\} - \uparrow - * = \left\{0, *\middle|0\right\} + \{*|0\} + *$.

If Right goes into the first game, Left goes into the second, giving the position $* + * = 0$ – a win. If Right goes into the second game, Left answers by $*$ in the first game – a win for the same reason. If Right goes into the third star, then Left goes to the star in the second game; if Right answers in the first game, then the position is $*$ – a win for Left – and if Right answers in the second star, Left goes to the first 0 – still a win.

The arguments if Left begins are similar. □

It is now clear that $\left\{0, *\middle|0\right\} + \left\{0, *\middle|0\right\}$ is positive, since it is equal to $\uparrow + * + \uparrow + * = \uparrow + \uparrow$. What make these games so small and close to fuzzy games? The main thing is to notice is that if all edges related to the ground in tricolor Hackenbush are green, then the game always end with no remaining bLue or Red edge.

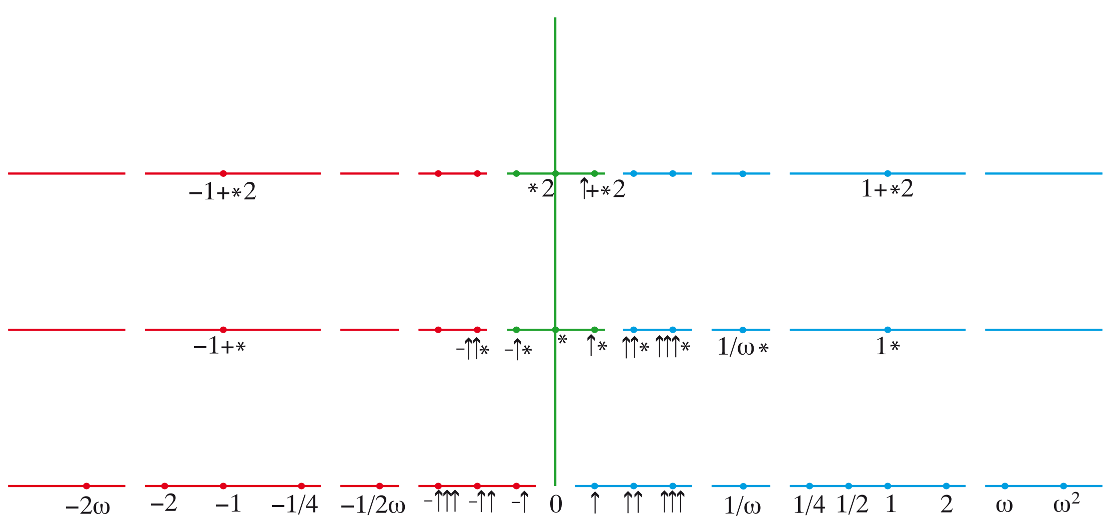
Definition 3.6. *A game is said to be* all small *if it cannot reach any number except* 0*.*

Proposition 3.7. *An all small game is smaller than any positive surreal number, and larger than any negative surreal number.*

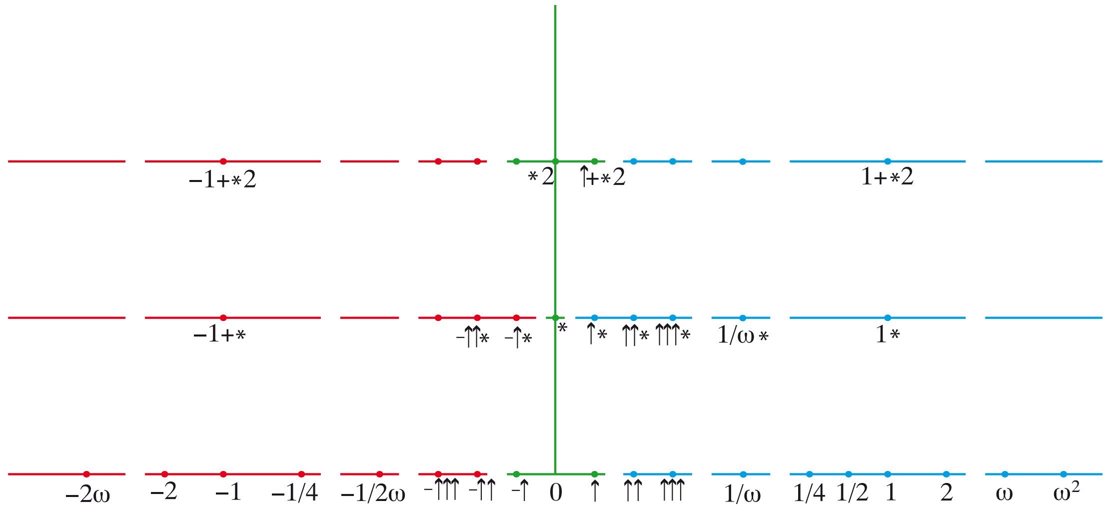
Proof. The argument is the same as in the proof of proposition 3.4. □

As we saw with $\uparrow + *$, all small games provide new fuzzy games, namely all games of the form $\uparrow + * n$ or $- \uparrow + * n$.

The following diagram represent the relative positions of all games we have seen so far. Positive games are in blue color, negative in red, and fuzzy in green.



Note that if we now compare games with $*$, we obtain different colors for $\uparrow$ and $\uparrow *$. On the following diagram, games larger than $*$ are in blue, lower than $*$ in red, and incomparable with $*$ are in green.



We have seen what happens when adding the stars and arrows. Note that when arrows arise as options, determine if two games are equal is all but trivial. For example, we let as exercises the following equalities:

Proposition 3.8. *We have* $\{\uparrow | - \uparrow\} = \{0 | - \uparrow\} = \{\uparrow | 0\} = *$

4. Hot games. Temperature

In all the games we have seen so far, players never gained significative advantages when playing. This situation is not typical: in many games, some good moves drastically improve your position.

From now on, we adopt the convention that players stop playing when they reach a number: if is positive, then they agree on Left's victory, if negative on Right's, and if zero on Second's.

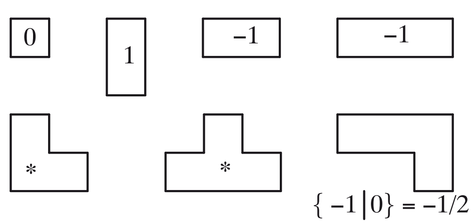
4.1. **Hot games.**

Rules 4.1 (Domineering)**.** Position: *A finite set of finite polyominos;*

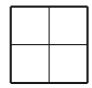
Moves: *Left places a vertical domino on two neighboring cells, deleted these cells; Right places a horizontal domino on two neighboring cells, deleting these cells;*

Loser: *As usual, who cannot play loses.*

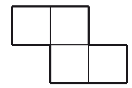
First one checks that playing on several polyominos is like playing the sum of the games associated to each one. Then some simple positions are easy to determine.



Turning to bigger polyominos, some unknown games appear.



$\{1 \mid -1\}$

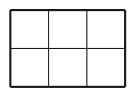


$\{0 \mid -1\}$

The game $\{1|-1\}$ is fuzzy, *i.e.* First player wins. But it is more than, since if we add or substract the game $\frac{1}{2}$, it is still fuzzy!

Definition 4.2. *A game is said* hot *if some Left option is strictly larger than some Right option. An example of hot game is the game* $\{x|y\}$*, where* x, y *are numbers satisfying* $x > y$*, we call it a* switch game*.*

The following position of domineering provides an example of non-centered switch.



Cancelling non optimal options, one checks that it is equal to the game

$$\left\{2, \{1|-1\}\middle| -\frac{1}{2}\right\} = \left\{2\middle| -\frac{1}{2}\right\}.$$

Proposition 4.3. *Let* $\{x|y\}$ *be a switch game, then for any number* z*,*

- *if* $z < y$*, then* $z < \{x|y\}$*;*
- *if* $y \leq z \leq x$*, then* $z \| \{x|y\}$*;*
- *if* $x < z$*, then* $\{x|y\} < z$*.*

Proof. Exercise. □

4.2. **Adding switches.** What happens when we add switches? First, we can note that it is always better to play in a switch than in a number, since playing in a switch improve your situation, while playing in a number make it worse.

Proposition 4.4 (Number avoidance theorem)**.** *If* $\{x|y\}$ *i a switch and* z *a number, then we have* $\{x|y\} + z = \left\{x + z\middle|y + z\right\}$*.*

Example 4.5. *What happens with the game* $\{2| - 2\} + \{1| - 1\}$*? First player can always win, the best move being in the first game.*

What happens with the game $\{2| - \frac{1}{2}\} + \{1| - 1\}$*? First player also always win. Note that even for Right, the best move is not in the* -1 *but in the* $-\frac{1}{2}$ *which is larger.*

Definition 4.6. *The* temperature *of a switch game* $\{x|y\}$ *is the number* $\frac{x-y}{2}$*.*

Thus the temperature represent the attraction of the game for both player: the hotter the game, the more the player want to play in it.

Proposition 4.7. *In a sum of switches, the best move is always in the switch with the largest temperature.*

Hence a sum of switches is always equal to some $z + \{a| - a\} + \{b| - b\} + \{c| - c\} + \dots$, with $a \geq b \geq c \geq \cdots \geq 0$. If Left begins, players will stop at $z + a - b + c - \dots$, and at $z - a + b - c + \dots$ if Right starts.

4.3. **Switches as options.** What is a game like $\left\{3\middle|\{1|0\}\right\}$ worth? Is it larger than 2? than $\{2|1\}$? It is easy to see that it cannot be compared with both games: First wins the difference. Still if we meet a complicated sum of games in which switches arise as options, we would like a strategy saying where the best move is. Unfortunately, this question is hard, namely

Theorem 4.8 (Yedwab-Moews, 1994)**.** *Deciding if Left has a winning strategy in a sum of games of the form $\left\{a\middle|\{b|c\}\right\}$, where $a \geq b \geq c$ are numbers, is NP-hard.*

Note that this problem is not proved to be in NP. Assume you have a strategy, verifying that it gives you the victory, whatever plays the opponent, is non trivial.

Then, we cannot hope for a complete classification, nor a strategy. Nevertheless we will end these notes by giving a nice heuristic for evaluating a position.

Example 4.9. *Denote by $\mathcal{H}$ the game $\left\{\{2|1\}\middle| -1\right\}$, then $\mathcal{H}$ is a fuzzy game: if Left starts, then the game goes to 1, and if Right starts, it goes to -1. Can we then say that $\mathcal{H}$ has mean value 0, whatever it means? Actually $\mathcal{H}$ satisfies the equation $\mathcal{H} + \mathcal{H} + \mathcal{H} + \mathcal{H} = 1$. To check this, note that both player prefer play in $\mathcal{H}$ than in any suboption. This implies that after four moves, the situation becomes $\{2|1\} + \{2|1\} + (-1) + (-1)$. At this point, players prefer to play switches, and after two more moves the situation is $2 + 1 + (-1) + (-1) = 1$.*

Then, the mean value of $\mathcal{H}$, if defined, should rather be $\frac{1}{4}$ than 0.

Definition 4.10. *Suppose that $\mathcal{G}$ is a game and n an integer such that there exists a real number x satisfying $n.\mathcal{G} = x$, then $\mathcal{G}$ is said to have* mean value $\frac{x}{n}$.

We want to show that for a game which is a composition of switches, the mean value is well defined, and give a way to compute it. Of course the mean value gives you information on $n.\mathcal{G}$ for n large is enough. It might not help for evaluating $\mathcal{G}$. But, at least it gives you some piece of information.

4.4. **Cooling down a game.** Since temperature represent the desire for both player to play a game, an idea for calming the game is to decrease this excitement, by adding a price to each move. Then players will only play an option if it significantly improve their situation.

Example 4.11. *Consider the game $\mathcal{G} = \{2|1\}$, and define $\mathcal{G}_t$ as the game $\left\{2 - t\middle|1 + t\right\}$. Then for $t < \frac{1}{2}$, $\mathcal{G}_t$ is still switch game, but its temperature is $1 - 2t$. For $\frac{1}{2} \leq t < 1$, it is the number $1\frac{1}{2}$, and for larger t it becomes 1, and then ultimately 0. Since we said that both players agree on stopping the game when a number is reached, we would like to define the mean value of $\mathcal{G}$ as the first encountered number when cooling down the game.*

Definition 4.12. *A game $\mathcal{G}$ is called a* recurswitch *if it is obtained by composing and adding switches.*

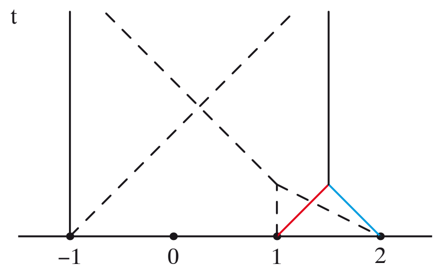
Let $\mathcal{G} = \left\{g_1^L, \dots | g_1^R, \dots\right\}$ be a recurswitch. Then define inductively the game $\mathcal{G}_t$ as the game

- $\left\{(g_1^L)_t - t, \dots | (g_1^R)_t + t, \dots\right\}$,
- *m if there exists $t_{\mathcal{G}} < t$ such that $\mathcal{G}_{t_{\mathcal{G}}}$ is equal to the number m.*

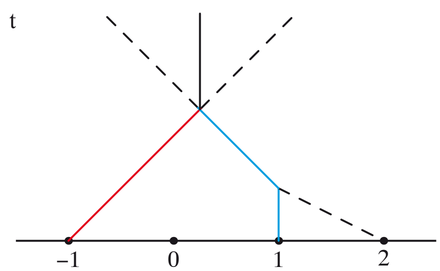
If such a $t_{\mathcal{G}}$ exists, we call it the temperature *of $\mathcal{G}$, and $m_{\mathcal{G}} = m$ the* mast *of $\mathcal{G}$.*

Note that if $\mathcal{G}$ is already a number, then $\mathcal{G}_t = \mathcal{G}$ for all t.

Example 4.13. *Let us see how this works with the recurswitch* $\mathcal{H} = \left\{\{2|1\}\middle| -1\right\}$*. Since the definition is recursive, we need first to compute* $\{2|1\}_t$*. If* $t < \frac{1}{2}$*, then* $\{2|1\}_t = \left\{2 - t\middle|1 + t\right\}$*. Since* $\left\{2 - \frac{1}{2}\middle|1 + \frac{1}{2}\right\} = 1\frac{1}{2}$*, for* $t \geq \frac{1}{2}$*, we have* $\{2|1\}_t = 1\frac{1}{2}$*.*



On this picture, called thermograph*, we see representations of the games* $(-1)_t$ *and* $\{2|1\}_t$*, as t varies. When issue depend on who starts, we draw in bLue the outcome if Left starts, and in Red if Right starts. In dotted lines we then indicate the games* $(-1)_t + t$ *and* $\{2|1\}_t - t$*, which are used for the computation of* $\mathcal{H}_t$*.*



In $\mathcal{H}_t$*, if Left starts, he goes to* $\{2|1\}_t - t$*, and Right to play. Thus the right frontier of the game* $\mathcal{H}_t$ *is obtained by considering the left frontier of the game* $\{2|1\}_t - t$*. Similarly, if Right starts, he goes to* $-1 + t$*, and Left to play. Hence the left frontier of* $\mathcal{H}_t$ *is obtained by considering the right frontier of* $-1 + t$*. These two lines crosses at temperature* $t = 1\frac{1}{4}$*, the associated game being* $\frac{1}{4}$*, yielding the above picture. The game* $\mathcal{H}_t$ *is therefore equal to*

- $\{1| - 1 + t\}$ *for* $t < \frac{1}{2}$*;*
- $\{1\frac{1}{2} - t| - 1 + t\}$ *for* $\frac{1}{2} \leq t < 1\frac{1}{4}$*;*
- $\frac{1}{4}$ *for* $1\frac{1}{4} \leq t$*.*

Although more complicated than a single switch, $\mathcal{H}$ *has a well-defined temperature (*$1\frac{1}{4}$*), and its mast (*$\frac{1}{4}$*) is equal to the mean we previously defined.*

Let us prove that these two remarks hold in the general case.

Definition 4.14. *The* thermograph *of a recurswitch* $\mathcal{G}$ *is the diagram associating to any temperature t the two* stop numbers s_t^L *(resp.* s_t^R*) at which* $\mathcal{G}_t$ *stops if Left (resp. Right) starts.*

Remember that both players agreed to stop the game whenever they reach any number x, giving the victory according to the sign of the x. Graphically, we represent the Left stops in bLue, the Right stops in Red, and in black if both are the same number (corresponding to the case $t \geq t_{\mathcal{G}}$).

Lemma 4.15. *For any games* $\mathcal{G}$ *and* $\mathcal{H}$*, we have* $(\mathcal{G} + \mathcal{H})_t = \mathcal{G}_t + \mathcal{H}_t$*.*

Proof. Exercise. Pay attention that the definition of $\mathcal{G}_t$ changes for t large enough when a number is encountered. □

Lemma 4.16. *Let $\mathcal{G}$ be a recurswitch. Then the Left stops and Right stops of $\mathcal{G}$ form piecewise affine lines in the thermograph, whose piecewise slopes are $\frac{1}{n}$ for some relative integer n.*

Proof. This is obviously true for numbers: the slope is $\infty = \frac{1}{0}$. Lemma 4.15 asserts that the sum of two thermographs is obtained by adding them. Since the sum of lines of slopes $\frac{1}{p}$ and $\frac{1}{q}$ is a line of slope $\frac{1}{p+q}$, the property remains true under addition.

When it comes to options, the Left boundary of the thermograph of $\mathcal{G}$ is obtained by taking the right boundary of $\mathcal{G}^R$, and adding t. Then if the right boundary of $\mathcal{G}^R$ was of slope $\frac{1}{p}$, it becomes $\frac{1}{p+1}$ when adding t. So the property is preserved. The same argument works for the right boundary. □

Proposition 4.17. *Any recurswitch $\mathcal{G}$ has a temperature and a mast.*

Proof. If $\mathcal{G}$ is the sum of recurswitches, then we see directly on the thermograph that the temperature is smaller than the maximal temperature of the summands, and that the mast is the sum of the masts.

If $\mathcal{G}^L$ and $\mathcal{G}^R$ are recurswitch, then for $t > t_{\mathcal{G}^L}$, $\mathcal{G}^L_t - t$ is a line of slope $+1$, and for $t > t_{\mathcal{G}^R}$, $\mathcal{G}^L_t + t$ is a line of slope -1. Therefore these two lines cross at some point $(m_\mathcal{G}, t_\mathcal{G})$. □

For any game $\mathcal{G}$ and any positive integer n, denote $n.\mathcal{G}$ the sum of n copies of $\mathcal{G}$. We then have the following:

Corollary 4.18. *For any recurwitch $\mathcal{G}$, we have*

$$n.m_\mathcal{G} - t_\mathcal{G} \leq n.\mathcal{G} \leq n.m_\mathcal{G} + t_\mathcal{G}.$$

Proof. The temperature of a sum is lower than the maximum of the temperature of summands, this implies that the temperature of $n.\mathcal{G}$ is at most $t_\mathcal{G}$. Since the mast is additive, the mast of $n.\mathcal{G}$ is $n.m_\mathcal{G}$. □

This corollary shows that, within a bounded error, many copies of $\mathcal{G}$ are equal to many copies of its mast. Although we did not succeed in defining a satisfactory value for a recurswitch, the mast value provide a way to evaluate what many copies of it are worth.

References

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