Dubna 2018: lines on cubic surfaces

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Lecture 3: twenty seven lines on smooth cubic surface

Every smooth cubic surface contains twenty seven lines

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$.

Theorem (Cayley, Salmon)

The surface S_3 contains exactly 27 lines.

Lines on the Fermat cubic surface I

- Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + z^3 + t^3 = 0$.
- Then S_3 is irreducible and smooth.

Put $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then S_3 contains lines

$$x+t=y+z=0, x+t=y+\omega z=0, x+t=y+\omega^2 z=0,$$

$$x+\omega t=y+z=0, x+\omega t=y+\omega z=0, x+\omega t=y+\omega^2 z=0,$$

$$x+\omega^2 t=y+z=0, x+\omega^2 t=y+\omega z=0, x+\omega^2 t=y+\omega^2 z=0,$$

$$y+t=x+z=0, y+t=x+\omega z=0, y+t=x+\omega^2 z=0,$$

$$y+\omega t=x+z=0, y+\omega t=x+\omega z=0, y+\omega t=x+\omega^2 z=0,$$

$$y+\omega^2 t=x+z=0, y+\omega^2 t=x+\omega z=0, y+\omega^2 t=x+\omega^2 z=0,$$

$$z+t=x+y=0, z+t=x+\omega y=0, z+t=x+\omega^2 y=0,$$

$$z+\omega t=x+y=0, z+\omega t=x+\omega y=0, z+\omega t=x+\omega^2 y=0,$$

$$z+\omega^2 t=x+y=0, z+\omega^2 t=x+\omega y=0, z+\omega^2 t=x+\omega^2 y=0.$$

Lines on the Fermat cubic surface II

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + z^3 + t^3 = 0$.

- Let L be a line in $\mathbb{P}^3_{\mathbb{C}}$ such that $L \subset S_3$.
- Let $P = [a : b : 0 : c]$ be the intersection of L with $z = 0$.
- Let $Q = [\alpha : \beta : \gamma : 0]$ be the intersection of L with $t = 0$.

We may assume that $P \neq Q$. Then L is given by

$$\lambda\big[a : b : 0 : c\big] + \mu\big[\alpha : \beta : \gamma : 0\big],$$

where $[\lambda : \mu]$ runs through all points in $\mathbb{P}^1_{\mathbb{C}}$. Then

$$\Big(\lambda a + \mu\alpha\Big)^3 + \Big(\lambda b + \mu\beta\Big)^3 + \lambda^3 c^3 + \mu^3\gamma^3 = 0$$

for every $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives

$$\lambda^3\big(a^3+b^3+c^3\big)+3\lambda^2\mu\big(a^2\alpha+b^2\beta\big)+3\lambda\mu^2\big(a\alpha^2+b\beta^2\big)+\mu^3\big(\alpha^3+\beta^3+\gamma^3\big) = 0$$

for every $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives

$$\boxed{a^3 + b^3 + c^3 = a^2\alpha + b^2\beta = a\alpha^2 + b\beta^2 = \alpha^3 + \beta^3 + \gamma^3 = 0.}$$

Let us use these equations to show that L is one of our 27 lines.

Lines on the Fermat cubic surface III

We have the line L that consists of the points

$$\left[\lambda a + \mu\alpha : \lambda b + \mu\beta : \mu\gamma : \lambda c\right]$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. We also have

$$a^3 + b^3 + c^3 = a^2\alpha + b^2\beta = a\alpha^2 + b\beta^2 = \alpha^3 + \beta^3 + \gamma^3 = 0.$$

Suppose that $a = 0$. Then

$$b^3 + c^3 = b^2\beta = \alpha^3 + \beta^3 + \gamma^3 = 0.$$

This gives $\beta = 0$, $b^3 + c^3 = 0$ and $\alpha^3 + \gamma^3 = 0$. Then

$$P = \left[0 : \omega^i : 0 : 1\right]$$

and $Q = [\omega^j : 0 : 1 : 0]$ for some i and j. Then L is one of the lines

$$y + t = x + z = 0, y + t = x + \omega z = 0, y + t = x + \omega^2 z = 0,$$

$$y + \omega t = x + z = 0, y + \omega t = x + \omega z = 0, y + \omega t = x + \omega^2 z = 0,$$

$$y + \omega^2 t = x + z = 0, y + \omega^2 t = x + \omega z = 0, y + \omega^2 t = x + \omega^2 z = 0.$$

Lines on the Fermat cubic surface IV

We may assume that $a \neq 0$. Then

$$P = \left[a : b : 0 : c\right] = \left[1 : \frac{b}{a} : 0 : \frac{c}{a}\right],$$

so that we may assume that $a = 1$. Then L consists of the points

$$\left[\lambda + \mu\alpha : \lambda b + \mu\beta : \mu\gamma : \lambda c\right]$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. We also have

$$1 + b^3 + c^3 = \alpha + b^2\beta = \alpha^2 + b\beta^2 = \alpha^3 + \beta^3 + \gamma^3 = 0.$$

Suppose that $b = 0$. Then $1 + c^3 = \beta = \alpha^3 + \beta^3 + \gamma^3 = 0$. This gives $\beta = 0$, $1 + c^3 = 0$ and $\alpha^3 + \gamma^3 = 0$. Then

$$P = \left[1 : 0 : 0 : \omega^i\right]$$

and $Q = [\omega^j : 0 : 0 : 1]$ for some i and j. Then L is one of the lines

$$x + t = y + z = 0, x + t = y + \omega z = 0, x + t = y + \omega^2 z = 0,$$

$$x + \omega t = y + z = 0, x + \omega t = y + \omega z = 0, x + \omega t = y + \omega^2 z = 0,$$

$$x + \omega^2 t = y + z = 0, x + \omega^2 t = y + \omega z = 0, x + \omega^2 t = y + \omega^2 z = 0.$$

Lines on the Fermat cubic surface V

Thus, we may assume that $b \neq 0$. Recall that

$$1 + b^3 + c^3 = \alpha + b^2\beta = \alpha^2 + b\beta^2 = \alpha^3 + \beta^3 + \gamma^3 = 0$$

This implies that $\beta \neq 0$ and $\alpha \neq 0$, since $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Then we may assume that $\beta = 1$, since

$$Q = \left[\alpha : \beta : \gamma : 0\right] = \left[\frac{\alpha}{\beta} : 1 : \frac{\gamma}{\beta} : 0\right].$$

Then $1 + b^3 + c^3 = \alpha + b^2 = \alpha^2 + b = \alpha^3 + 1 + \gamma^3 = 0$. Now using $\alpha + b^2 = \alpha^2 + b = 0$, we get $b^3 = \alpha^3 = -1$. Then $c = \gamma = 0$, since $1 + b^3 + c^3 = \alpha^3 + 1 + \gamma^3 = 0$. Then

$$P = \left[1 : \omega^i : 0 : 0\right]$$

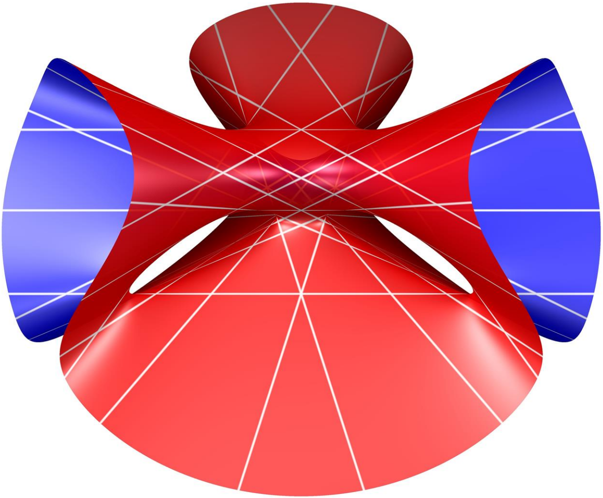
and $Q = [\omega^j : 1 : 0 : 1]$ for some i and j. Then L is one of the lines

$$z + t = x + y = 0, z + t = x + \omega y = 0, z + t = x + \omega^2 y = 0,$$

$$z + \omega t = x + y = 0, z + \omega t = x + \omega y = 0, z + \omega t = x + \omega^2 y = 0,$$

$$z + \omega^2 t = x + y = 0, z + \omega^2 t = x + \omega y = 0, z + \omega^2 t = x + \omega^2 y = 0.$$

Twenty seven lines on smooth cubic surface



Lines on $x^3 + y^3 + z^2t + t^3 = 0$ using brute force I

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + z^2t + t^3 = 0$.

- Let L be a line in $\mathbb{P}^3_{\mathbb{C}}$ such that $L \subset S_3$.
- Let $P = [0 : a : b : c]$ be the intersection of L with $x = 0$.
- Let $Q = [\alpha : 0 : \beta : \gamma]$ be the intersection of L with $y = 0$.

Suppose that $P = Q$. Then $P = Q$ is one of the three points

$$\Big[0 : 0 : 1 : 0\Big], \Big[0 : 0 : 1 : i\Big], \Big[0 : 0 : 1 : -i\Big].$$

Let Π be the tangent plane to S_3 at P. Then $\boxed{L \subseteq \Pi \cap S_3}$.

- If $P = [0 : 0 : 1 : 0]$, then Π is given by $t = 0$.
- If $P = [0 : 0 : 1 : i]$, then Π is given by $z + it = 0$.
- If $P = [0 : 0 : 1 : -i]$, then Π is given by $z - it = 0$.

Put $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. This gives us 9 lines on S_3 given by

$$x + y = t = 0, x + \omega y = t = 0, x + \omega^2 y = t = 0,$$

$$x + y = z + it = 0, x + \omega y = z + it = 0, x + \omega^2 y = z + it = 0,$$

$$x + y = z - it = 0, x + \omega y = z - it = 0, x + \omega^2 y = z - it = 0.$$

Lines on $x^3 + y^3 + z^2 t + t^3 = 0$ using brute force II

Now we assume that $P \neq Q$, and neither P nor Q is among

$$\big[0:0:1:0\big], \big[0:0:1:i\big], \big[0:0:1:-i\big].$$

Then the line L is given by

$$\lambda\big[0:a:b:c\big] + \mu\big[\alpha:0:\beta:\gamma\big],$$

where $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$ and $a \neq 0$ and $\alpha \neq 0$. Then

$$\boxed{\mu^3\alpha^3 + \lambda^3 a^3 + \Big(\lambda c + \mu\gamma\Big)\Big(\lambda b + \mu\beta\Big)^2 + \Big(\lambda c + \mu\gamma\Big)^3 = 0}$$

for every $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives

$$a^3+b^2c+c^3 = 2\beta bc+\gamma b^2+3\gamma c^2 = c\beta^2+2bc\beta+3c\gamma^2 = \alpha^3+\beta^2\gamma+\gamma^3 = 0.$$

Let us use these equations to find the remaining lines on S_3.

Lines on $x^3 + y^3 + z^2t + t^3 = 0$ using brute force III

The line L in $\mathbb{P}^3_{\mathbb{C}}$ consists of the points

$$\left[\mu\alpha : \lambda a : \lambda b + \mu\beta : \lambda c + \mu\gamma\right],$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$ and $a \neq 0$ and $\alpha \neq 0$ and

$$a^3+b^2c+c^3 = 2\beta bc+\gamma b^2+3\gamma c^2 = c\beta^2+2bc\beta+3c\gamma^2 = \alpha^3+\beta^2\gamma+\gamma^3 = 0.$$

Then $c \neq 0$ and $\gamma \neq 0$. Thus, we can put $c = \gamma = 1$. Then

$$a^3 + b^2 + 1 = 2\beta b + b^2 + 3 = \beta^2 + 2b\beta + 3 = \alpha^3 + \beta^2 + 1 = 0.$$

Then $b \neq 0$ and $\beta \neq 0$. Then $\beta = -\frac{3+b^2}{2b}$, so that

$$\left(-\frac{3+b^2}{2b}\right)^2 + 2b\left(-\frac{3+b^2}{2b}\right)^2 + 3 = \beta^2 + 2b\beta + 3 = 0,$$

which gives $b^4 - 2b^2 - 3 = 0$. Then either $b = \pm\sqrt{3}$ or $b = \pm i$.
If $b = \pm i$, then $a = 0$. By assumption, this is not the case.

- We have $b = \pm\sqrt{3}$, $\beta = \mp\sqrt{3}$, $a^3 = -4$ and $\alpha^3 = -4$.

Lines on $x^3 + y^3 + z^2t + t^3 = 0$ using brute force IV

Put $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then S_3 contains 9 lines

$$x + y = t = 0, x + \omega y = t = 0, x + \omega^2 y = t = 0,$$

$$x + y = z + it = 0, x + \omega y = z + it = 0, x + \omega^2 y = z + it = 0,$$

$$x + y = z - it = 0, x + \omega y = z - it = 0, x + \omega^2 y = z - it = 0.$$

For every i and j in $\{0, 1, 2\}$, the surface S_3 contains the line

$$\Big[- \mu\sqrt[3]{4}\omega^i : -\lambda\sqrt[3]{4}\omega^j : \pm\sqrt{3}\big(\lambda - \mu\big) : \lambda + \mu\Big],$$

where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives us 18 lines

$$\boxed{6y - \sqrt{3}\sqrt[3]{4}\omega^i z - 3\sqrt[3]{4}\omega^i t = 3\omega^i x + 3y\omega^j - \sqrt{3}\sqrt[3]{4}\omega^{j+i} z = 0,}$$

$$\boxed{6y + \sqrt{3}\sqrt[3]{4}\omega^i z - 3\sqrt[3]{4}\omega^i t = 3\omega^i x + 3y\omega^j + \sqrt{3}\sqrt[3]{4}\omega^{j+i} z = 0.}$$

Thus, we proved that S_3 does not contain other lines.

- This approach is not easy to apply in general.

Twenty seven lines on Clebsch cubic surface

Lines on $x^3 + y^3 + z^2 t + t^3 = 0$ using conics I

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + z^2 t + t^3 = 0$.

- The surface S_3 contains the line L given by $t = x + y = 0$.

Let Π be a plane in $\mathbb{P}^3_{\mathbb{C}}$ that contains the line L. Then

$$S_3 \cap \Pi = L \cup C,$$

where C is a conic in Π. The plane Π is given by

$$\lambda(x + y) + \mu t = 0$$

for some $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. Put $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

- If $[\lambda : \mu] = [0 : 1]$, then C splits as a union of the line

$$x + \omega y = t = 0$$

  and the line $x + \omega^2 y = t = 0$.
- If $[\lambda : \mu] = [1 : 0]$, then C splits as a union of the line

$$x + y = z + it = 0$$

  and the line $x + y = z - it = 0$.

Lines on $x^3 + y^3 + z^2 t + t^3 = 0$ using conics II

Let Π be a plane in $\mathbb{P}^3_{\mathbb{C}}$ given by $t = \lambda(x+y)$ for $\lambda \in \mathbb{C}$. Then

$$\Pi \cap S_3 = L \cup C,$$

where L is the line $t = x + y = 0$ and C is a conic in Π. Then

$$\begin{cases} t = \lambda(x+y) \\ x^3 + y^3 + \lambda z^2 (x+y) + \lambda^3 (x+y)^3 = 0 \end{cases}$$

defines the intersection $\Pi \cap S_3$. Then C is given by

$$\begin{cases} t = \lambda(x+y) \\ x^2 - xy + y^2 + \lambda z^2 + \lambda^3 (x+y)^2 = 0 \end{cases}$$

The conic C is isomorphic to the conic in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$(1+\lambda^3)x^2 + (2\lambda^3 - 1)xy + (1+\lambda^3)y^2 + \lambda z^2 = 0$$

Then C splits as a union of two lines if and only if

$$\begin{vmatrix} 1+\lambda^3 & \frac{2\lambda^3-1}{2} & 0 \\ \frac{2\lambda^3-1}{2} & 1+\lambda^3 & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda\Big(3\lambda^3 + \frac{3}{4}\Big) = 0.$$

Lines on $x^3 + y^3 + z^2 t + t^3 = 0$ using conics III

- Let Π be a plane in $\mathbb{P}^3_{\mathbb{C}}$ given by $t = \lambda(x + y)$ for $\lambda \in \mathbb{C}$.
- Then $\Pi \cap S_3$ is a union of the line $t = x + y = 0$ and conic

$$x^2 - xy + y^2 + \lambda z^2 + \lambda^3 (x + y)^2 = t - \lambda(x + y) = 0.$$

- This conic is reducible $\iff$ $\lambda = \infty,\ 0,\ -\frac{1}{\sqrt[3]{4}},\ -\frac{1}{\sqrt[3]{4}\omega},\ -\frac{1}{\sqrt[3]{4}\omega^2}$.

Thus, the line $t = x + y = 0$ gives us 10 more lines

1. $x + y = z + it = 0$,
2. $x + y = z - it = 0$,
3. $x + \omega y = t = 0$,
4. $x + \omega^2 y = t = 0$,
5. $\sqrt{3}x - \sqrt{3}y - z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + y) = 0$,
6. $\sqrt{3}x - \sqrt{3}y + z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + y) = 0$,
7. $\sqrt{3}x - \sqrt{3}y - z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + y) = 0$,
8. $\sqrt{3}x - \sqrt{3}y + z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + y) = 0$,
9. $\sqrt{3}x - \sqrt{3}y - z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + y) = 0$,
10. $\sqrt{3}x - \sqrt{3}y + z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + y) = 0$.

Lines on $x^3 + y^3 + z^2t + t^3 = 0$ using conics IV

- Let Π be a plane in $\mathbb{P}^3_{\mathbb{C}}$ given by $t = \lambda(x + \omega y)$ for $\lambda \in \mathbb{C}$.
- Then $\Pi \cap S_3$ is a union of the line $t = x + \omega y = 0$ and conic

$$x^2 - \omega xy + \omega^2 y^2 + \lambda z^2 + \lambda^3 (x + \omega y)^2 = t - \lambda(x + \omega y) = 0.$$

- This conic is reducible $\iff$ $\lambda = \infty$, 0, $-\frac{1}{\sqrt[3]{4}}$, $-\frac{1}{\sqrt[3]{4}\omega}$, $-\frac{1}{\sqrt[3]{4}\omega^2}$.

Thus, the line $t = x + \omega y = 0$ gives us 10 more lines

1. $x + \omega y = z + it = 0$,
2. $x + \omega y = z - it = 0$,
3. $x + \omega^2 y = t = 0$,
4. $x + y = t = 0$,
5. $\sqrt{3}x - \sqrt{3}\omega y - z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + \omega y) = 0$,
6. $\sqrt{3}x - \sqrt{3}\omega y + z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + \omega y) = 0$,
7. $\sqrt{3}x - \sqrt{3}\omega y - z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + \omega y) = 0$,
8. $\sqrt{3}x - \sqrt{3}\omega y + z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + \omega y) = 0$,
9. $\sqrt{3}x - \sqrt{3}\omega y - z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + \omega y) = 0$,
10. $\sqrt{3}x - \sqrt{3}\omega y + z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + \omega y) = 0$.

Lines on $x^3 + y^3 + z^2t + t^3 = 0$ using conics V

- Let Π be a plane in $\mathbb{P}^3_{\mathbb{C}}$ given by $t = \lambda(x + \omega^2 y)$ for $\lambda \in \mathbb{C}$.
- Then $\Pi \cap S_3$ is a union of the line $t = x + \omega^2 y = 0$ and conic

  $$x^2 - \omega^2 xy + \omega y^2 + \lambda z^2 + \lambda^3 (x + \omega^2 y)^2 = t - \lambda(x + \omega^2 y) = 0.$$

- This conic is reducible $\iff$ $\lambda = \infty,\ 0,\ -\frac{1}{\sqrt[3]{4}},\ -\frac{1}{\sqrt[3]{4}\omega},\ -\frac{1}{\sqrt[3]{4}\omega^2}$.

Thus, the line $t = x + \omega^2 y = 0$ gives us 10 more lines

1. $x + \omega^2 y = z + it = 0$,
2. $x + \omega^2 y = z - it = 0$,
3. $x + \omega y = t = 0$,
4. $x + y = t = 0$,
5. $\sqrt{3}x - \sqrt{3}\omega^2 y - z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + \omega^2 y) = 0$,
6. $\sqrt{3}x - \sqrt{3}\omega^2 y + z\sqrt[3]{4} = t + \frac{1}{\sqrt[3]{4}}(x + \omega^2 y) = 0$,
7. $\sqrt{3}x - \sqrt{3}\omega^2 y - z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + \omega^2 y) = 0$,
8. $\sqrt{3}x - \sqrt{3}\omega^2 y + z\sqrt[3]{4}\omega = t + \frac{1}{\sqrt[3]{4}\omega}(x + \omega^2 y) = 0$,
9. $\sqrt{3}x - \sqrt{3}\omega^2 y - z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + \omega^2 y) = 0$,
10. $\sqrt{3}x - \sqrt{3}\omega^2 y + z\sqrt[3]{4}\omega^2 = t + \frac{1}{\sqrt[3]{4}\omega^2}(x + \omega^2 y) = 0$.

Lines on $x^3 + y^3 + z^2 t + t^3 = 0$ using conics VI

Let Π be the plane $t = 0$. Then $\Pi \cap S_3$ is a union of 3 lines

$$\boxed{t = x + y = 0}, \boxed{t = x + \omega y = 0}, \boxed{t = x + \omega^2 y = 0}.$$

- We found 10 lines in S_3 that intersect $t = x + y = 0$.
- We found 10 lines in S_3 that intersect $t = x + \omega y = 0$.
- We found 10 lines in S_3 that intersect $t = x + \omega^2 y = 0$.
- This gives us 27 lines.

Let ℓ be a line in $\mathbb{P}^3_{\mathbb{C}}$ that is contained in S_3.

Lemma

ℓ intersects $t = x + y = 0$, $t = x + \omega y = 0$ or $t = x + \omega^2 y = 0$.

Proof.

Either $\ell \subset \Pi$ or $\ell \cap \Pi$ consists of one point. $\square$

- Thus, the 27 lines we found are all lines on the surface S_3.

Twenty seven lines on smooth cubic surface

Let S_3 be any smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$.

Theorem (Cayley, Salmon)

The surface S_3 contains exactly 27 lines.

Proof.

- Show that S_3 contains a line L_1.
- Find a plane $\Pi \subset \mathbb{P}^3_{\mathbb{C}}$ such that

  $$\Pi \cap S_3 = L_1 \cup L_2 \cup L_3$$

  for two more lines L_3 and L_3.
- Find all lines in S_3 that intersect L_1.
- Find all lines in S_3 that intersect L_2.
- Find all lines in S_3 that intersect L_3.
- This gives us all lines that are contained in S_3.
- Since S_3 is smooth, this gives 27 lines.

□

Smooth cubic surfaces as blow ups of the plane

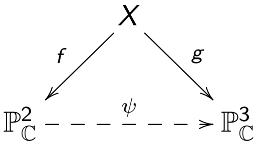
Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Let

$$\begin{cases} A(x,y,z) = \big(6\omega + 3\big)y^2z + 2ixz^2 - ix^2y, \\ B(x,y,z) = \big(3\omega - 3\big)y^2z + i\omega x^2y - 2ixz^2, \\ C(x,y,z) = (3\omega - 3)yz^2 + (3\omega + 6)xy^2 + ix^2z, \\ D(x,y,z) = i(3\omega - 3)yz^2 + x^2z. \end{cases}$$

Let $\phi \colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^3_{\mathbb{C}}$ be a map that it given by

$$\big[x : y : z\big] \mapsto \Big[A(x,y,z) : B(x,y,z) : C(x,y,z) : D(x,y,z)\Big].$$

Then ϕ is not defined at 6 points. There is a commutative diagram



where f blows up these 6 points, and g is well defined.

- The image of g is the surface given by $x^3 + y^3 + z^2t + t^3 = 0$.