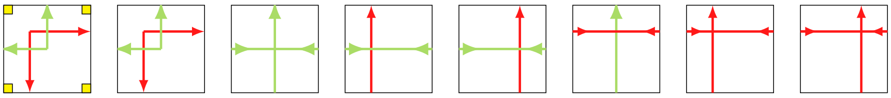
3.e Exercises for Lecture 3

Here are all the exercises from the lecture notes, reorganized and renumbered to make an exercise sheet. There are also a few new questions, to keep things interesting. Solve only what you like.

Robinson tileset

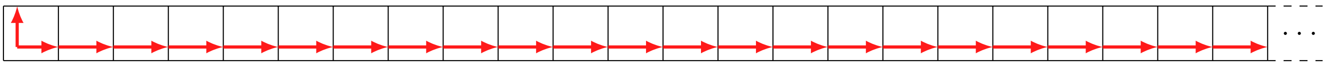
$R =$ 

Exercise 3.e.1. (a) Prove that there are infinitely many Robinson tilings of the plane.

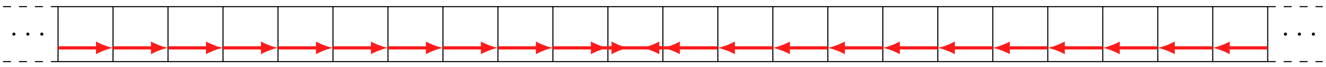
(b) Prove that there are *uncountably* many Robinson tilings of the plane.

(c) Prove that no Robinson tiling has a period: thus, the Robinson tileset is aperiodic.

(d) Show that there exists a Robinson tiling with a right-infinite red arrow:



(e) Show that there exists a Robinson tiling with a bi-infinite red arrow:



Exercise 3.e.2. An *emulation* of R by Wang tiles is a pair (W, f) where W is a Wang tileset and f is a map $W \to R$ such that for any valid tiling T by W, the tiling $f(T)$ by R is also valid. (Apply f on each tile of T to get $f(T)$.) Design an emulation of R by Wang tiles, having:

(a) ≤ 64 tiles;

(b) 56 tiles.

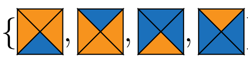
Hierarchy

If X and Y are two rectangles of tiles with the same height, then we define $\oplus$ as $X \oplus Y :=$ [X | Y].

If X and Y are two rectangles of tiles with the same width, then we define $\ominus$ as: $X \ominus Y :=$ [X over Y].

Given a tileset A, an $(n \times m)$-*substitution* is a function $s : A \to A^{n\times m}$ which maps each tile to a valid block of size $n \times m$, so that for all tiles x, y in A:

- $x \ominus y$ is valid if and only if $s(x) \ominus s(y)$ is valid;
- $x \oplus y$ is valid if and only if $s(x) \oplus s(y)$ is valid.

Exercise 3.e.3. Consider the tileset $T = \{$$\}$:

(a) find an (1×3)-subsitution for T;

(b) find an $(n \times 3)$-substitution for T, for all $n \geq 2$.

Exercise 3.e.4. Let A be a tileset, s be a substitution $s : A \to A^{n\times m}$ and T be a tiling of the plane by A. We say that T is a *substitution tiling* for s if and only if, for any block C appearing in T, there exists an integer n and a tile a such that C appears in $s^n(a)$.

(a) Design a substitution tiling for the (2×3)-substitution from Exercise 3.e.3.

(b) Prove that there exist no substitution tiling for the (1×3)-substitution from Exercise 3.e.3.

(c) Prove that, for any tileset A and any substitution $s : A \to A^{1\times n}$, there exist no substitution tiling for A and s. Same question if s is $s : A \to A^{n\times 1}$.

Exercise 3.e.5. Let A denote a tileset, s denote a substitution and T denote a substitution tiling for A and s. We say that T has an *unique derivation* if there is *exactly one* tiling U such that $T = s(U)$.

(a) Show that the tiling you found for Exercise 3.e.4(a) doesn't have unique derivation.

(b) Show that there is no tiling with unique derivation for the (2×3)-substitution from Exercise 3.e.3.

(c) Let T_0 denote a substitution tiling with substitution s. Prove that if there is an *unique* infinite sequence of tilings $T_1, T_2, T_3, \ldots$ such that:

$$T_n = s(T_{n+1})$$

for all n in $\mathbb{N}$ (in particular, each T_n has an unique derivation), then T_0 is aperiodic.

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