Dubna 2018: lines on cubic surfaces

Ivan Cheltsov

24th July 2018

Lecture 4: rational parametrizations of cubic surfaces

Yuri Manin, Cubic Forms

ПРЕДИСЛОВИЕ

I

Любой математик, неравнодушный к теории чисел, испытал на себе очарование теоремы Ферма о сумме двух натуральных квадратов. Психолог юнговской школы нашел бы, вероятно, что такие диофантовы задачи в высшей степени архитипичны.

Замысел предлагаемой книги возник из попытки разобраться, что происходит с суммами трех рациональных кубов. Излишне говорить, что результат далек от простоты, фундаментальности и завершенности классических образцов. Автор обобщал задачу всеми способами, которые приходили ему на ум, и применял все технические средства, какие только умел. Получившееся в итоге нагромождение неассоциативных законов композиции, моноидальных преобразований и когомологий Галуа составило эту книжку.

II

Задача о суммах трех кубов имеет почтенную историю. Вот основной результат, оставленный классиками (см. Диксон [1]).

Теорема. *Любое рациональное число является суммой трех кубов рациональных чисел.*

Первое доказательство (Райли, 1825; Ричмонд, 1930):

$$a = \left(\frac{a^3 - 3^6}{3^2a^7 + 3^4a + 3^6}\right)^3 + \left(\frac{-a^3 + 3^5a + 3^6}{3^2a^2 + 3^4a + 3^6}\right)^3 + \left(\frac{a^2 + 3^4a}{3^2a^2 + 3^4a + 3^6}\right)^3.$$

Rational curves

Example (Pythagoras)

Let m, n, k be any integers. Then

$$\Big(k(m^2-n^2)\Big)^2+\Big(2kmn\Big)^2=\Big(k(m^2+n^2)\Big)^2,$$

which gives all integral solutions to $x^2+y^2=z^2$.

- Let $\mathcal{C}$ be a circle in $\mathbb{R}^2$ given by $x^2+y^2=1$.
- All points in $\mathcal{C}\setminus(1,0)$ with rational coordinates are given by

$$\left(\frac{m^2-k^2}{m^2+k^2},\frac{2mk}{m^2+k^2}\right)$$

for some integers m and k such that $(m,k)\neq(0,0)$.

- All points in $\mathcal{C}\setminus(1,0)$ with rational coordinates are given by

$$\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)$$

for some $t\in\mathbb{Q}$.

Non-rational curves

Theorem

Let $x(t)$, $y(t)$, $z(t)$ *be coprime polynomials in* $\mathbb{C}[t]$ *such that*

$$\boxed{x^3(t) + y^3(t) = z^3(t).}$$

Then all $x(t)$, $y(t)$, $z(t)$ *are constant.*

- The proof of this theorem is easy and elementary.

Theorem

Let $x(t)$, $y(t)$, $z(t)$ *be coprime polynomials in* $\mathbb{C}[t]$ *such that*

$$\boxed{x^n(t) + y^n(t) = z^n(t)}$$

for some $n \geqslant 3$. *Then* $x(t)$, $y(t)$, $z(t)$ *are constant.*

- The proof of this theorem is also easy and elementary.

Infinite descent

Let $x(t)$, $y(t)$, $z(t)$ be coprime non-zero polynomials in $\mathbb{C}[t]$ such that

$$\boxed{x^3(t) + y^3(t) = z^3(t)}$$

and $x(t)$, $y(t)$, $z(t)$ are coprime polynomials in $\mathbb{C}[t]$.
Then $x(t)$, $y(t)$, and $z(t)$ are pairwise coprime in $\mathbb{C}[t]$.
Let d_x, d_y, d_z be the degrees of $x(t)$, $y(t)$, $z(t)$, respectively.
Put $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then

$$\left(x(t) + y(t)\right)\left(x(t) + \omega y(t)\right)\left(x(t) + \omega^2 y(t)\right) = z^3(t),$$

and $x(t) + y(t)$, $x(t) + \omega y(t)$, $x(t) + \omega^2 y(t)$ are pairwise coprime.
Then there are polynomials $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ such that

$$\boxed{x(t) + y(t) = \alpha^3(t)}, \boxed{x(t) + \omega y(t) = \beta^3(t)}, \boxed{x(t) + \omega^2 y(t) = \gamma^3(t)}.$$

Then $-\omega\alpha^3(t) + (\omega + 1)\beta^3(t) = \gamma^3(t)$. Then

$$\boxed{\left(\sqrt[3]{-\omega}\,\alpha(t)\right)^3 + \left(\sqrt[3]{\omega + 1}\,\beta(t)\right)^3 = \gamma^3(t)}$$

and the degree of α is $\frac{d_z}{3}$. Now iterate.

Fermat cubic is non-rational

Theorem

Let $x(t)$ and $y(t)$ be rational functions in $\mathbb{C}(t)$ such that

$$\boxed{x^3(t) + y^3(t) = 1}.$$

Then both $x(t)$ and $y(t)$ are constant.

Proof.

We may assume that neither $x(t) = 0$ nor $y(t) = 0$.
There are coprime $a(t)$ and $b(t)$ in $\mathbb{C}[t]$ such that $x(t) = \frac{a(t)}{b(t)}$.
There are coprime $c(t)$ and $d(t)$ in $\mathbb{C}[t]$ such that $y(t) = \frac{c(t)}{d(t)}$.
Since $x^3(t) + y^3(t) = 1$, we have

$$\boxed{a^3(t)d^3(t) + c^3(t)b^3(t) = b^3(t)d^3(t).}$$

Then $b^3(t)|d^3(t)|b^3(t)$. Then $b(t) = \lambda d(t)$ for some $\lambda \in \mathbb{C}^*$.
This implies that $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are constant. □

Rational parametrization of the unit sphere

Let S_2 be the quadric surface in $\mathbb{C}^3$ that is given by

$$x^2 + y^2 + z^2 = 1.$$

Then S_2 has rational parametrization:

$$\left(\frac{1 - u^2 - v^2}{1 + u^2 + v^2}, \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2} \right).$$

When (v, u) runs through $\mathbb{C}^2$, we obtain $S_2 \setminus (-1, 0, 0)$.

Question

What is a rational parametrization of the sphere S_2?

The sphere S_2 also has rational parametrization:

$$\left(\frac{1 - (u^2)^2 - (v^4)^2}{1 + (u^2)^2 + (v^4)^2}, \frac{2(u^2)}{1 + (u^2)^2 + (v^4)^2}, \frac{2(v^4)}{1 + (u^2)^2 + (v^4)^2} \right).$$

When (v, u) runs through $\mathbb{C}^2$, we also obtain $S_2 \setminus (-1, 0, 0)$.

Rational parametrization of smooth quadrics

Let S_2 be the quadric surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$x^2 + y^2 + z^2 = t^2.$$

Then S_2 has rational parametrization:

$$\left[w^2 - u^2 - v^2 : 2uw : 2vw : w^2 + u^2 + v^2\right].$$

When $[v : u : w]$ runs through $\mathbb{P}^2_{\mathbb{C}}$ without $w = 0$, we obtain

$$S_2 \setminus \Big(L_1 \cup L_2\Big),$$

where L_1 and L_2 are the lines $w = u + iv = 0$ and $w = u - iv = 0$.

Question

What is a rational parametrization of the surface S_2?

- A dominant rational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow S_2$.
- A birational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow S_2$.

Rational and unirational varieties

Let X be an irreducible projective variety of dimension n.

Definition

X is rational if $\exists$ birational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.

Definition

X is unirational if $\exists$ dominant rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.

- If X is rational, then X is unirational.

Example

Irreducible conics in $\mathbb{P}^2_{\mathbb{C}}$ are rational.

Example

Smooth cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ are not unirational.

Let S_d be a smooth surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $d \geqslant 1$.

Theorem

If $d \geqslant 4$, then S_d is not unirational.

- If $d = 1$ or $d = 2$, then S_d is rational.

Lüroth Problem

Question

Are there unirational varieties of dimension n that are not rational?

Theorem (Lüroth, 1876)

Every subfield of $\mathbb{C}(x)$ that contains $\mathbb{C}$ is isomorphic to $\mathbb{C}(x)$.

Corollary

Every one-dimensional complex unirational variety is rational.

Theorem (Castelnuovo)

Every two-dimensional complex unirational variety is rational.

Theorem (Iskovskikh & Manin, 1971)

Every smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is not rational.

Theorem (Clemens & Griffiths, 1972)

Every smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is not rational.

- Some smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ are unirational.
- All smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ are unirational.

Rationality of smooth cubic surfaces

Theorem

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$. Then S_3 is rational.

Proof.

Define a map $\phi\colon \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ by

$$\Big([\alpha : \beta] : [\gamma : \delta]\Big) \to \Big[\alpha\gamma : \alpha\delta : \beta\gamma : \beta\delta\Big].$$

The image of ϕ is the quadric $S_2 \subset \mathbb{P}^3_{\mathbb{C}}$ given by $xt = yz$.
Let L_1 and L_2 be two lines in S_3 such that $L_1 \cap L_2 = \varnothing$.
Since $L_1 \cong L_2 \cong \mathbb{P}^1_{\mathbb{C}}$, we can identify $L_1 \times L_2 = S_2$ via ϕ.
Define a map $\psi\colon S_2 \dashrightarrow S_3$ as follows:

- Let (P, Q) be a general point in $L_1 \times L_2 = S_2$.
- Let ℓ be the line in $\mathbb{P}^3_{\mathbb{C}}$ that contains P and Q.
- Let $\phi\big((P, Q)\big)$ be the third point in $\ell \cap S_3$.

Then $\phi\colon S_2 \dashrightarrow S_3$ is a birational map.
Since S_2 is rational, the surface S_3 is also rational. □

Rational parametrization of $x^3+y^3+t+t^3=0$

Let S_3 be the surface in $\mathbb{C}^3$ that is given by $x^3+y^3+t+t^3=0$. Let L_1 and L_2 be the lines in $\mathbb{C}^3$ given by $x+y=t=0$ and

$$\omega x+y=t-i=0,$$

respectively. Here $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then $L_1\subset S_3$ and $L_2\subset S_3$. Put $P=(a,-a,0)$ and $Q=(b,-\omega b,i)$. Then $P\in L_1$ and $Q\in L_2$. Let ℓ be the line in $\mathbb{C}^3$ that contains P and Q. Then ℓ is given by

$$\Big(a+\lambda(b-a),-a+\lambda(a-\omega b),\lambda i\Big),$$

where $\lambda\in\mathbb{C}$. Then $\ell\cap S_3$ consists of the points P, Q and

$$\left(\frac{(6\omega+3)a^2b^2+2ia-ib}{(3\omega-3)a^2b+(3\omega+6)ab^2+i},\right.$$
$$\frac{(3\omega-3)a^2b^2+i\omega b-2ia}{(3\omega-3)a^2b+(3\omega+6)ab^2+i},$$
$$\left.\frac{i(3\omega-3)a^2b+1}{(3\omega-3)a^2b+(3\omega+6)ab^2+i}\right).$$

Rationality of $x^3 + y^3 + t + t^3 = 0$

Let S_3 be the surface in $\mathbb{C}^3$ that is given by $x^3 + y^3 + t + t^3 = 0$. Then there is a birational map $\mathbb{C}^2 \dashrightarrow S_3$ given by

$$(a, b) \mapsto \left(\frac{(6\omega + 3)a^2b^2 + 2ia - ib}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i}, \frac{(3\omega - 3)a^2b^2 + i\omega b - 2ia}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i}, \frac{i(3\omega - 3)a^2b + 1}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i} \right).$$

Compose it with the map $\mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ given by $(a, b) \mapsto \left(\frac{1}{a}, b\right)$. Then we obtain a birational map $\mathbb{C}^2 \dashrightarrow S_3$ given by

$$(a, b) \mapsto \left(\frac{(6\omega + 3)b^2 + 2ia - ia^2b}{(3\omega - 3)b + (3\omega + 6)ab^2 + ia^2}, \frac{(3\omega - 3)b^2 + i\omega a^2b - 2ia}{(3\omega - 3)b + (3\omega + 6)ab^2 + ia^2}, \frac{i(3\omega - 3)b + a^2}{(3\omega - 3)b + (3\omega + 6)ab^2 + ia^2} \right).$$

Rationality of the surface $x^3 + y^3 + z^2 t + t^3 = 0$

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + z^2 t + t^3 = 0$.
There is a birational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow S_3$ that maps $[a : b : c]$ to

$$\Big[(6\omega + 3)b^2 c + 2iac^2 - ia^2 b : (3\omega - 3)b^2 c + i\omega a^2 b - 2iac^2 :$$

$$: (3\omega - 3)bc^2 + (3\omega + 6)ab^2 + ia^2 c : i(3\omega - 3)bc + a^2 c\Big].$$

This map is undefined in the points

$$\begin{cases} (6\omega + 3)b^2 c + 2iac^2 - ia^2 b = 0, \\ (3\omega - 3)b^2 c + i\omega a^2 b - 2iac^2 = 0, \\ (3\omega - 3)bc^2 + (3\omega + 6)ab^2 + ia^2 c = 0, \\ i(3\omega - 3)bc^2 + a^2 c = 0. \end{cases}$$

This system of equations gives us are exactly 6 points in $\mathbb{P}^2_{\mathbb{C}}$.

- The inverse map $S_3 \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ is well defined.
- It contracts 6 disjoint lines in S_3 to the points above.

Serge's Theorem

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ that is defined over $\mathbb{Q}$.

Theorem

The surface S_3 is unirational over $\mathbb{Q}$ $\iff$ S_3 has a rational point.

Suppose that S_3 contains a rational point P.

- Let Π be the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_3 in P.
- Put $C = S_3 \cap \Pi$. Then C is a singular cubic curve.
- Then C is defined over $\mathbb{Q}$, since P is defined over $\mathbb{Q}$.
- Suppose that C is irreducible. Then C is rational over $\mathbb{Q}$.
- This gives us a infinitely many rational points in S_3.
- Pick one of them $Q \neq P$ and repeat the construction.
- This gives singular cubic curve $Z \subset S_3$ defined over $\mathbb{Q}$.

Now we can construct a dominant rational map

$$C \times Z \dashrightarrow S_3$$

as in the proof of rationality of complex smooth cubic surfaces.

Cubic Forms I

Theorem

Every rational number is a sum of three cubes of rational numbers.

Proof.

Let q be a rational number. Let us put

$$\alpha = \frac{1}{36}\frac{512q^4 - 1600q^3 + 108440q^2 - 173691q - 729}{128q^3 - 416q^2 + 8082q - 243}.$$

Note that $128q^3 - 416q^2 + 8082q - 243 \neq 0$. Put

$$\beta = -\frac{q(64q^2 - 1648q - 7263)}{128q^3 - 416q^2 + 8082q - 243}.$$

Similarly, let us put

$$\gamma = -\frac{1}{36}\frac{512q^4 - 1600q^3 - 15976q^2 + 246213q - 729}{128q^3 - 416q^2 + 8082q - 243}.$$

Using Maple, one can check that $\alpha^3 + \beta^3 + \gamma^3 = q$. $\square$

Cubic Forms II

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by

$$x^3 + y^3 + z^3 - qt^3 = 0,$$

where q is a non-zero rational number. Then S_3 is smooth.
Then S_3 is unirational over $\mathbb{Q}$ by Segre's Theorem.
Let us show this. To do this, replace S_3 by its affine part $z \neq 0$.
Thus, we may assume that S_3 is the surface in $\mathbb{Q}^3$ given by

$$x^3 + y^3 + 1 - qt^3 = 0.$$

Let ℓ be the line in $\mathbb{Q}^3$ that is given by

$$\Big(-1 + 2\lambda, \lambda, 0\Big),$$

where $\lambda \in \mathbb{Q}$. Then $\ell \cap S_3 = (-1, 0, 0)$ over $\mathbb{Q}$.
Over $\mathbb{Q}(\sqrt{-2})$ the intersection $\ell \cap S_3$ contains two more points:

$$\left(\frac{1 \pm 2\sqrt{-2}}{3}, \frac{2 \pm \sqrt{-2}}{3}, 0\right).$$

Cubic Forms III

Put $\widehat{x} = x - \frac{1+2\sqrt{-2}}{3}$, $\widehat{y} = y - \frac{2+\sqrt{-2}}{3}$, $\widehat{t} = t$. Then S_3 is given by

$$\Big(-\frac{7}{3}+\frac{4}{3}\sqrt{-2}\Big)\widehat{x}+\Big(\frac{2}{3}+\frac{4}{3}\sqrt{-2}\Big)\widehat{y}+ \\ +\big(1+2\sqrt{-2}\big)\widehat{x}^2+\big(2+\sqrt{-2}\big)\widehat{y}^2+\widehat{y}^3+\widehat{x}^3-q\widehat{t}^3=0.$$

Let Π be the tangent plane in $\mathbb{C}^3$ to S_3 at P. Then Π is given by

$$\widehat{y}=\frac{7-4\sqrt{-2}}{4\sqrt{-2}+2}\widehat{x}.$$

Thus, the intersection $\Pi\cap S_3$ is given by

$$\Big(-10\sqrt{-2}-31\Big)\widehat{x}^3+\Big(36-18\sqrt{-2}\Big)\widehat{x}^2+8q\widehat{t}^3=0.$$

Intersecting this curve with the line $t=\lambda x$ in Π, we get the point

$$\left(\frac{2-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}},\frac{36\lambda-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}},\frac{-27\sqrt{-2}-54}{31-8q\lambda^3+10\sqrt{-2}}\right).$$

Cubic Forms IV

We see that the surface S_3 contain the point

$$\left(\frac{2-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{36\lambda-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{-27\sqrt{-2}-54}{31-8q\lambda^3+10\sqrt{-2}}\right)$$

in coordinates $\widehat{x}=x-\frac{1+2\sqrt{-2}}{3}$, $\widehat{y}=y-\frac{2+\sqrt{-2}}{3}$, $\widehat{t}=t$.
Rewriting this point in coordinated x, y and t, we obtain the point

$$\left(-\frac{2\sqrt{-2}+1}{3}\cdot\frac{8q\lambda^3+20\sqrt{-2}-19}{31-8q\lambda^3+10\sqrt{-2}},\right.$$
$$\frac{2\sqrt{-2}+4}{3}\cdot\frac{-4q\lambda^3+5\sqrt{-2}-25}{31-8q\lambda^3+10\sqrt{-2}},$$
$$\left.\frac{\lambda(36-18\sqrt{-2})}{31-8q\lambda^3+10\sqrt{-2}}\right)$$

contained in S_3 for any $\lambda\in\mathbb{C}$ such that $31-8q\lambda^3+10\sqrt{-2}\neq 0$.

- Main trick: put $\boxed{\lambda=a+b\sqrt{-2}}$.

Cubic Forms V

Recall that S_3 is the surface in $\mathbb{Q}^3$ given by $x^3+y^3+1=qt^3$. Put

$$x_1=\frac{1}{3}\,\frac{(2\sqrt{-2}+1)(-16\sqrt{-2}b^3q-48ab^2q+24\sqrt{-2}a^2bq+8a^3q+20\sqrt{-2}-19)}{-16\sqrt{-2}b^3q-48ab^2q+24\sqrt{-2}a^2bq+8a^3q-10\sqrt{-2}-31},$$

$$y_1=\frac{2}{3}\,\frac{(\sqrt{-2}+2)(-8\sqrt{-2}b^3q-24ab^2q+12\sqrt{-2}a^2bq+4a^3q-5\sqrt{-2}+25)}{-16\sqrt{-2}b^3q-48ab^2q+24\sqrt{-2}a^2bq+8a^3q-10\sqrt{-2}-31},$$

$$t_1=\frac{18(a+b\sqrt{-2})(\sqrt{-2}-2)}{-16\sqrt{-2}b^3q-48ab^2q+24\sqrt{-2}a^2bq+8a^3q-10\sqrt{-2}-31}.$$

Then $(x_1,y_1,t_1)\in S_3$ for every rational a and b such that

$$-16\sqrt{-2}b^3q-48ab^2q+24\sqrt{-2}a^2bq+8a^3q-10\sqrt{-2}-31\neq 0.$$

The complex conjugate point $(\overline{x}_1,\overline{y}_1,\overline{t}_1)$ also lies in S_3. Put

$$x_2=\frac{1}{3}\,\frac{(-2\sqrt{-2}+1)(16\sqrt{-2}b^3q-48ab^2q-24\sqrt{-2}a^2bq+8a^3q-20\sqrt{-2}-19)}{16\sqrt{-2}b^3q-48ab^2q-24\sqrt{-2}a^2bq+8a^3q+10\sqrt{-2}-31},$$

$$y_2=\frac{2}{3}\,\frac{(-\sqrt{-2}+2)(8\sqrt{-2}b^3q-24ab^2q-12\sqrt{-2}a^2bq+4a^3q+5\sqrt{-2}+25)}{16\sqrt{-2}b^3q-48ab^2q-24\sqrt{-2}a^2bq+8a^3q+10\sqrt{-2}-31},$$

$$t_2=\frac{18(a-b\sqrt{-2})(\sqrt{-2}-2)}{16\sqrt{-2}b^3q-48ab^2q-24\sqrt{-2}a^2bq+8a^3q+10\sqrt{-2}-31}.$$

Then $(x_2,y_2,t_2)=(\overline{x}_1,\overline{y}_1,\overline{t}_1)$ is contained in S_3.

Cubic Forms VI

Let L be the line that contains (x_1, y_1, t_1) and (x_2, y_2, t_2). Then L is defined over $\mathbb{Q}$.
The intersection $L \cap S_3$ consists of (x_1, y_1, t_1), (x_2, y_2, t_2) and $\left(\frac{\theta_1}{\epsilon}, \frac{\theta_2}{\epsilon}, \frac{\theta_3}{\epsilon}\right)$, where

$$\theta_1 = -512a^{12}q^4 + 6144a^{10}b^2q^4 + 30720a^8b^4q^4 + 81920a^6b^6q^4 + 122880a^4b^8q^4 + 98304a^2b^1 0q^4 + \\ + 32768b^1 2q^4 - 1600a^9q^3 + 1920a^8bq^3 + 10240a^6b^3q^3 + 38400a^5b^4q^3 + 15360a^4b^5q^3 + 102400a^3b^6q^3 + \\ + 76800ab^8q^3 - 10240b^9q^3 + 108440a^6q^2 + 30048a^5bq^2 - 317760a^4b^2q^2 - 760192a^3b^3q^2 + 1192800a^2b^4q^2 + \\ + 120192ab^5q^2 - 496000b^6q^2 - 173691a^3q + 633582a^2bq - 729324ab^2q + 286200b^3q - 729.$$

$$\theta_2 = 2304a^9q^3 + 34560a^8bq^3 + 184320a^6b^3q^3 - 55296a^5b^4q^3 + 276480a^4b^5q^3 - \\ - 147456a^3b^6q^3 - 110592ab^8q^3 - 184320b^9q^3 - 59328a^6q^2 - 146880a^5bq^2 + 100224a^4b^2q^2 + 419328a^3b^3q^2 - \\ - 200448a^2b^4q^2 - 587520ab^5q^2 + 474624b^6q^2 - 261468a^3q + 801900a^2bq - 793152ab^2q + 252720b^3q.$$

$$\theta_3 = -4608a^{10}q^3 - 4608a^9bq^3 - 27648a^8b^2q^3 - 36864a^7b^3q^3 - 36864a^6b^4q^3 - \\ - 110592a^5b^5q^3 + 73728a^4b^6q^3 - 147456a^3b^7q^3 + 221184a^2b^8q^3 - 73728ab^9q^3 + 147456b^{10}q^3 + \\ + 14976a^7q^2 - 19584a^6bq^2 + 165888a^5b^2q^2 - 105984a^4b^3q^2 + 281088a^3b^4q^2 - 207360a^2b^5q^2 + 18432ab^6q^2 - \\ - 147456b^7q^2 - 290952a^4q + 255960a^3bq + 820368a^2b^2q - 1402272ab^3q + 616896b^4q + 8748a - 8748b.$$

$$\epsilon = 512a^{12}q^4 + 6144a^{10}b^2q^4 + 30720a^8b^4q^4 + 81920a^6b^6q^4 + 122880a^4b^8q^4 + \\ + 98304a^2b^1 0q^4 + 32768b^1 2q^4 - 1600a^9q^3 + 1920a^8bq^3 + 10240a^6b^3q^3 + 38400a^5b^4q^3 + 15360a^4b^5q^3 + \\ + 102400a^3b^6q^3 + 76800ab^8q^3 - 10240b^9q^3 - 15976a^6q^2 - 343200a^5bq^2 + 55488a^4b^2q^2 + 608384a^3b^3q^2 + \\ + 446304a^2b^4q^2 - 1372800ab^5q^2 + 499328b^6q^2 + 246213a^3q - 626130a^2bq + 530388ab^2q - 133704b^3q - 729.$$

Cubic Forms VII

For every rational a and b such that $\epsilon \neq 0$, we have

$$\left(\frac{\theta_1}{\epsilon}\right)^3 + \left(\frac{\theta_2}{\epsilon}\right)^3 + 1 = q\left(\frac{\theta_3}{\epsilon}\right)^3.$$

Thus, for every rational a and b such that $\theta_3 \neq 0$, we have

$$q = \left(\frac{\theta_1}{\theta_3}\right)^3 + \left(\frac{\theta_2}{\theta_3}\right)^3 + \left(\frac{\epsilon}{\theta_3}\right)^3.$$

For example, put $a = 1$ and $b = 0$. Then

$$\frac{\theta_1}{\theta_3} = \frac{1}{36}\frac{512q^4 - 1600q^3 + 108440q^2 - 173691q - 729}{128q^3 - 416q^2 + 8082q - 243},$$

$$\frac{\theta_2}{\theta_3} = -\frac{q(64q^2 - 1648q - 7263)}{128q^3 - 416q^2 + 8082q - 243},$$

$$\frac{\epsilon}{\theta_3} = -\frac{1}{36}\frac{512q^4 - 1600q^3 - 15976q^2 + 246213q - 729}{128q^3 - 416q^2 + 8082q - 243}.$$

Non-rational unirational cubic surfaces

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ that is defined over $\mathbb{Q}$.

Theorem (Segre, 1943)

Suppose that for every curve $C \subset S_3$ defined over $\mathbb{Q}$ one has

$$C = S_3 \cap F$$

for some surface F in $\mathbb{P}^3_{\mathbb{C}}$. Then S_3 is not rational over $\mathbb{Q}$.

Example

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$2x^3 + 3y^3 + 5z^3 + 7t^3 = 0.$$

Then for every curve $C \subset S_3$ defined over $\mathbb{Q}$ one has

$$C = S_3 \cap F$$

for some surface $F \subset \mathbb{P}^3_{\mathbb{C}}$. But $[1 : 1 : -1 : 0] \in S_3$.
Thus, the surface S_3 is unirational and non-rational over $\mathbb{Q}$.