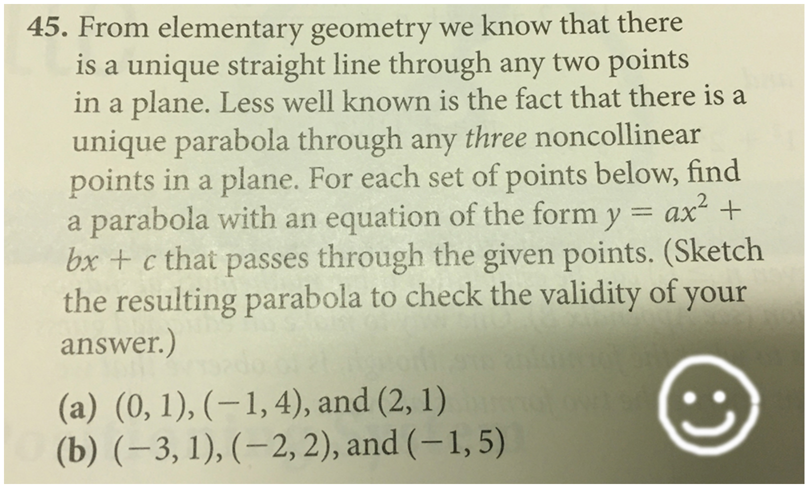
EXERCISES

DUBNA 2018: LINES ON CUBIC SURFACES

Exercise 1. The following problem is from *Linear Algebra, A Modern Introduction* by David Poole (2014).



45. From elementary geometry we know that there is a unique straight line through any two points in a plane. Less well known is the fact that there is a unique parabola through any *three* noncollinear points in a plane. For each set of points below, find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through the given points. (Sketch the resulting parabola to check the validity of your answer.)

(a) $(0,1), (-1,4)$, and $(2,1)$

(b) $(-3,1), (-2,2)$, and $(-1,5)$

The sentence "Less well known is the fact that there is a unique parabola through any three noncollinear points in a plane" is mathematically wrong. In this problem, Poole assumes that parabola is the curve in $\mathbb{R}^2$ that is given by the equation

$$y = ax^2 + bx + c$$

for some real numbers a, b and c. This assumption is a bit weird, since parabolas were used long before René Descartes introduced Cartesian coordinates. Moreover, this definition of parabola discriminates x-coordinate, which is not appropriate ☺. The goal of this exercise is to solve this problem using *good* definition of parabola: parabola is a subset in $\mathbb{R}^2$ such that there exists a composition of rotations and translations that maps it to the curve given by

$$y = px^2,$$

where p is a positive real number.

(a) Find all parabolas in $\mathbb{R}^2$ that pass through the points $(0,1)$, $(-1,4)$, $(2,1)$, $(19,20)$.
(b) Find all parabolas in $\mathbb{R}^2$ that pass through the points $(0,1)$, $(-1,4)$, $(2,1)$, $(9,10)$.
(c) Describe all parabolas in $\mathbb{R}^2$ that pass through the points $(0,1)$, $(-1,4)$, $(2,1)$.
(d) Let P be a point in $\mathbb{R}^2$ that is different from $(0,1)$, $(-1,4)$, $(2,1)$. Explain when there exists a parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and P.

Exercise 2. Let Σ be a subset in $\mathbb{P}^2_{\mathbb{C}}$ such that Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$.

(a) Suppose that $|\Sigma| \leqslant 6$. Prove that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.
(b) Suppose that $|\Sigma| = 7$. Prove that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.
(c) Suppose that $|\Sigma| = 8$. Prove that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

Exercise 3. Do the following:

(a) Find all lines in $\mathbb{P}^2_{\mathbb{C}}$ that contains exactly 2 points among

$$[0:0:1], [0:1:1], [1:1:-1], [1:3:1], [2:5:1], [1:1:1], [1:4:2].$$

(b) Find a smooth conic $C \subset \mathbb{P}^2_{\mathbb{C}}$ such that C contains the points

$$[0:0:1], [0:1:0], [1:0:0],$$

the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents the conic C at the point $[1:0:0]$ is given by $y-z=0$, and the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents C at the point $[0:0:1]$ is given by $y+2x=0$.

(c) Find all smooth conics in $\mathbb{P}^2_{\mathbb{C}}$ that passes through

$$[1:0:2], [3:1:2], [1:2:1], [1:1:1],$$

and tangent to the line $x+2y+z=0$.

Exercise 4. Observe that no three points among the four points $[1:2:3]$, $[1:0:-1]$, $[2:5:1]$ and $[-1:7:1]$ in $\mathbb{P}^2_{\mathbb{C}}$ are collinear.

(a) Find the projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([1:2:3]) = [1:0:0]$, $\phi([1:0:-1]) = [0:1:0]$, $\phi([2:5:1]) = [0:0:1]$ and $\phi([-1:7:1]) = [1:1:1]$.

(b) Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$-xy + 2y^2 - 3xz + 7yz + 3z^2 = 0.$$

Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by $xy = 0$.

(c) Let $\mathcal{C}$ be the conic in $\mathbb{P}^2$ that is given by

$$x^2 + xy - 2y^2 + 3xz + 3yz + z^2 = 0.$$

Then $\mathcal{C}$ contains the point $[-2:1:3]$. Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([-2:1:3]) = [0:0:1]$ and $\phi(\mathcal{C})$ is given by $xz + y^2 = 0$.

Exercise 5. Let λ be a complex number. Put

$$f(x,y,z) = x^3 + y^3 + z^3 + \lambda xyz.$$

Let C be a subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $f(x,y,z) = 0$. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so that $\omega^3 = 1$. Denote by Σ the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of the following 9 points:

$$[1:-1:0], [1:-\omega:0], [1:-\omega^2:0],$$
$$[1:0:-1], [1:0:-\omega], [1:0:-\omega^2],$$
$$[0:1:-1], [0:1:-\omega], [0:1:-\omega^2].$$

(a) Check that C contains Σ. Show that the set Σ is not contained in any line in $\mathbb{P}^2_{\mathbb{C}}$. Going through all pairs of points in Σ, one can see that every line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that passes through two points in Σ contains another point in Σ. Check this in some cases.

(b) Suppose that $\lambda^3 \neq -27$. Show that there is no point $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Use Bezout theorem to show that the homogeneous polynomial $f(x,y,z)$ is irreducible. Conclude that C is a smooth irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 3. Pick a point $P \in \Sigma$. Find the equation of the line $L_P \subset \mathbb{P}^2_{\mathbb{C}}$ that is tangent to the curve C at the point P. Show that $L_P \cap C = P$.

(c) Suppose that $\lambda^3 = -27$. Show that there are 3 points $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Use Bezout theorem to deduce that the curve C is a union of 3 different lines in $\mathbb{P}^2_{\mathbb{C}}$. Conclude that $f(x,y,z)$ is a product of 3 different polynomials in $\mathbb{C}[x,y,z]$ of degree 1. Find these 3 polynomials explicitly.

Exercise 6. Let $\mathcal{C}$ be the conic in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$4x^2 - 4xy + y^2 - 4xz - 13yz + 12z^2 = 0.$$

Let $P_1 = [0:1:1]$, $P_2 = [-1:4:1]$, $P_3 = [2:1:1]$. Then $\mathcal{C}$ contains the points P_1, P_2, P_3. Let $Q_1 = [19:20:1]$, $Q_2 = [1:2:0]$, $Q_3 = [57:37:49]$. Then $\mathcal{C}$ contains Q_1, Q_2, Q_3.

(a) Show that $\mathcal{C}$ is irreducible. Find the intersection of the conic $\mathcal{C}$ and the line $z = 0$.

(b) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by

$$xz + y^2 = 0.$$

Compute $\phi(P_1)$, $\phi(P_2)$, $\phi(P_3)$, $\phi(Q_1)$, $\phi(Q_2)$ and $\phi(Q_3)$.

(c) Let L_{12}, L_{13}, L_{23}, L_{21}, L_{31}, L_{32} be the lines in $\mathbb{P}^2_{\mathbb{C}}$ defined as follows:

- L_{12} contains P_1 and Q_2; L_{13} contains P_1 and Q_3; L_{23} contains P_2 and Q_3;
- L_{21} contains P_2 and Q_1; L_{31} contains P_3 and Q_1; L_{32} contains P_3 and Q_2.

Find the defining equations of the lines L_{12}, L_{13}, L_{23}, L_{21}, L_{31} and L_{32}. Show that the points $L_{12} \cap L_{21}$, $L_{13} \cap L_{31}$ and $L_{23} \cap L_{32}$ are collinear.

Exercise 7. Put $f(x,y,z) = xy^3 + yz^3 + zx^3$. Let C be a subset in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$f(x,y,z) = 0.$$

(a) Show that there is no point $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Use Bezout theorem to show that $f(x,y,z)$ is irreducible.

(b) Let L be the tangent line to C at $[0:0:1]$. Find $L \cap C$.

(c) Denote by $g(x,y,z)$ the determinant of the matrix

$$\begin{pmatrix} \frac{\partial^2 f(x,y,z)}{\partial x \partial x} & \frac{\partial^2 f(x,y,z)}{\partial x \partial y} & \frac{\partial^2 f(x,y,z)}{\partial x \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial y \partial x} & \frac{\partial^2 f(x,y,z)}{\partial y \partial y} & \frac{\partial^2 f(x,y,z)}{\partial y \partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial z \partial x} & \frac{\partial^2 f(x,y,z)}{\partial z \partial y} & \frac{\partial^2 f(x,y,z)}{\partial z \partial z} \end{pmatrix}.$$

Denote by Z the subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $g(x,y,z) = 0$. Show that $3 \leqslant |C \cap Z| \leqslant 24$.

Exercise 8. Let C_4 be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

(a) Show that the curve C_4 has at most 3 singular points.

(b) Suppose that the curve C_4 has a singular point P such that

$$\mathrm{mult}_P\big(C_4\big) = 3.$$

Show that the curve C_4 does not have other singular points.

(c) Give an example of a singular irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

Exercise 9. Let S_2 be an algebraic subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $f_2(x,y,z,t)=0$, where

$$f_2(x,y,z,t)=2x^2-4tx-ty+xy+2xz-y^2+yz.$$

Put $P=[1:-1:0:0]$.

(a) Show that $f_2(x,y,z,t)$ is irreducible. Prove that S_2 is smooth.

(b) Check that $P\in S_2$. Find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. Find $[A:B:C:D]\in\mathbb{P}^3_{\mathbb{C}}$ such that the equation

$$Ax+By+Cz+Dt=0$$

defines a plane $\Pi\subset\mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_2 at the point P. Describe $\Pi\cap S_2$.

(c) Find a projective transformation $\phi\colon\mathbb{P}^2_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy=zt$. Use this to describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2.

Exercise 10. Let S_2 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $f_2(x,y,z,t)=0$, where

$$f_2(x,y,z,t)=t^2+tx-2ty+tz+xy+xz-y^2+yz.$$

Put $P=[1:-2:1:1]$.

(a) Show that $f_2(x,y,z,t)$ is irreducible. Prove that S_2 is smooth.

(b) Check that $P\in S_2$. Find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. Find $[A:B:C:D]\in\mathbb{P}^3_{\mathbb{C}}$ such that the equation

$$Ax+By+Cz+Dt=0$$

defines a plane $\Pi\subset\mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_2 at the point P. Describe $\Pi\cap S_2$.

(c) Find a projective transformation $\phi\colon\mathbb{P}^2_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy=zt$. Use this to describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2.

Exercise 11. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=txz+xy^2+y^3$.

(a) Show that $f_3(x,y,z,t)$ is irreducible.

(b) Find all singular points (if any) of the cubic surface S_3.

(c) Find all lines on S_3.

Exercise 12. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=xyz+xyt+xzt+yzt$.

(a) Show that $f_3(x,y,z,t)$ is irreducible.

(b) Find all singular points (if any) of the cubic surface S_3.

(c) Find all lines on S_3.

Exercise 13. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=txz+y^2z+x^3+\lambda z^3$ for some complex number λ.

(a) Show that $f_3(x,y,z,t)$ is irreducible.

(b) Find all singular points (if any) of the cubic surface S_3.

(c) Find all lines on S_3.

Exercise 14. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = tz^2 + zx^2 + y^2x + \lambda t^3$ for some complex number λ.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Exercise 15. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = x^3 + y^2z + z^2t$.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Exercise 16. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = x^3 + y^3 + z^3 + t^3 - (x + y + z + t)^3$.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Exercise 17. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = txz + y^2z + x^3$.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Exercise 18. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = xyz - t^3$.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.