Dubna 2018: lines on cubic surfaces

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Lecture 1: projective plane

Complex plane

Definition

A line in $\mathbb{C}^2$ is a subset that is given by

$$\boxed{\mathbf{a}x + \mathbf{b}y + \mathbf{c} = 0}$$

for some complex numbers $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that $(\mathbf{a}, \mathbf{b}) \neq (0, 0)$.

- Here x and y are coordinates on $\mathbb{C}^2$.

Lemma

There is a unique line in $\mathbb{C}^2$ passing through two distinct points.

Proof.

Let $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ be two distinct points. Then

$$\left(\mathbf{y}_2 - \mathbf{y}_1\right)\left(x - \mathbf{x}_1\right) = \left(\mathbf{x}_2 - \mathbf{x}_1\right)\left(y - \mathbf{y}_1\right)$$

defines the line that contains $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$. □

Intersection of two lines

- Let L_1 be a line in $\mathbb{C}^2$ that is given by

$$\mathbf{a}_1 x + \mathbf{b}_1 y = \mathbf{c}_1,$$

where $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$ are complex numbers and $(\mathbf{a}_1, \mathbf{b}_1) \neq (0,0)$.

- Let L_2 be a line in $\mathbb{C}^2$ that is given by

$$\mathbf{a}_2 x + \mathbf{b}_2 y = \mathbf{c}_2$$

where $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{c}_2$ are complex numbers and $(\mathbf{a}_2, \mathbf{b}_2) \neq (0,0)$.

Lemma

Suppose that $L_1 \neq L_2$. Then $L_1 \cap L_2$ consists of at most one point.

Proof.

If $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \neq 0$, then $L_1 \cap L_2$ consists of the point

$$\boxed{\left(\frac{\mathbf{b}_2\mathbf{c}_1 - \mathbf{b}_1\mathbf{c}_2}{\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1}, \frac{\mathbf{a}_1\mathbf{c}_2 - \mathbf{a}_2\mathbf{c}_1}{\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1}\right).}$$

If $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 = 0$, then $L_1 \cap L_2 = \varnothing$. □

Conics

Definition

A conic in $\mathbb{C}^2$ is a subset that is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.

The conic is said to be *irreducible* if the polynomial

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f}$$

is *irreducible*. Otherwise the conic is called *reducible*.

- If $\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f}$ is *reducible*, then

  $$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = (\alpha x + \beta y + \gamma)(\alpha' x + \beta' y + \gamma')}$$

  for some complex numbers α, β, γ, α', β', γ'.
- In this case the conic is a union of two lines.

Matrix form

Let C be a conic in $\mathbb{C}^2$ that is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.

- We can rewrite the equation of the conic C as

$$\boxed{\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.}$$

- Denote this 3×3 matrix by M.

Lemma

The conic C is irreducible if and only if $\det(M) \neq 0$.

Example

The conic $xy - 1 = 0$ is *irreducible*.

Intersection of a line and a conic

Let L be a line in $\mathbb{C}^2$. Let C be an *irreducible* conic in $\mathbb{C}^2$.

Lemma

The intersection $L \cap C$ consists of at most 2 points.

Proof.

The line L is given by

$$\alpha x + \beta y + \gamma = 0$$

for some complex numbers α, β, γ such that $(\alpha, \beta) \neq (0, 0)$.
The conic C is given by a polynomial equation

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ are complex numbers and $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (0, 0, 0)$.
Then the intersection $L \cap C$ is given by

$$\begin{cases} \alpha x + \beta y + \gamma = 0, \\ \mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0. \end{cases}$$

Five points determine a conic

Let P_1, P_2, P_3, P_4, P_5 be distinct points in $\mathbb{C}^2$.

- Suppose that no 4 points among them are collinear.

Theorem

There is a unique conic in $\mathbb{C}^2$ that contains P_1, P_2, P_3, P_4, P_5.

Proof.

Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$, $P_4 = (x_4, y_4)$, $P_5 = (x_5, y_5)$.
Find complex numbers **a**, **b**, **c**, **d**, **e**, **f** such that

$$\begin{cases} \mathbf{a}x_1^2 + \mathbf{b}x_1y_1 + \mathbf{c}y_1^2 + \mathbf{d}x_1 + \mathbf{e}y_1 + \mathbf{f} = 0, \\ \mathbf{a}x_2^2 + \mathbf{b}x_2y_2 + \mathbf{c}y_2^2 + \mathbf{d}x_2 + \mathbf{e}y_2 + \mathbf{f} = 0, \\ \mathbf{a}x_3^2 + \mathbf{b}x_3y_3 + \mathbf{c}y_3^2 + \mathbf{d}x_3 + \mathbf{e}y_3 + \mathbf{f} = 0, \\ \mathbf{a}x_4^2 + \mathbf{b}x_4y_4 + \mathbf{c}y_4^2 + \mathbf{d}x_4 + \mathbf{e}y_4 + \mathbf{f} = 0, \\ \mathbf{a}x_5^2 + \mathbf{b}x_5y_5 + \mathbf{c}y_5^2 + \mathbf{d}x_5 + \mathbf{e}y_5 + \mathbf{f} = 0. \end{cases}$$

Then the conic containing P_1, P_2, P_3, P_4, P_5 is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}x + \mathbf{e}y + \mathbf{f} = 0.$$

Complex projective plane

- Let (x, y, z) be a point in $\mathbb{C}^3$ such that $(x, y, z) \neq (0, 0, 0)$.
- Let $[x : y : z]$ be the subset in $\mathbb{C}^3$ such that

$$\big(a, b, c\big) \in \big[x : y : z\big] \iff \begin{cases} a = \lambda x \\ b = \lambda y \\ c = \lambda z \end{cases}$$

for some non-zero complex number λ.

Definition

The projective plane $\mathbb{P}^2_{\mathbb{C}}$ is the set of of all possible $[x : y : z]$.

- We refer to the elements of $\mathbb{P}^2_{\mathbb{C}}$ as points.
- We have $[1 : 2 : 3] = [7 : 14 : 21] = [2 - i : 4 - 4i : 3 - 3i]$.
- We have $[1 : 2 : 3] \neq [3 : 2 : 1]$ and $[0 : 0 : 1] \neq [0 : 1 : 0]$.
- There is no such point as $[0 : 0 : 0]$.

How to live in projective plane?

Let U_z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : z]$ with $z \neq 0$.

Lemma

The map $U_z \to \mathbb{C}^2$ given by

$$\left[x : y : z\right] = \left[\frac{x}{z} : \frac{y}{z} : 1\right] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$$

is a bijection (one-to-one and onto).

- Thus, we can identify $U_z = \mathbb{C}^2$.
- Put $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.
- Then we can consider $\overline{x}$ and $\overline{y}$ as coordinates on $U_z = \mathbb{C}^2$.

Question

What is $\mathbb{P}^2_{\mathbb{C}} \setminus U_z$?

- The subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $[x : y : 0]$.
- We can identify $\mathbb{C}^2 \setminus U_z$ and $\mathbb{P}^1_{\mathbb{C}}$.
- This is a line at infinity.

Line at infinity

Three charts

$\mathbb{P}^2_{\mathbb{C}}$ consists of 3-tuples $[x:y:z]$ with $(x,y,z) \neq (0,0,0)$ such that

★ $[x:y:z] = [\lambda x : \lambda y : \lambda z]$ for every non-zero $\lambda \in \mathbb{C}$.

Let U_x be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $x = 0$.

- Then $U_x = \mathbb{C}^2$ with coordinates $\widetilde{y} = \frac{y}{x}$ and $\widetilde{z} = \frac{z}{x}$.

Let U_y be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $y = 0$.

- Then $U_y = \mathbb{C}^2$ with coordinates $\widehat{x} = \frac{x}{y}$ and $\widehat{z} = \frac{z}{y}$.

Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$.

- Then $U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

Then $\mathbb{P}^2_{\mathbb{C}}$ is a union of the charts U_x, U_y, U_z patched together by

$$\boxed{\widetilde{y} = \frac{1}{\widehat{x}} = \frac{\overline{y}}{\overline{x}}, \widetilde{z} = \frac{\widehat{z}}{\widehat{x}} = \frac{1}{\overline{x}}}$$

$$\boxed{\widehat{x} = \frac{\overline{x}}{\overline{y}} = \frac{1}{\widetilde{y}}, \widehat{z} = \frac{1}{\overline{y}} = \frac{\widetilde{z}}{\widetilde{y}}}$$

$$\boxed{\overline{x} = \frac{1}{\widetilde{z}} = \frac{\widehat{z}}{\widehat{x}}, \overline{y} = \frac{\widetilde{y}}{\widetilde{z}} = \frac{1}{\widehat{z}}}$$

What is a line?

Definition

A line in $\mathbb{P}^2_{\mathbb{C}}$ is the subset given by

$$Ax + By + Cz = 0$$

for some (fixed) point $[A : B : C] \in \mathbb{P}^2_{\mathbb{C}}$.

Example

Let $P = [5 : 0 : -2]$. Let $Q = [1 : -1 : 1]$. Then the line

$$2x - 3y + 5z = 0$$

contains P and Q. It is the only line in $\mathbb{P}^2_{\mathbb{C}}$ that contains P and Q.

Example

Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$x + 2y + 3z = 0.$$

Let L' be the line given by $x - y = 0$. Then $L \cap L' = [1 : 1 : -1]$.

Lines and points in projective plane

- Let P and Q be two points in $\mathbb{P}^2_{\mathbb{C}}$ such that $P \neq Q$.

Theorem

There is a unique line in $\mathbb{P}^2_{\mathbb{C}}$ that contains P and Q.

Proof.

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $Ax + By + Cz = 0$.
If $P = [a : b : c] \in L$ and $Q = [a' : b' : c'] \in L$, then

$$\begin{cases} Aa + Bb + Cc = 0, \\ Aa' + Bb' + Cc' = 0. \end{cases}$$

The rank–nullity theorem implies that L exists and is unique. $\square$

- Let L and L' be two lines in $\mathbb{P}^2_{\mathbb{C}}$ such that $L \neq L'$.

Theorem

The intersection $L \cap L'$ consists of one point in $\mathbb{P}^2_{\mathbb{C}}$.

Conics

Definition

A conic in $\mathbb{P}^2_{\mathbb{C}}$ is a subset that is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

The conic is said to be *irreducible* if the polynomial

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$$

is *irreducible*. Otherwise the conic is called *reducible*.

If $\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$ is *reducible*, then

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = (\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z)}$$

for some complex numbers α, β, γ, α', β', γ'.

In this case the conic is a union of two lines.

Matrix form

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ that is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- Rewrite the equation of the conic $\mathcal{C}$ in the matrix form:

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

- Denote this 3×3 matrix by M.

Lemma

The conic $\mathcal{C}$ is irreducible if and only if $\det(M) \neq 0$.

Example

The conic $xy - z^2 = 0$ is *irreducible*.

Intersection of a line and a conic

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$. Let C be an *irreducible* conic in $\mathbb{P}^2_{\mathbb{C}}$.

Lemma

The intersection $L \cap C$ consists of 2 points (counted with multiplicities).

Proof.

The line L is given by

$$\alpha x + \beta y + \gamma z = 0$$

for complex numbers α, β, γ such that $(\alpha, \beta, \gamma) \neq (0, 0, 0)$.
The conic C is given by a polynomial equation

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.
Then the intersection $L \cap C$ is given by

$$\begin{cases} \alpha x + \beta y + \gamma z = 0, \\ \mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0. \end{cases}$$

Five points determine a conic

Let P_1, P_2, P_3, P_4, P_5 be distinct points in $\mathbb{P}^2_{\mathbb{C}}$.

- Suppose that no 4 points among them are collinear.

Theorem

There is a unique conic in $\mathbb{P}^2_{\mathbb{C}}$ that contains P_1, P_2, P_3, P_4, P_5.

Proof.

Let $[x_1 : y_1 : z_1]$, $[x_2 : y_2 : z_2]$, $[x_3 : y_3 : z_3]$, $[x_4 : y_4 : z_4]$, $[x_5 : y_5 : z_6]$ be our points. Find complex numbers **a**, **b**, **c**, **d**, **e**, **f** such that

$$\begin{cases} \mathbf{a}x_1^2 + \mathbf{b}x_1y_1 + \mathbf{c}y_1^2 + \mathbf{d}x_1z_1 + \mathbf{e}y_1z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_2^2 + \mathbf{b}x_2y_2 + \mathbf{c}y_2^2 + \mathbf{d}x_2z_1 + \mathbf{e}y_2z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_3^2 + \mathbf{b}x_3y_3 + \mathbf{c}y_3^2 + \mathbf{d}x_3z_1 + \mathbf{e}y_3z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_4^2 + \mathbf{b}x_4y_4 + \mathbf{c}y_4^2 + \mathbf{d}x_4z_1 + \mathbf{e}y_4z_1 + \mathbf{f}z_1^2 = 0, \\ \mathbf{a}x_5^2 + \mathbf{b}x_5y_5 + \mathbf{c}y_5^2 + \mathbf{d}x_5z_1 + \mathbf{e}y_5z_1 + \mathbf{f}z_1^2 = 0. \end{cases}$$

Then the conic containing P_1, P_2, P_3, P_4, P_5 is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0.$$

□

Complex irreducible plane curves

Definition

An irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geqslant 1$ is a subset given by

$$f(x, y, z) = 0$$

for an irreducible homogeneous polynomial $f(x, y, z)$ of degree d.

Let us give few examples. The equation

$$2x^2 - y^2 + 2z^2 = 0$$

defines an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$. The equation

$$zy^2 - x(x - z)(x + z) = 0$$

defines an irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$. The equation

$$\Big(2x^2 - y^2 + 2z^2\Big)\Big(zy^2 - x(x - z)(x + z)\Big) = 0$$

defines the union of the two curves above.

Projective transformations

Let $\mathbf{M}$ be a complex 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be the map given by

$$\left[x : y : z\right] \mapsto \left[a_{11}x+a_{12}y+a_{13}z : a_{21}x+a_{22}y+a_{23}z : a_{31}x+a_{32}y+a_{33}z\right].$$

Recall that there is no such point in $\mathbb{P}^2_{\mathbb{C}}$ as $[0 : 0 : 0]$.

Question

When ϕ is well-defined?

The map ϕ is well-defined $\iff \det(\mathbf{M}) \neq 0$.

Definition

If $\det(\mathbf{M}) \neq 0$, we say that ϕ a projective transformation.

Projective linear group

Projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ form a group.

- Let $\mathbf{M}$ be a matrix in $\mathrm{GL}_3(\mathbb{C})$.
- Denote by $\phi_{\mathbf{M}}$ the corresponding projective transformation.

Question

When $\phi_{\mathbf{M}}$ is an identity map?

The map $\phi_{\mathbf{M}}$ is an identity map $\iff$ $\mathbf{M}$ is scalar.
Recall that $\mathbf{M}$ is said to be scalar if

$$M = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

for some complex number λ.

Corollary

Let $\mathbf{G}$ be a subgroup in $\mathrm{GL}_3(\mathbb{C})$ consisting of scalar matrices. The group of projective transformations of $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to

$$\mathrm{PGL}_3(\mathbb{C}) = \mathrm{GL}_3(\mathbb{C})/\mathbf{G}.$$

Four points in the plane

Let P_1, P_2, P_3, P_4 be four points in $\mathbb{P}^2_{\mathbb{C}}$ such that

- no three points among them are collinear.

Then there is a projective transformation $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$P_1 \mapsto [1:0:0], P_2 \mapsto [0:1:0], P_3 \mapsto [0:0:1], P_4 \mapsto [1:1:1].$$

Let $P_1 = [a_{11} : a_{12} : a_{13}]$, $P_2 = [a_{21} : a_{22} : a_{23}]$, $P_3 = [a_{31} : a_{32} : a_{33}]$.
Let ϕ be the projective transformation

$$\big[x : y : z\big] \mapsto \big[a_{11}x+a_{21}y+a_{31}z : a_{12}x+a_{22}y+a_{32}z : a_{13}x+a_{23}y+a_{33}x\big].$$

Then $\phi([1:0:0]) = P_1$, $\phi([0:1:0]) = P_2$, $\phi([0:0:1]) = P_3$.
Let ψ be the inverse of the map ϕ. Write $\psi(P_4) = [\alpha : \beta : \gamma]$.
Let τ be the projective transformation

$$[x : y : z] \mapsto \left[\frac{x}{\alpha} : \frac{y}{\beta} : \frac{z}{\gamma}\right] = \big[\beta\gamma x : \alpha\gamma y : \alpha\beta z\big].$$

Then $\tau \circ \psi$ is the required projective transformation.

Conics and their tangent lines

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$, and let $\mathcal{C}$ be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$.

Question

When $|L \cap \mathcal{C}| = 1$?

We may assume that $[0:0:1] \in L \cap \mathcal{C}$. Then $\mathcal{C}$ is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz = 0$$

for some $[\mathbf{a}:\mathbf{b}:\mathbf{c}:\mathbf{d}:\mathbf{e}] \in \mathbb{P}^4_{\mathbb{C}}$.
We may assume that L is given by $x = 0$. Then

$$L \cap \mathcal{C} = [0:0:1] \cup [0:\mathbf{e}:-\mathbf{c}].$$

Thus, we have $|L \cap \mathcal{C}| = 1 \iff \mathrm{e} = 0$.

- Let U_z be the complement in $\mathbb{P}^2_{\mathbb{C}}$ to the line $z = 0$.
- Identify U_z and $\mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.

Then $U_z \cap \mathcal{C}$ is given by $\mathbf{a}\overline{x}^2 + \mathbf{b}\overline{xy} + \mathbf{c}\overline{y}^2 + \mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$.

- $\mathbf{d}\overline{x} + \mathbf{e}\overline{y} = 0$ is the tangent line to $U_z \cap \mathcal{C}$ at $(0,0)$.
- $\mathbf{d}x + \mathbf{e}y = 0$ is the tangent line to $\mathcal{C}$ at $[0:0:1]$.

Then $|L \cap \mathcal{C}| = 1 \iff L$ is tangent to $\mathcal{C}$ at the point $L \cap \mathcal{C}$.

Smooth complex plane curves

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d given by

$$f(x,y,z)=0,$$

where $f(x,y,z)$ is a homogeneous polynomial of degree d.

Definition

A point $[a:b:c]\in\mathbb{P}^2_{\mathbb{C}}$ is a singular point of the curve C if

$$\boxed{\frac{\partial f(a,b,c)}{\partial x}=\frac{\partial f(a,b,c)}{\partial y}=\frac{\partial f(a,b,c)}{\partial z}=0.}$$

- Denote by $\mathrm{Sing}(C)$ the set of singular points of the curve C.
- Non-singular points of the curve C are called smooth.
- The curve C is said to be smooth if $\mathrm{Sing}(C)=\varnothing$

Example

1. If $f=zx^{d-1}-y^d$ and $d\geqslant 3$, then $\mathrm{Sing}(C)=[0:0:1]$.
2. If $f=x^d+y^d+z^d$, then $\mathrm{Sing}(C)=\varnothing$.

Tangent lines

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d given by

  $$f(x, y, z) = 0,$$

  where $f(x, y, z)$ is a homogeneous polynomial of degree d.
- Let $P = [\alpha : \beta : \gamma]$ be a smooth point in C. Then the line

  $$\boxed{\frac{\partial f(\alpha, \beta, \gamma)}{\partial x} x + \frac{\partial f(\alpha, \beta, \gamma)}{\partial y} y + \frac{\partial f(\alpha, \beta, \gamma)}{\partial z} z = 0}$$

  is the tangent line to the curve C at the point P.

Remark

We may assume that $P = [0 : 0 : 1]$. Then

$$\boxed{f(x, y, z) = z^{d-1} h_1(x, y) + z^{d-2} h_2(x, y) + \cdots + z h_{d-1}(x, y) + h_d(x, y) = 0,}$$

where $h_i(x, y)$ is a homogenous polynomial of degree i.

Then $\boxed{h_1(x, y) = 0}$ is the tangent line to C at the point P.

Conics and projective transformation

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ is given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

Theorem

There is a projective transformations ϕ such that $\phi(\mathcal{C})$ is given by

1. *either $xy = z^2$ (an irreducible smooth conic),*
2. *or $xy = 0$ (a union of two lines in $\mathbb{P}^2_{\mathbb{C}}$),*
3. *or $x^2 = 0$ (a line in $\mathbb{P}^2_{\mathbb{C}}$ taken with multiplicity 2).*

Example

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$ given by $(x - 3y + z)(x + 7y - 5z) = 0$.
Let $\phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a projective transformations given by

$$[x : y : z] \mapsto \left[x - 3y + z : x + 7y - 5z : z\right].$$

Then $\phi(\mathcal{C})$ is a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $xy = 0$.

Irreducible conics

Let $\mathcal{C}$ be a conic in $\mathbb{P}^2_{\mathbb{C}}$. Then $\mathcal{C}$ that is given by

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- Denote this 3×3 matrix by $\mathcal{M}$.

Lemma

The conic $\mathcal{C}$ is irreducible if and only if $\det(\mathcal{M}) \neq 0$.

Proof.

Let $\phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a projective transformation given by matrix $\mathbf{M}$. Let $\mathbf{N} = \mathbf{M}^{-1}$. Then the conic $\phi(\mathcal{C})$ is given by

$$\boxed{\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{N}^T \begin{pmatrix} \mathbf{a} & \frac{\mathbf{b}}{2} & \frac{\mathbf{d}}{2} \\ \frac{\mathbf{b}}{2} & \mathbf{c} & \frac{\mathbf{e}}{2} \\ \frac{\mathbf{d}}{2} & \frac{\mathbf{e}}{2} & \mathbf{f} \end{pmatrix} \mathbf{N} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.}$$

Classification of irreducible conics

Let $\mathcal{C}$ be an irreducible conic in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

1. Pick a point in $\mathcal{C}$ and map it to $[0 : 0 : 1]$. This kills $\mathbf{f}$.
2. Map the tangent line $\mathbf{d}x + \mathbf{e}y = 0$ to $x = 0$. This kills $\mathbf{e}$.
3. Map the line $z = 0$ to the line

$$z + \alpha y + \beta z$$

   for appropriate α and β to kill $\mathbf{a}$ and $\mathbf{b}$.
4. Scale x, y, and z appropriately to get $\mathbf{b} = 1$ and $\mathbf{c} = -1$.

This gives a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$\boxed{xz = y^2}$$

defines the curve $\phi(\mathcal{C})$.

The conic $x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0$

Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\boxed{x^2 + y^2 - 2xy + xz - 3yz + 2z^2 = 0.}$$

1. Note that $[0 : 1 : 1] \in \mathcal{C}$. Let $\mathbf{y} = y - z$. Then $\mathcal{C}$ is given by
$$x^2 + \mathbf{y}^2 - \mathbf{y}z - 2x\mathbf{y} - xz = 0.$$
2. To map the tangent line $x + \mathbf{y} = 0$ to the line $x = 0$, let
$$\mathbf{x} = x + \mathbf{y}.$$
Then $\mathcal{C}$ is given by $\mathbf{x}^2 + 4\mathbf{y}^2 - 4\mathbf{x}\mathbf{y} - \mathbf{x}z = 0$.
3. Let $\mathbf{z} = z + \mathbf{x} - 4\mathbf{y}$. Then $\mathcal{C}$ is given by $4\mathbf{y}^2 - \mathbf{x}\mathbf{z} = 0$.

Since $\mathbf{x} = x + y - z$, $\mathbf{y} = y - z$, and $\mathbf{z} = x - 3y + 4z$, the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & -3 & 4 \end{pmatrix}$$

gives a projective transformation that maps $\mathcal{C}$ to $xz = 4y^2$.

Intersection of two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{C} \neq \mathcal{C}'$.

Theorem

One has $1 \leqslant |\mathcal{C} \cap \mathcal{C}'| \leqslant 4$.

Proof.

We may assume that $\mathcal{C}$ is given by $xy = z^2$. Then $\mathcal{C}'$ is given by

$$\boxed{\mathbf{a}x^2 + \mathbf{b}xy + \mathbf{c}y^2 + \mathbf{d}xz + \mathbf{e}yz + \mathbf{f}z^2 = 0}$$

for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ in $\mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}) \neq (0, 0, 0, 0, 0, 0)$.

- Let L be the line $y = 0$. Then $L \cap \mathcal{C} \cap \mathcal{C}' \subset [1 : 0 : 0]$.
- One has $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0] \iff \mathbf{a} = 0$.
- Let $U_y = \mathbb{P}^2_{\mathbb{C}} \setminus L$. Then $U_y \cap \mathcal{C} \cap \mathcal{C}'$ is given by

$$y - 1 = x - z^2 = \mathbf{a}z^4 + \mathbf{d}z^3 + (\mathbf{b} + \mathbf{f})z^2 + \mathbf{e}z + \mathbf{c} = 0.$$

If $\mathbf{a} = 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = [1 : 0 : 0]$ and $0 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 3$.
If $\mathbf{a} \neq 0$, then $L \cap \mathcal{C} \cap \mathcal{C}' = \varnothing$ and $1 \leqslant |U_y \cap \mathcal{C} \cap \mathcal{C}'| \leqslant 4$. □

Intersection of two conics: four points

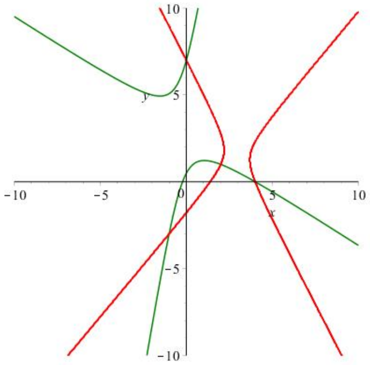
Let $\mathcal{C}$ be the irreducible conic

$$511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2 = 0.$$

Let $\mathcal{C}$ be the irreducible conic

$$1217x^2 - 394xy - 541y^2 - 6555xz + 2823yz + 6748z^2 = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $[4:0:1]$, $[1:3:-1]$, $[0:7:1]$, $[2:1:1]$.



Intersection of two conics: three points

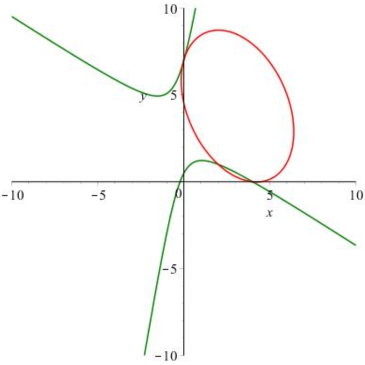
Let $\mathcal{C}$ be the irreducible conic

$$511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2 = 0.$$

Let $\mathcal{C}$ be the irreducible conic

$$42049x^2+21271xy+23536y^2-355005xz-271500yz+747236z^2 = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $[4:0:1]$, $2 \times [0:7:1]$, $[2:1:1]$.



Intersection of two conics: two points (2+2)

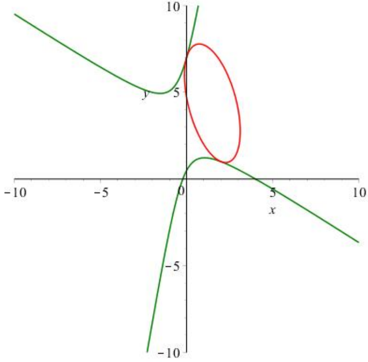
Let $\mathcal{C}$ be the irreducible conic $f(x, y, z) = 0$, where

$$f(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x - 853y + 5971z)(821x - 3779y + 2137z) - 9700f(x, y, z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $2 \times [0 : 7 : 1]$ and $2 \times [2 : 1 : 1]$.



Intersection of two conics: two points (3+1)

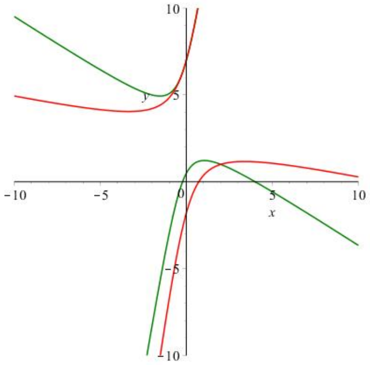
Let $\mathcal{C}$ be the irreducible conic $f(x, y, z) = 0$, where

$$f(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x - 853y + 5971z)(6x + 2y - 14z) - 50f(x, y, z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $3 \times [0 : 7 : 1]$ and $[2 : 1 : 1]$.



Intersection of two conics: one point

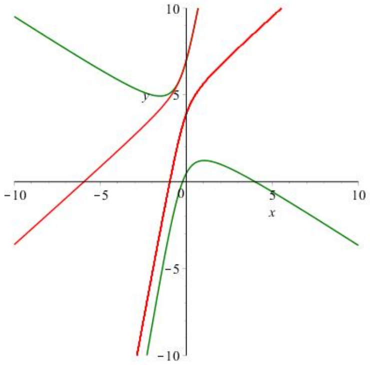
Let $\mathcal{C}$ be the irreducible conic $f(x, y, z) = 0$, where

$$f(x, y, z) = 511x^2 + 709xy - 131y^2 - 1932xz + 981yz - 448z^2.$$

Let $\mathcal{C}$ be the irreducible conic

$$(3031x - 853y + 5971z)^2 - 5000f(x, y, z) = 0.$$

Then $\mathcal{C} \cap \mathcal{C}$ consists of $4 \times [0 : 7 : 1]$.



Transversal intersection of two conics

Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$.

Question

When the intersection $\mathcal{C} \cap \mathcal{C}'$ consists of 4 points?

Let P be a point in $\mathcal{C} \cap \mathcal{C}'$.

- $\exists$ unique line $L \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L$ and $|L \cap \mathcal{C}| = 1$.
- $\exists$ unique line $L' \subset \mathbb{P}^2_{\mathbb{C}}$ such that $P \in L'$ and $|L' \cap \mathcal{C}| = 1$.

The lines L and L' are tangent lines to $\mathcal{C}$ and $\mathcal{C}'$ at P, respectively.

Definition

We say that $\mathcal{C}$ intersects $\mathcal{C}'$ transversally at P if $L \neq L'$.

- The answer to the question above is given by

Theorem

The following two conditions are equivalent:

1. *the intersection* $\mathcal{C} \cap \mathcal{C}'$ *consists of* 4 *points,*
2. $\mathcal{C}$ *intersects* $\mathcal{C}'$ *transversally at every point of* $\mathcal{C} \cap \mathcal{C}'$.

Bezout's theorem

- Let $f(x, y, z)$ be a homogeneous polynomial of degree d.
- Let $g(x, y, z)$ be a homogeneous polynomial of degree $\widehat{d}$.

Consider the system of equations

$$\boxed{\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}} \qquad (\bigstar)$$

Question

How many solutions in $\mathbb{P}^2_{\mathbb{C}}$ does $(\bigstar)$ has?

- Infinite if $f(x, y, z)$ and $g(x, y, z)$ have a common factor.

Theorem (Bezout)

Suppose that $f(x, y, z)$ and $g(x, y, z)$ have no common factors. Then the number of solutions to $(\bigstar)$ depends only on d and $\widehat{d}$.

- Here we should count solutions with multiplicities.

Bezout's theorem: baby case

- Let $f(x, y, z)$ be a homogeneous polynomial of degree d.
- Let $g(x, y, z)$ be a homogeneous polynomial of degree 1.

Suppose that $g(x, y, z)$ does not divide $f(x, y, z)$.

- We may assume that $g(x, y, z) = z$.

We have to solve the system

$$\boxed{\begin{cases} z = 0, \\ f(x, y, z) = 0. \end{cases}}$$

Theorem (Fundamental Theorem of Algebra)

There are linear polynomials $h_1(x, y), \ldots, h_d(x, y)$ such that

$$f(x, y, 0) = \prod_{i=1}^{d} h_i(x, y).$$

- This gives d points in $\mathbb{P}^2_{\mathbb{C}}$ counted with multiplicities.

Bezout's theorem: algebraic version

- Let $f(x, y, z)$ be a homogeneous polynomial of degree d.
- Let $g(x, y, z)$ be a homogeneous polynomial of degree $\widehat{d}$.

Suppose that $f(x, y, z)$ and $g(x, y, z)$ do not have common factors.

- Let C be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x, y, z) = 0$.
- Let Z be the subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $g(x, y, z) = 0$.

For every $P \in C \cap Z$, define a positive integer $(f, g)_P$ as follows:

- Assume that $P \in U_z = \mathbb{C}^2$ with coordinates $\overline{x} = \frac{x}{z}$ and $\overline{y} = \frac{y}{z}$.
- Let $\mathbf{R}$ be a subring in $\mathbb{C}(\overline{x}, \overline{y})$ consisting of all fractions

$$\frac{a(\overline{x}, \overline{y})}{b(\overline{x}, \overline{y})}$$

  with $a(\overline{x}, \overline{y})$ and $b(\overline{x}, \overline{y})$ in $\mathbb{C}[\overline{x}, \overline{y}]$ such that $b(P) \neq 0$.
- Let $\mathbf{I}$ be the ideal in $\mathbf{R}$ generated by $f(\overline{x}, \overline{y}, 1)$ and $g(\overline{x}, \overline{y}, 1)$.
- Let $(f, g)_P = \dim_{\mathbb{C}}(\mathbf{R}/\mathbf{I}) \geqslant 1$.

Then Bezout's theorem says that

$$\sum_{P \in C \cap Z} \big(f, g\big)_P = d\widehat{d}.$$

Intersection multiplicity

- Let $f(x, y, z)$ be a homogeneous polynomial.
- Let $g(x, y, z)$ be a homogeneous polynomial.

Suppose that $f(x, y, z)$ and $g(x, y, z)$ do not have common factors. Fix $P \in \mathbb{P}^2_{\mathbb{C}}$ such that $f(P) = g(P) = 0$. Then

$$\boxed{\left(f, g\right)_P = \left(g, f\right)_P \geqslant 1.}$$

- Let $h(x, y, z)$ be a homogeneous polynomial.

Suppose that $f(x, y, z)$ and $h(x, y, z)$ do not have common factors.

- If $h(P) = 0$, then

$$\boxed{\left(f, gh\right)_P = \left(f, g\right)_P + \left(f, h\right)_P.}$$

- If $h(P) \neq 0$, then

$$\boxed{\left(f, gh\right)_P = \left(f, g\right)_P.}$$

Bezout's theorem: geometric version

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by
$$f(x,y,z)=0,$$
where f is a homogeneous irreducible polynomial of degree d.
- Let Z be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by
$$g(x,y,z)=0,$$
where g is a homogeneous irreducible polynomial of degree $\widehat{d}$.

Theorem (Bezout)

Suppose that $f(x,y,z)\neq\lambda g(x,y,z)$ for any $\lambda\in\mathbb{C}^*$. Then

$$\boxed{1\leqslant\big|C\cap Z\big|\leqslant\sum_{P\in C\cap Z}\big(C\cdot Z\big)_P=d\widehat{d}}$$

where $(C\cdot Z)_P=\big(f,g\big)_P$ is the intersection multiplicity.

Corollary

$C=Z \iff f(x,y,z)=\lambda g(x,y,z)$ *for some* $\lambda\in\mathbb{C}^*$.

Intersection of two cubics

Let $\mathcal{C}$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$-5913252577x^3+30222000280x^2y-21634931915xy^2+$$
$$+5556266591y^3-73906985473x^2z+102209537669xyz-37300172365y^2z+$$
$$+1389517162xz^2-88423819400yz^2+204616284808z^3=0.$$

Let $\mathcal{C}$ be the irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$-4844332x^3-8147864x^2y-4067744xy^2-$$
$$-1866029y^3+32668904x^2z-28226008xyz+41719157y^2z+$$
$$+252639484xz^2+126319742yz^2-960898976z^3=0$$

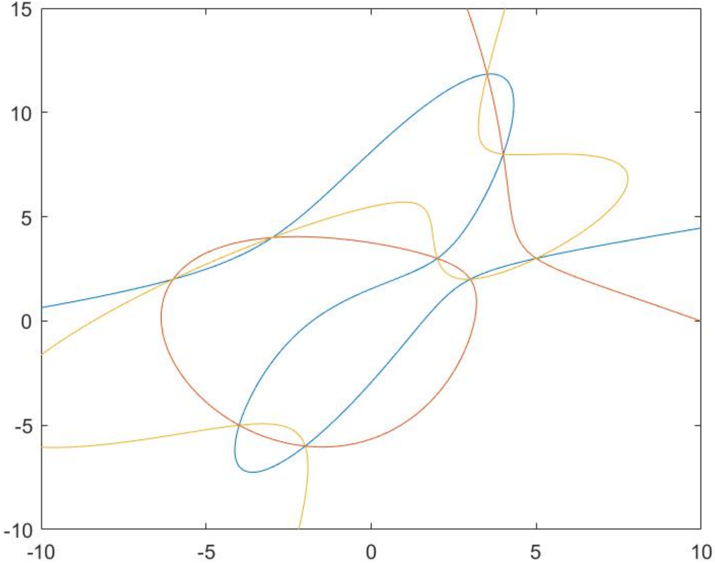
Then the intersection $\mathcal{C}\cap\mathcal{C}$ consists of the eight points

$[2:3:1],[-3:4:1],[4:5:-1],[-6:2:1],[5:3:1],[3:2:1],[2:6:-11],[4:8:1]$

and the ninth point

$$\left[1439767504290697562:4853460637572644276:4099420541047597 19\right].$$

Intersection of three cubics



How to find the intersection $\mathcal{C} \cap \mathcal{C}$?

1. Let $f(x, y)$ be the polynomial

$$-5913252577x^3 + 30222000280x^2y - 21634931915xy^2 + 5556266591y^3 - 73906985473x^2 + \\ + 102209537669xy - 37300172365y^2 + 1389517162x - 88423819400y + 204616284808.$$

2. Let $g(x, y)$ be the polynomial

$$-4844332x^3 - 8147864x^2y - 4067744xy^2 - 1866029y^3 + 32668904x^2 - \\ - 28226008xy + 41719157y^2 + 252639484x + 126319742y - 960898976.$$

3. Consider $f(x, y)$ and $g(x, y)$ as polynomials in y with coefficients in $\mathbb{C}[x]$.
4. Their resultant $R(f, g, y)$ is the polynomial:

$$319168411614335505141855887784472124841956719232716 9x^9 - \\ - 80179076502326448020959208485535781077792914885854 93x^8 - \\ - 19951895461883394788720945351923685301295332302821 5633x^7 + \\ + 56880707484802669486621609640000274581156521359635 9157x^6 + \\ + 38806142666086015230321945019845701520691647539989 33464x^5 - \\ - 11708714303403885204269002049013593498191154175608 876232x^4 - \\ - 27936678172063675450258473952703104020433424068758 015952x^3 + \\ + 86672526536406322333733242006002412277456517441705 929808x^2 + \\ + 61609026384389751204137037731562203601860663683619 173632x - \\ - 19370174572297727746873020967216261287511627800617 0799360.$$

5. Its roots are 2, 3, 4, 5, -6, -4, -3, -2 and $\frac{1439767504290697562}{409942054104759719}$.

Resultant

One has $f(x,y) = a_3 y^3 + a_2 y^2 + a_1 y + a_0$, where

$$\begin{cases} a_3 = 5556266591, \\ a_2 = -21634931915x - 37300172365, \\ a_1 = 30222000280x^2 + 102209537669x - 88423819400, \\ a_0 = 5913252577x^3 - 73906985473x^2 + 1389517162x + 204616284808. \end{cases}$$

One has $g(x,y) = b_3 y^3 + b_2 y^2 + b_1 y + b_0$, where

$$\begin{cases} b_3 = -1866029, \\ b_2 = -4067744x + 41719157, \\ b_1 = -8147864x^2 - 28226008x + 126319742, \\ b_0 = -4844332x^3 + 32668904x^2 + 252639484x - 960898976. \end{cases}$$

The resultant of $f(x,y)$ and $g(x,y)$ (considered as polynomials in y) is

$$R(f,g,y) = \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix} = \det \begin{pmatrix} f(x,y) & a_1 & a_2 & a_3 & 0 & 0 \\ yf(x,y) & a_0 & a_1 & a_2 & a_3 & 0 \\ y^2 f(x,y) & 0 & a_0 & a_1 & a_2 & a_3 \\ g(x,y) & b_1 & b_2 & b_3 & 0 & 0 \\ yg(x,y) & b_0 & b_1 & b_2 & b_3 & 0 \\ y^2 g(x,y) & 0 & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

This shows that $R(f,g,y) = A(x,y)f(x,y) + B(x,y)g(x,y)$ for some polynomials $A(x,y)$ and $B(x,y)$.

Intersection multiplicity and transversal intersection

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d.
- Let Z be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $\widehat{d}$.

Pick $P \in C \cap Z$.

Definition

We say that C intersects the curve Z transversally at P if

1. both curves C and Z are smooth at the point P,
2. and the tangent lines to C and Z are P are different.

Then $(C \cdot Z)_P = 1 \iff C$ intersects Z transversally at P.

Corollary

The following two conditions are equivalent:

1. $|C \cap Z| = d\widehat{d}$,
2. *C intersects Z transversally at every point of $C \cap Z$.*

Corollary

If $|C \cap Z| = d\widehat{d}$, then $\mathrm{Sing}(C) \cap Z = \varnothing = C \cap \mathrm{Sing}(Z)$.

Intersection multiplicity and singular points

- Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree d.

Let $P = [0:0:1]$. Then C is given by the equation

$$z^d h_0(x,y) + z^{d-1}h_1(x,y) + z^{d-2}h_2(x,y) + \cdots + h_d(x,y) = 0,$$

where $h_i(x,y)$ is a homogenous polynomial of degree i. Let

$$\mathrm{mult}_P\big(C\big) = \min\Big\{i \mid h_i(x,y) \text{ is not a zero polynomial}\Big\}$$

- $\mathrm{mult}_P(C) \geqslant 1 \iff P \in C$.
- $\mathrm{mult}_P(C) \geqslant 2 \iff P \in \mathrm{Sing}(C)$.

We say that C has multiplicity $\mathrm{mult}_P(C)$ at the point P.

- Let Z be another irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$.

Lemma

Suppose that $C \neq Z$ and $P \in C \cap Z$. Then

$$\big(C \cdot Z\big)_P \geqslant \mathrm{mult}_P\big(C\big)\mathrm{mult}_P\big(Z\big).$$

Bezout's theorem: first application

Let $f(x, y, z)$ be a homogeneous polynomial of degree $d \geqslant 1$.

Lemma

Suppose that the system

$$\frac{\partial f(x,y,z)}{\partial x} = \frac{\partial f(x,y,z)}{\partial y} = \frac{\partial f(x,y,z)}{\partial z} = 0$$

has no solutions in $\mathbb{P}^2_{\mathbb{C}}$. Then $f(x, y, z)$ is irreducible.

Proof.

Suppose that $f(x, y, z)$ is not irreducible. Then

$$f\big(x, y, z\big) = g(x, y, z)h(x, y, z),$$

where g and h are homogeneous polynomials of positive degrees. There is $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ with $g(a, b, c) = h(a, b, c) = 0$. Then

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial g(a,b,c)}{\partial x} h(a,b,c) + g(a,b,c)\frac{\partial h(a,b,c)}{\partial x} = 0.$$

Bezout's theorem: second application

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geqslant 2$.

Theorem

Let P and Q be two different points in C. Then

$$\mathrm{mult}_P(C) + \mathrm{mult}_Q(C) \leqslant d.$$

Proof.

Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P and Q. Then

$$d = \sum_{O \in L \cap C} \left(L \cdot C\right)_O \geqslant \left(L \cdot C\right)_P + \left(L \cdot C\right)_Q \geqslant \mathrm{mult}_P(C) + \mathrm{mult}_Q(C).$$

□

Corollary

Let P be a point in C. Then $\mathrm{mult}_P(C) < d$.

Corollary

Suppose that $d = 3$. Then C has at most one singular point.

Bezout's theorem: third application

Let C be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

Lemma

The curve C has at most 3 singular points.

Proof.

Suppose that C has at least 4 singular points.
Denote four singular points of C as P_1, P_2, P_3, P_4.
Let Q be a point in C that is different from these 4 points.
There is a homogeneous polynomial $f(x, y, z)$ of degree 2 such that

$$f(P_1) = f(P_2) = f(P_3) = f(P_4) = f(Q) = 0.$$

Let Z the curve in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x, y, z) = 0$.
Since C is irreducible, we can apply Bezout's theorem to C and Z:

$$8 = \sum_{O \in C \cap Z} \big(C \cdot Z\big)_O \geqslant \sum_{i=1}^{4} \big(C \cdot Z\big)_{P_i} + \big(C \cdot Z\big)_Q \geqslant \sum_{i=1}^{4} \mathrm{mult}_{P_i}(C) + 1.$$

□

Bezout's theorem: fourth application

Let $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ be a map given by

$$\big[x : y : z\big] \mapsto \big[f(x,y,z) : g(x,y,z) : h(x,y,z)\big]$$

for some homogeneous polynomials f, g, h of degree d such that

$$f(x,y,z) = g(x,y,z) = h(x,y,z) = 0$$

does not have solutions in $\mathbb{P}^2_{\mathbb{C}}$. Suppose that ϕ is bijection.

- Let $[A : B : C]$ and $[A' : B' : C']$ be general points in $\mathbb{P}^2_{\mathbb{C}}$.
- Let L be a line given by $Ax + By + Cz = 0$.
- Let L' be a line given by $A'x + B'y + C'z = 0$.

The preimage of $L \cap L'$ via ϕ is 1 point. But it is given by

$$\begin{cases} Af(x,y,z) + Bg(x,y,z) + Ch(x,y,z) = 0, \\ A'f(x,y,z) + B'g(x,y,z) + C'h(x,y,z) = 0. \end{cases}$$

One can show that this system has d^2 solutions. Then $d = 1$.