L-FUNCTIONS AND THE RIEMANN HYPOTHESIS

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1. The zeta-function and Dirichlet L-functions

For real $s > 1$, the infinite series

$$\sum_{n\geq 1} \frac{1}{n^s} \tag{1.1}$$

converges by the integral test. We want to use this series when s is a complex number. First we describe a simple convergence test for infinite series of complex numbers and then we explain what n^s means when $s \in \mathbf{C}$.

Definition 1.1. An infinite series of complex numbers $\sum_{n\geq 1} z_n$ is defined, like an infinite series of real numbers, as the limit of its partial sums:

$$\sum_{n\geq 1} z_n = \lim_{N\to\infty} \sum_{n=1}^{N} z_n.$$

For an infinite series of real numbers $\sum_{n\geq 1} x_n$ where the terms are not all positive, the most important convergence test is the absolute convergence test: if the nonnegative series $\sum_{n\geq 1} |x_n|$ converges then the original series $\sum_{n\geq 1} x_n$ converges.[1] The absolute convergence test also works for an infinite series of complex numbers.

Theorem 1.2. *If $\sum_{n\geq 1} |z_n|$ converges in $\mathbf{R}$ then $\sum_{n\geq 1} z_n$ converges in $\mathbf{C}$*

Proof. Let $s_N = \sum_{n=1}^{N} z_n$. To prove the numbers s_N converge in $\mathbf{C}$, the idea is to prove they form a Cauchy sequence. For $N > M \geq 1$,

$$|s_N - s_M| = \left| \sum_{n=M+1}^{N} z_N \right| \leq \sum_{n=M+1}^{N} |z_N|. \tag{1.2}$$

Since the series of real numbers $\sum_{n\geq 1} |z_n|$ is assumed to converge, the sequence of its partial sums $\sum_{n=1}^{N} |z_n|$ is a Cauchy sequence in $\mathbf{R}$, so the numbers $\sum_{n=M+1}^{N} |z_N|$ become arbitrarily small when M and N are large enough. That means by (1.2) that $|s_N - s_M|$ is arbitrarily small when M and N are large enough, so the numbers s_N are a Cauchy sequence in $\mathbf{C}$ and thus converge in $\mathbf{C}$. □

Remark 1.3. Every rearrangement of the terms in an absolutely convergent series in $\mathbf{R}$ also converges to the same value, and the same property is true for absolutely convergent series in $\mathbf{C}$.

[1] The converse is false: $\sum_{n\geq 1} (-1)^{n-1}/n$ converges but $\sum_{n\geq 1} |(-1)^{n-1}/n| = \sum_{n\geq 1} 1/n$ does not converge.

We put these ideas to work to define the exponential function on $\mathbf{C}$. From calculus,

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}$$

for all $x \in \mathbf{R}$. We use the right side to define the exponential function on complex numbers.

Definition 1.4. For $s \in \mathbf{C}$, $e^s := \displaystyle\sum_{n\geq 0} \frac{s^n}{n!}$.

This series converges by the absolute convergence test (Theorem 1.2): $\displaystyle\sum_{n\geq 0} \left|\frac{s^n}{n!}\right| = \sum_{n\geq 0} \frac{|s|^n}{n!}$, which is finite since it is $e^{|s|}$. The important algebraic property $e^x e^y = e^{x+y}$ for $x, y \in \mathbf{R}$ holds for the complex exponential function: $e^z e^w = e^{z+w}$ for all $z, w \in \mathbf{C}$. In particular, $e^z e^{-z} = e^{z-z} = e^0 = 1$, so $e^z \neq 0$: *the complex exponential function is never zero.*

The next theorem says the absolute value (modulus) of e^s depends on s only by its real part.

Theorem 1.5. *If* $s \in \mathbf{C}$ *then* $|e^s| = e^{\operatorname{Re}(s)}$.

Proof. For every complex number $z = x + iy$, $|z|^2 = x^2 + y^2 = z\overline{z}$. Therefore

$$|e^s|^2 = e^s\overline{e^s} = e^s e^{\overline{s}} = e^{s+\overline{s}} = e^{2\operatorname{Re}(s)} = (e^{\operatorname{Re}(s)})^2,$$

so $|e^s| = e^{\operatorname{Re}(s)}$ since $|e^s|$ and $e^{\operatorname{Re}(s)}$ are both positive numbers. □

In particular, e^s has a constant absolute value along every vertical line in $\mathbf{C}$: $|e^{x+iy}| = e^x$ for all $x, y \in \mathbf{R}$.

Definition 1.6. For $a > 0$ in $\mathbf{R}$, set $a^s := e^{s\ln a}$.

This function a^s with positive base a has properties similar to the function e^s: $a^z a^w = a^{z+w}$ (so $a^z \neq 0$ for all $z \in \mathbf{C}$) and $|a^s| = a^{\operatorname{Re}(s)}$. We will *not* define complex powers of general complex numbers, only complex powers of positive numbers.

For a positive integer n we have $|n^s| = n^{\operatorname{Re}(s)}$ for $s \in \mathbf{C}$, so the series (1.1) with $s \in \mathbf{C}$ is absolutely convergent when $\operatorname{Re}(s) > 1$, and thus it is convergent by Theorem 1.2.

Definition 1.7. For $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 1$, the *Riemann zeta-function* at s is

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

There is a connection between $\zeta(s)$ and the prime numbers, first discovered by Euler in the 1700s for real s.

Theorem 1.8. *For* $s \in \mathbf{C}$ *with* $\operatorname{Re}(s) > 1$,

$$\zeta(s) = \prod_p \frac{1}{1 - 1/p^s} = \frac{1}{1-1/2^s}\frac{1}{1-1/3^s}\frac{1}{1-1/5^s}\cdots.$$

Proof. The idea is to expand each factor $1/(1 - 1/p^s)$ into a geometric series and multiply together all those geometric series. What is a geometric series in $\mathbf{C}$? It is $\sum_{n\geq 0} z^n$ for a complex number z, and just like a geometric series of real numbers, this series converges if and only if $|z| < 1$, in which case

$$\sum_{n \geq 0} z^n = \frac{1}{1-z}.$$

Taking $z = 1/p^s$ for a prime p, we have $|1/p^s| < 1$ when $1/p^{\mathrm{Re}(s)} < 1$, which means $\mathrm{Re}(s) > 0$. Therefore

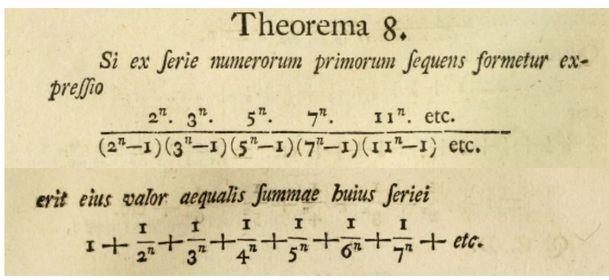
$$\mathrm{Re}(s) > 0 \Longrightarrow \frac{1}{1-1/p^s} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \tag{1.3}$$

When we multiply together this series for $p = 2$ and $p = 3$ we get

$$\left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{27^s} + \cdots\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$$

where the right side is the sum of all $1/n^s$ where n has only prime factors 2 or 3 (or both). If we multiply together the series (1.3) as p runs over *all* prime numbers, we need $\mathrm{Re}(s) > 1$ rather than just $\mathrm{Re}(s) > 0$ to justify the calculations and we get the sum of $1/n^s$ where n runs over *all* positive integers by the uniqueness of prime factorization. The sum of all $1/n^s$ is $\zeta(s)$, so we are done. □

The product in Theorem 1.8 is called the *Euler product* representation of $\zeta(s)$. Here is how it appeared in Euler's paper, where he wrote $1/(1-1/p^s)$ as $p^s/(p^s-1)$.


Theorema 8.

Si ex ſerie numerorum primorum ſequens formetur expreſſio

$$\frac{2^n.\ 3^n.\ 5^n.\ 7^n.\ 11^n.\ \text{etc.}}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\ \text{etc.}}$$

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$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \textit{etc.}$$

Each factor in the Euler product is nonzero, and from this $\boxed{\zeta(s) \neq 0 \text{ when } \mathrm{Re}(s) > 1}$. This property is *not* obvious if we only use the series that defines $\zeta(s)$ (how can you tell when an infinite series is nonzero?).

In the 1830s, Dirichlet introduced a generalization of the Riemann zeta-function, where the coefficients in the series for $\zeta(s)$ are not all 1.

Definition 1.9. For $m \geq 1$, $(\mathbf{Z}/m\mathbf{Z})^\times$ denotes the invertible numbers modulo m. A function $\chi\colon (\mathbf{Z}/m\mathbf{Z})^\times \to \mathbf{C}^\times$ is called a *Dirichlet character*, or a *Dirichlet character mod* m when we want to specify the modulus, if it is multiplicative[2]: $\chi(ab) = \chi(a)\chi(b)$ for all a and b in $(\mathbf{Z}/m\mathbf{Z})^\times$.

Example 1.10. The character χ_4 on $(\mathbf{Z}/4\mathbf{Z})^\times$ is defined by the rule

$$\chi_4(a \bmod 4) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 4, \\ -1, & \text{if } a \equiv 3 \bmod 4. \end{cases}$$

This is multiplicative since $3^2 \equiv 1 \bmod 4$ and $\chi_4(3)^2 = (-1)^2 = 1 = \chi_4(3^2)$.

[2] In the language of abstract algebra, we call this a *group homomorphism*.

It does *not* make sense to define a character χ on $(\mathbf{Z}/4\mathbf{Z})^\times$ by

$$\chi(a \bmod 4) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 4, \\ i, & \text{if } a \equiv 3 \bmod 4. \end{cases}$$

since this is not multiplicative: $3^2 \equiv 1 \bmod 4$ but $\chi(3)^2 = i^2 = -1$ while $\chi(1) = 1$.

Example 1.11. In $(\mathbf{Z}/5\mathbf{Z})^\times$, every number is a power of 2: $1 \equiv 2^0 \bmod 5$, $2 \equiv 2^1 \bmod 5$, $3 \equiv 2^3 \bmod 5$, and $4 \equiv 2^2 \bmod 5$. A power $2^k \bmod 5$ depends on k modulo 4 since $2^{k+4\ell} = 2^k 16^\ell \equiv 2^k \bmod 5$ for all $\ell \in \mathbf{Z}$. Therefore we can define a character χ_5 on $(\mathbf{Z}/5\mathbf{Z})^\times$ having values in the 4th roots of unity in $\mathbf{C}^\times$ by the rule

$$\chi_5(2^k \bmod 5) = i^k.$$

Here is an explicit formula for the values of this character:

$$\chi_5(a) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 5, \\ i, & \text{if } a \equiv 2 \bmod 5, \\ -i, & \text{if } a \equiv 3 \bmod 5, \\ -1, & \text{if } a \equiv 4 \bmod 5. \end{cases}$$

A Dirichlet character mod m is a multiplicative function defined on the integers that are relatively prime to m. If χ is 1 on all of $(\mathbf{Z}/m\mathbf{Z})^\times$, we call χ the *trivial* Dirichlet character mod m and write $\chi = \mathbf{1}_m$. There is a trivial Dirichlet character for each modulus.

Each element of $(\mathbf{Z}/m\mathbf{Z})^\times$ has finite order: $a^{\varphi(m)} \equiv 1 \bmod m$ for all a in $(\mathbf{Z}/m\mathbf{Z})^\times$. Therefore the values of a Dirichlet character χ on $(\mathbf{Z}/m\mathbf{Z})^\times$ have to be roots of unity in $\mathbf{C}$:

$$a^{\varphi(m)} \equiv 1 \bmod m \Longrightarrow \chi(a)^{\varphi(m)} = 1 \text{ in } \mathbf{C}.$$

We can consider χ as a function of period m on all integers, not just on integers relatively prime to m, by defining $\chi(n) = 0$ if $\gcd(n, m) > 1$. As a function on $\mathbf{Z}$, χ remains multiplicative: $\chi(ab) = \chi(a)\chi(b)$ for *all* integers a and b.

Example 1.12. The character χ_4 from Example 1.10 is defined on all integers by

$$\chi_4(a) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 4, \\ -1, & \text{if } a \equiv 3 \bmod 4, \\ 0, & \text{if } a \text{ is even.} \end{cases}$$

Example 1.13. The character χ_5 from Example 1.11 is defined on all integers by

$$\chi_5(a) = \begin{cases} 1, & \text{if } a \equiv 1 \bmod 5, \\ i, & \text{if } a \equiv 2 \bmod 5, \\ -i, & \text{if } a \equiv 3 \bmod 5, \\ -1, & \text{if } a \equiv 4 \bmod 5, \\ 0, & \text{if } a \text{ is a multiple of } 5. \end{cases}$$

Example 1.14. The trivial character mod m is defined on all integers by

$$\mathbf{1}_m(n) = \begin{cases} 1, & \text{if } (n, m) = 1, \\ 0, & \text{if } (n, m) > 1, \end{cases}$$

Definition 1.15. For a Dirichlet character χ, the *Dirichlet L-function*[3] of χ for $\mathrm{Re}(s) > 1$ is

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}. \tag{1.4}$$

Why do we say $\mathrm{Re}(s) > 1$ in the definition of $L(s,\chi)$? Since $\chi(n)$ is either a root of unity or 0, $|\chi(n)| = 1$ or 0. Therefore $|\chi(n)| \leq 1$ for all positive integers n, so the series (1.4) converges for $\mathrm{Re}(s) > 1$ because it is absolutely convergent:

$$\sum_{n\geq 1}\left|\frac{\chi(n)}{n^s}\right| \leq \sum_{n\geq 1}\frac{1}{n^{\mathrm{Re}(s)}} < \infty \Longrightarrow \sum_{n\geq 1}\left|\frac{\chi(n)}{n^s}\right| \text{ converges} \Longrightarrow \sum_{n\geq 1}\frac{\chi(n)}{n^s} \text{ converges.}$$

Example 1.16. The L-function of χ_4 is

$$L(s,\chi_4) = \sum_{n\geq 1}\frac{\chi_4(n)}{n^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} + \frac{1}{13^s} - \frac{1}{15^s} + \cdots$$

when $\mathrm{Re}(s) > 1$, with alternating signs in the numerators and powers of odd numbers in the denominators.

Example 1.17. The L-function of χ_5 is

$$L(s,\chi_5) = \sum_{n\geq 1}\frac{\chi_5(n)}{n^s} = 1 + \frac{i}{2^s} - \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \frac{i}{7^s} - \frac{i}{8^s} - \frac{1}{9^s} + \frac{1}{11^s} + \cdots$$

when $\mathrm{Re}(s) > 1$.

Example 1.18. The L-function of the trivial character mod m is

$$L(s,\mathbf{1}_m) = \sum_{n\geq 1}\frac{\mathbf{1}_m(n)}{n^s} = \sum_{(n,m)=1}\frac{1}{n^s},$$

which looks like the zeta-function without terms at integers that have a factor in common with m.

Since a Dirichlet character χ is multiplicative, $L(s,\chi)$ has an Euler product: if $\mathrm{Re}(s) > 1$,

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)/p^s} = \frac{1}{1-\chi(2)/2^s}\frac{1}{1-\chi(3)/3^s}\frac{1}{1-\chi(5)/5^s}\frac{1}{1-\chi(7)/7^s}\cdots.$$

Proving this infinite product equals the series (1.4) is similar to the proof that $\zeta(s)$ has an Euler product, and is left to the reader. Like the zeta-function, from the Euler product we have $\boxed{L(s,\chi) \neq 0 \text{ when } \mathrm{Re}(s) > 1}$.

Example 1.19. For $\mathrm{Re}(s) > 1$,

$$L(s,\chi_4) = \prod_p \frac{1}{1-\chi_4(p)/p^s} = \frac{1}{1+1/3^s}\frac{1}{1-1/5^s}\frac{1}{1+1/7^s}\frac{1}{1+1/11^s}\frac{1}{1-1/13^s}\cdots.$$

The Euler factor at $p = 2$ is 1 since $\chi_4(2) = 0$. Although $\chi_4(n)$ has alternating values $1, -1, 1, -1, \ldots$ on odd numbers, it does *not* have alternating values on odd prime numbers!

[3] It is not known why Dirichlet denoted his functions with an L.

Example 1.20. For $\text{Re}(s) > 1$,

$$L(s, \chi_5) = \prod_p \frac{1}{1 - \chi_5(p)/p^s} = \frac{1}{1 - i/2^s} \frac{1}{1 + i/3^s} \frac{1}{1 - i/7^s} \frac{1}{1 - 1/11^s} \frac{1}{1 + i/13^s} \cdots.$$

The Euler factor at $p = 5$ is 1 since $\chi_5(5) = 0$.

Example 1.21. For $m > 1$,

$$L(s, \mathbf{1}_m) = \prod_p \frac{1}{1 - \mathbf{1}_m(p)/p^s} = \prod_{(p,m)=1} \frac{1}{1 - 1/p^s}.$$

This is the Euler product for the zeta-function with the factors at primes dividing m removed.

The importance of $\zeta(s)$ and the functions $L(s, \chi)$ in number theory is that important theorems about prime numbers depend on properties of these functions, but these properties involve the functions *outside* the region $\text{Re}(s) > 1$ where they are initially defined. Our next goal is to explain how to extend $\zeta(s)$ and $L(s, \chi)$ to the whole complex plane, except at $s = 1$ in the case of the zeta-function.

2. The Γ-function

To define $\zeta(s)$ and $L(s, \chi)$ beyond $\text{Re}(s) > 1$ we will use an idea of Riemann, which involves the Γ-function, which is defined by an improper integral depending on a parameter.

In 1729, Euler essentially discovered that

$$n! = \int_0^\infty x^n e^{-x}\, dx$$

for integers $n \geq 0$. The right side makes sense even if n is not an integer: check as an exercise that the improper integral $\int_0^\infty x^t e^{-x}\, dx$ converges for all real $t > -1$. Since $|x^s e^{-x}| = x^{\text{Re}(s)} e^{-x}$, the complex-valued integral $\int_0^\infty x^s e^{-x}\, dx$ makes sense for all s with $\text{Re}(s) > -1$ (this is analogous to the absolute convergence test for infinite series of complex numbers).

Definition 2.1. For $s \in \mathbf{C}$ with $\text{Re}(s) > 0$, we define

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}.$$

Notice the $1/x$ at the end of the integrand. That is why the integral converges when $|x^s/x| = x^{\text{Re}(s)-1}$ has exponent greater than -1, which means $\text{Re}(s) > 0$. We have $\boxed{\Gamma(n+1) = n!}$ for all integers $n \geq 0$.

Remark 2.2. When we integrate a complex-valued function, there is no geometric interpretation of its value as an area, volume, and so on. Integrals of complex-valued functions could be defined by integrating the real and imaginary parts: if $f(x) = u(x) + iv(x)$ where $u(x)$ and $v(x)$ are real-valued functions then $\int_a^b f(x)\, dx = \int_a^b u(x) + i \int_a^b v(x)\, dx$. However, it is better to define these integrals as limits of Riemann sums, just like in calculus, but using limits of complex numbers instead of limits of real numbers.

Using integration by parts, which is valid for complex-valued functions, check that

$$\Gamma(s+1) = s\Gamma(s) \tag{2.1}$$

when $\mathrm{Re}(s) > 0$. Rewriting this formula as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \tag{2.2}$$

the right side makes sense when $\mathrm{Re}(s+1) > 0$, meaning when $\mathrm{Re}(s) > -1$, except at $s = 0$. Therefore we use (2.2) to *define* $\Gamma(s)$ for $-1 < \mathrm{Re}(s) \le 0$ except when $s = 0$. At $s = 0$ there is definitely a problem, since the numerator $\Gamma(s+1)$ is $\Gamma(1) = 0! = 1$ while the denominator is 0: we have to set $\Gamma(0) = \infty$. Now (2.2) is true for all s with $\mathrm{Re}(s) > -1$.

If $\mathrm{Re}(s) > -1$, so $\mathrm{Re}(s+1) > 0$, then (2.2) implies $\Gamma(s+1) = \Gamma(s+2)/(s+1)$, so

$$\mathrm{Re}(s) > -1 \Longrightarrow \Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)/(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)}.$$

The expression at the end makes sense when $\mathrm{Re}(s) > -2$ except at $s = 0$ and $s = -1$. Therefore we can define $\Gamma(s)$ for $-2 < \mathrm{Re}(s) \le -1$ by the formula

$$\Gamma(s) = \frac{\Gamma(s+2)}{s(s+1)},$$

and that makes (2.2) true for all s with $\mathrm{Re}(s) > -2$ except at $s = 0$ and $s = -1$, where $\Gamma(0) = \Gamma(-1) = \infty$. We can use (2.2) to extend $\Gamma(s)$ consistently to $\mathrm{Re}(s) > -3$ by

$$\Gamma(s) = \frac{\Gamma(s+3)}{s(s+1)(s+3)}$$

except at $s = 0, -1$, and -2, where $\Gamma(s) = \infty$, and more generally $\Gamma(s)$ extends to $\mathrm{Re}(s) > -k$ for each $k \in \mathbf{Z}^+$ by

$$\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)\cdots(s+k-1)}. \tag{2.3}$$

except at $s = 0, -1, \ldots, -(k-1)$, where $\Gamma(s) = \infty$. In this way, $\Gamma(s)$ can be defined on all of $\mathbf{C}$ except at 0 and the negative integers, where $\Gamma(s) = \infty$. Equation (2.1) is now true for all $s \in \mathbf{C}$.

Although it is not obvious, it turns out that $\boxed{\Gamma(s) \neq 0 \text{ for all } s \in \mathbf{C}}$. Knowing the Γ-function never vanishes and that it becomes infinite at $0, -1, -2, \ldots$ and nowhere else will be important later when we extend $\zeta(s)$ and $L(s, \chi)$ to $s \in \mathbf{C}$ and ask where these functions are 0.

Remark 2.3. The original definition of $\Gamma(s)$ for $\mathrm{Re}(s) > 0$ as an integral uses 0 as the lower bound of integration. If the lower bound of integration were a positive number, the integral would make sense from the beginning on the whole complex plane: check as an exercise that for $c > 0$, the integral $\int_c^\infty x^t e^{-x}\, dx/x$ converges for *all* $t \in \mathbf{R}$, so $\int_c^\infty x^s e^{-x}\, dx/x$ converges for *all* $s \in \mathbf{C}$ (analogue of absolute convergence test for complex series).

3. Extending the zeta-function to C

Riemann used $\Gamma(s)$ to extend the definition of $\zeta(s)$ beyond the region $\mathrm{Re}(s) > 1$. Here is part of what he showed in his only paper on number theory. (Riemann's primary interests were in geometry, analysis, and mathematical physics.)

Theorem 3.1 (Riemann, 1859)**.** *The function*

$$Z(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

can be extended from $\mathrm{Re}(s) > 1$ *to the whole complex plane, except at* $s = 0$ *and* $s = 1$*, and it satisfies the functional equation*

$$Z(s) = Z(1-s).$$

The function $Z(s)$ is called the *completed Riemann zeta-function.* It is definitely not obvious why $\pi^{-s/2}\Gamma(s/2)$ is a reasonable factor to use here! Nearly 100 years later, Tate's thesis (1950) explained this, but it requires ideas beyond the scope of these lectures (p-adic numbers and adeles).

Proof. For $\mathrm{Re}(s) > 0$, we rewrite $\pi^{-s/2}\Gamma(s/2)$ using a change of variables:

$$\Gamma(s) = \int_0^\infty x^s e^{-x}\,\frac{dx}{x} \Longrightarrow \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = \int_0^\infty \frac{x^{s/2}}{\pi^{s/2}}e^{-x}\,\frac{dx}{x} = \int_0^\infty t^{s/2}e^{-\pi t}\,\frac{dt}{t},$$

where $t = x/\pi$. (Observe for $c > 0$ that $d(cx)/(cx) = dx/x$. We will use this again below.)

Now we multiply this by $\zeta(s)$. For $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \sum_{n\geq 1}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} \\
&= \sum_{n\geq 1}\int_0^\infty \frac{t^{s/2}}{n^s}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1}\int_0^\infty \frac{t^{s/2}}{(n^2)^{s/2}}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1}\int_0^\infty y^{s/2}e^{-\pi n^2 y}\,\frac{dy}{y},
\end{aligned}$$

where $y = t/n^2$. Interchanging the sum and integral (this can be justified), we get

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \sum_{n\geq 1} y^{s/2}e^{-\pi n^2 y}\,\frac{dy}{y} = \int_0^\infty \left(\sum_{n\geq 1} e^{-\pi n^2 y}\right) y^{s/2}\,\frac{dy}{y}. \tag{3.1}$$

Set

$$h(y) = \sum_{n\geq 1} e^{-\pi n^2 y} = e^{-\pi y} + e^{-4\pi y} + e^{-9\pi y} + \cdots$$

for $y > 0$. For large y we have $\boxed{h(y) \approx e^{-\pi y}}$. The series $h(y)$ runs over positive integers. The related series over *all* integers

$$\theta(y) = \sum_{n\in\mathbf{Z}} e^{-\pi n^2 y} = 1 + 2e^{-\pi y} + 2e^{-4\pi y} + 2e^{-9\pi y} + \cdots = 1 + 2h(y)$$

is a famous function in analysis, and Riemann knew a remarkable connection (found by Jacobi) between its values at y and $1/y$:

$$\boxed{\theta\left(\frac{1}{y}\right) = \sqrt{y}\,\theta(y)}. \tag{3.2}$$

A proof of this formula needs Fourier analysis and the Poisson summation formula; we do not discuss it here, but see numerical data in the table below. For large y, $\theta(y)$ is nearly 1 and $\theta(1/y)$ is nearly $\sqrt{y}$.

y	2	3	4	5
$\theta(y)$	1.003734...	1.000161...	1.000006...	1.000000...
$\theta(1/y)$	1.419495...	1.732330...	2.000013...	2.236068...

In terms of $h(y)$, (3.2) says

$$h\left(\frac{1}{y}\right) = \frac{1}{2}(\sqrt{y}(1+2h(y))-1) = \frac{\sqrt{y}-1}{2} + \sqrt{y}h(y). \tag{3.3}$$

Returning to (3.1), break up the integral over $(0,\infty)$ into integrals over $(0,1)$ and $(1,\infty)$, and then replace y with $1/y$ to write the integral over $(0,1)$ as an integral over $(1,\infty)$, noting $d(1/y)/(1/y) = -dy/y$:

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty h(y)y^{s/2}\,\frac{dy}{y} \\
&= \int_0^1 h(y)y^{s/2}\,\frac{dy}{y} + \int_1^\infty h(y)y^{s/2}\,\frac{dy}{y} \\
&= \int_1^\infty h\left(\frac{1}{y}\right)y^{-s/2}\,\frac{dy}{y} + \int_1^\infty h(y)y^{s/2}\,\frac{dy}{y}.
\end{aligned}$$

Since the second integral has a positive lower bound of integration and $h(y) \approx e^{-\pi y}$ for large y, the second integral converges for *all* $s \in \mathbf{C}$. Recall we said earlier that the integral defining the Γ-function would converge for all s if the lower bound of integration were positive (Remark 2.3); the same thing is happening here when the lower bound of integration is 1.

By (3.3), the first integral above is

$$\int_1^\infty \left(\frac{\sqrt{y}-1}{2} + \sqrt{y}h(y)\right)y^{-s/2}\,\frac{dy}{y} = \frac{1}{2}\int_1^\infty (\sqrt{y}-1)y^{-s/2}\,\frac{dy}{y} + \int_1^\infty h(y)y^{(1-s)/2}\,\frac{dy}{y}$$

and the second integral above converges for all s.

Check as an exercise that

$$\operatorname{Re}(a) > 0 \Longrightarrow \int_a^\infty \frac{1}{y^a}\,\frac{dy}{y} = \frac{1}{a},$$

so for $\operatorname{Re}(s) > 1$,

$$\begin{aligned}
\frac{1}{2}\int_1^\infty (\sqrt{y}-1)y^{-s/2}\,\frac{dy}{y} &= \frac{1}{2}\int_1^\infty \left(\frac{1}{y^{(s-1)/2}} - \frac{1}{y^{s/2}}\right)\frac{dy}{y} \\
&= \frac{1}{2}\left(\frac{1}{(s-1)/2} - \frac{1}{s/2}\right) = \frac{1}{s-1} - \frac{1}{s}.
\end{aligned}$$

We therefore have obtained the following formula for $\pi^{-s/2}\Gamma(s/2)\zeta(s)$:

$$\begin{aligned}
\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_1^\infty h(y)y^{s/2}\,\frac{dy}{y} + \int_1^\infty h(y)y^{(1-s)/2}\,\frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s} \\
&= \int_1^\infty h(y)(y^{s/2} + y^{(1-s)/2})\,\frac{dy}{y} - \frac{1}{1-s} - \frac{1}{s}. \qquad (3.4)
\end{aligned}$$

Up to this point, $\mathrm{Re}(s) > 1$. The formula (3.4) makes sense for all $s \in \mathbf{C}$ other than 0 and 1: the integral converges for all s and the terms $1/s$ and $1/(1-s)$ are meaningful when $s \notin \{0, 1\}$. Therefore we use (3.4) to extend the meaning of $Z(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ to all of $\mathbf{C} - \{0, 1\}$. The formula in (3.4) is unchanged when we replace s with $1 - s$, so $Z(s) = Z(1 - s)$. □

The most important step in Riemann's proof is using the formula (3.2). In a certain sense, (3.2) is equivalent to the functional equation for $Z(s)$.

Remark 3.2. We only proved the functional equation $Z(s) = Z(1 - s)$ *after* we found the formula (3.4) that let us extend $Z(s)$ from $\mathrm{Re}(s) > 1$ to $\mathbf{C} - \{0, 1\}$. It would be absurd to try to prove $Z(s) = Z(1 - s)$ if the domain for $Z(s)$ does not include both s and $1 - s$!

Corollary 3.3. *For* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(s) < 0$*,* $Z(s) \neq 0$*.*

Proof. First we will treat the case $\mathrm{Re}(s) > 1$ and then we will use the functional equation $Z(s) = Z(1 - s)$ to treat the case $\mathrm{Re}(s) < 0$.

If $\mathrm{Re}(s) > 1$ then $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and the three factors are each nonzero when $\mathrm{Re}(s) > 1$: $\zeta(s) \neq 0$ by the Euler product, $\Gamma(s/2) \neq 0$ because the Γ-function is finite and nonzero on $\mathrm{Re}(s) > 0$, and $\pi^{s/2} \neq 0$ because $\pi^{s/2} \neq 0$ for all $s \in \mathbf{C}$.

If $\mathrm{Re}(s) < 0$ then $\mathrm{Re}(1 - s) > 1$ so $Z(s) = Z(1 - s) \neq 0$ by what we first showed when the real part is greater than 1. □

Corollary 3.4. *The Riemann zeta-function extends from* $\mathrm{Re}(s) > 1$ *to all* $s \in \mathbf{C}$ *except at* $s = 1$*, where* $\zeta(1) = \infty$*. We have* $\zeta(0) = -1/2$ *and* $\zeta(s) = 0$ *when* s *is a negative even integer.*

Proof. For $s \in \mathbf{C}$, the definition $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ motivates us to define

$$\zeta(s) = \frac{\pi^{s/2}Z(s)}{\Gamma(s/2)}. \tag{3.5}$$

This formula is consistent with the original definition of $\zeta(s)$ when $\mathrm{Re}(s) > 1$. The function $Z(s)$ makes sense everywhere except at 0 and 1, while $\Gamma(s/2) = \infty$ for $s \in \{0, -2, -4, \dots\}$ and $\Gamma(s/2) \neq 0$ for all s, so the above definition of $\zeta(s)$ makes sense everywhere except perhaps at $s = 0$, $s = 1$, and $s \in \{-2, -4, -6, \dots\}$. We now look more closely at these possibilities.

If $s \in \{-2, -4, -6, \dots\}$ then $Z(s) \neq 0$ by Corollary 3.3 while $\Gamma(s/2) = \infty$, so it is natural to interpret (3.5) as saying $\zeta(s) = 0$.

What happens to (3.5) when $s = 0$ and $s = 1$? When $s = 1$, $\pi^{s/2}/\Gamma(s/2) = \pi^{1/2}/\Gamma(1/2)$ is finite and nonzero[4], while $Z(1) = \infty$, so $\zeta(1) = \infty$. The case $s = 0$ is more subtle, since $Z(s)$ and $\Gamma(s/2)$ are both infinite at $s = 0$. Recall $s\Gamma(s) = \Gamma(s + 1)$ for $s \in \mathbf{C}$. This tells us

$$\frac{s}{2}\Gamma\left(\frac{s}{2}\right) = \Gamma\left(\frac{s}{2} + 1\right),$$

and when $s = 0$ we have $\Gamma(s/2 + 1) = \Gamma(1) = 1$. Therefore we write

$$\zeta(s) = \frac{\pi^{s/2}Z(s)}{\Gamma(s/2)} = \frac{\pi^{s/2}(s/2)Z(s)}{(s/2)\Gamma(s/2)} = \frac{\pi^{s/2}sZ(s)/2}{\Gamma(s/2 + 1)}.$$

[4]In fact this value is 1: $\Gamma(1/2) = \sqrt{\pi}$.

By (3.4),

$$sZ(s) = s\int_1^\infty h(y)(y^{s/2} + y^{(1-s)/2})\,\frac{dy}{y} - \frac{s}{1-s} - 1 \Longrightarrow \lim_{s\to 0} sZ(s) = -1.$$

Therefore as $s \to 0$,

$$\zeta(s) = \frac{\pi^{s/2} sZ(s)/2}{\Gamma(s/2+1)} \to \frac{\pi^0(-1)/2}{\Gamma(1)} = -\frac{1}{2},$$

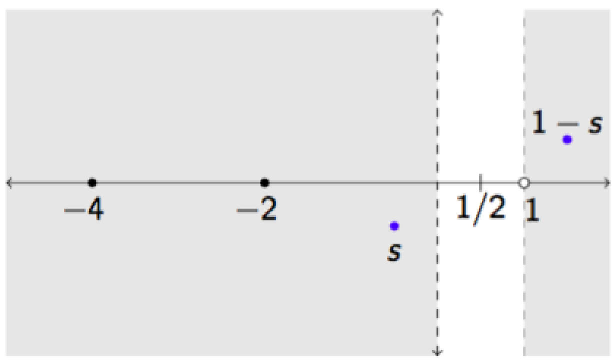
so we set $\zeta(0) = -1/2$. □

From now on, we consider $\zeta(s)$ as a function on $\mathbf{C}$ that is finite everywhere except at $s = 1$. Remember that its original definition as a series *only* makes sense when $\mathrm{Re}(s) > 1$.

Remark 3.5. It is possible to write the functional equation $Z(s) = Z(1-s)$ as a functional equation relating $\zeta(s)$ and $\zeta(1-s)$, but it looks awful:

$$\zeta(1-s) = \frac{1}{\pi}(2\pi)^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

When you think about s and $1-s$, have the picture below in mind: s and $1-s$ are symmetric around the point $1/2$, which is the midpoint of the line between them.



We know $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$ by the Euler product, and the formula

$$\zeta(s) = \frac{\pi^{s/2} Z(s)}{\Gamma(s/2)}$$

implies $\zeta(s) \neq 0$ for $\mathrm{Re}(s) < 0$ except at negative even integers. Negative even integers are called *trivial zeros* of $\zeta(s)$. Other zeros are called *nontrivial* and satisfy $0 \leq \mathrm{Re}(s) \leq 1$.

It is not that hard to show $\zeta(s) < 0$ when $0 < s < 1$, so there are no real nontrivial zeros. It is much harder to show $\zeta(s) \neq 0$ along the whole line $\mathrm{Re}(s) = 1$: this is equivalent to the prime number theorem! Using the functional equation, it follows that $\zeta(s) \neq 0$ along the whole line $\mathrm{Re}(s) = 0$.

One of the most famous problems in mathematics is about the location of nontrivial zeros of $\zeta(s)$.

Riemann Hypothesis: *Every nontrivial zero of the zeta-function has real part* $\frac{1}{2}$.

The first few nontrivial zeros of $\zeta(s)$ with positive imaginary part have the form $1/2 + it$ for the following approximate values of t:

$$14.1347, \quad 21.0220, \quad 25.0108.$$

There are infinitely many nontrivial zeros of $\zeta(s)$, and unlike the trivial zeros, there is no simple formula for any of them.

4. Extending Dirichlet L-functions to $\mathbf{C}$

Following the ideas from the previous section, we will extend each $L(s,\chi)$, for nontrivial χ, from the region $\mathrm{Re}(s) > 1$ to the whole complex plane. A property of a Dirichlet character, called its parity (being even or odd) will be important. Since $(\chi(-1))^2 = \chi((-1)^2) = \chi(1) = 1$, we have $\chi(-1) = 1$ or $\chi(-1) = -1$.

Definition 4.1. A Dirichlet character χ is called *even* if $\chi(-1) = 1$ and it is called *odd* if $\chi(-1) = -1$.

Since $\chi(-a) = \chi(-1)\chi(a)$, whether χ is even or odd as a character is the same as whether χ is even or odd as a function on $\mathbf{Z}$.

Example 4.2. Every trivial character is even.

Example 4.3. The characters χ_4 and χ_5 from Section 1 are odd.

Example 4.4. For an odd prime p, the Legendre symbol $(\frac{a}{p})$, where

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \equiv \Box \bmod p, \\ -1, & \text{if } a \not\equiv \Box \bmod p \end{cases}$$

for $a \not\equiv 0 \bmod p$, is a character on $(\mathbf{Z}/p\mathbf{Z})^\times$. Here is a table of its values when $p = 7$.

a	1	2	3	4	5	6
$(\frac{a}{7})$	1	1	-1	1	-1	-1

For example, $(\frac{2}{7}) = 1$ since $2 \equiv 9 \bmod 7$.

The multiplicativity of the Legendre symbol, $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$, is not obvious. It says in particular that a product of two nonsquares mod p is a square mod p, and this property is often false when the modulus is composite. For example, the only squares in $(\mathbf{Z}/15\mathbf{Z})^\times = \{1, 2, 4, 7, 8, 11, 13, 14 \bmod 15\}$ are 1 and 4, so the product of two nonsquares such as 2 and 7 is 14, which is also not a square.

A basic theorem in number theory says for odd primes p, $-1 \equiv \Box \bmod p$ if and only if $p \equiv 1 \bmod 4$, so the Legendre symbol mod p is even if $p \equiv 1 \bmod 4$ and odd if $p \equiv 3 \bmod 4$.

Definition 4.5. For a nontrivial character χ mod m, its *completed L-function* is

$$\Lambda(s,\chi) = m^{s/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi)$$

if χ is even and

$$\Lambda(s,\chi) = m^{(s+1)/2}\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)$$

if χ is odd. To write these in a unified notation, let $\delta \in \{0,1\}$ be the integer such that $\chi(-1) = (-1)^\delta$, so $\delta = 0$ for even χ and $\delta = 1$ for odd χ. Then both formulas above say

$$\Lambda(s,\chi) = m^{(s+\delta)/2}\pi^{-(s+\delta)/2}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi).$$

Note the modulus m of χ plays a direct role in the definition of $\Lambda(s,\chi)$.

Theorem 4.6. *For a nontrivial Dirichlet character χ, $\Lambda(s,\chi)$ can be extended from $\mathrm{Re}(s) > 1$ to all of $\mathbf{C}$, with finite values everywhere, and satisfies the functional equation*

$$\Lambda(s,\chi) = w_\chi\Lambda(1-s,\overline{\chi}).$$

for a complex number w_χ such that $|w_\chi| = 1$.

There are few things to say about Theorem 4.6 before we discuss its proof.

- While $Z(s)$ is infinite at $s = 0$ and $s = 1$, $\Lambda(s, \chi)$ for nontrivial χ is finite everywhere. This is an important difference between the case of trivial characters (whose L-functions are the zeta-function with at most finitely many Euler factors removed) and nontrivial characters.
- Unless $\overline{\chi} = \chi$, which means χ takes values in $\{\pm 1\}$, the functional equation does *not* relate $\Lambda(s, \chi)$ with $\Lambda(1 - s, \chi)$, but with the completed L-function of the conjugate character $\overline{\chi}$.
- When χ takes values in $\{\pm 1\}$ it turns out that $w_\chi = 1$, so the functional equation says $\Lambda(s, \chi) = \Lambda(1-s, \chi)$, which looks like the functional equation $Z(s) = Z(1-s)$.
- We are avoiding a technical issue in our statement of Theorem 4.6: the functional equation is not true for some nontrivial characters! It is only true when χ has an additional property called being *primitive*. Primitivity is not a property we want to discuss here (you can read about it in analytic number theory books), and in practice it is a mild condition: every Dirichlet character that is not primitive can be associated to a primitive Dirichlet character, and their L-functions are closely related (they differ in a finite number of Euler factors), so properties of L-functions for nonprimitive characters often can be reduced to the case of L-functions of primitive characters. We will indicate in the proof of Theorem 4.6 where the "missing" property of primitivity is needed. The characters χ_4, χ_5, and $(\frac{\cdot}{p})$ are all primitive.

Proof. In the proof of Theorem 3.1, we saw for $\text{Re}(s) > 0$ that $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{s/2}e^{-\pi t}\,\frac{dt}{t}$. By similar reasoning, for $\text{Re}(s) > 0$

$$\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right) = \int_0^\infty \frac{x^{(s+1)/2}}{\pi^{(s+1)/2}}e^{-x}\,\frac{dx}{x} = \int_0^\infty t^{(s+1)/2}e^{-\pi t}\,\frac{dt}{t}, \tag{4.1}$$

where $t = x/\pi$. We will use these formulas to write $\Lambda(s, \chi)$ for $\text{Re}(s) > 1$ as an integral.

When $\boxed{\chi \bmod m \text{ is even}}$ and $\text{Re}(s) > 1$,

$$\begin{aligned}
\Lambda(s,\chi) &= m^{s/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)L(s,\chi) \\
&= \sum_{n\geq 1} m^{s/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\frac{\chi(n)}{n^s} \\
&= \sum_{n\geq 1}\chi(n)\int_0^\infty m^{s/2}\frac{t^{s/2}}{n^s}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1}\chi(n)\int_0^\infty \frac{m^{s/2}t^{s/2}}{(n^2)^{s/2}}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1}\chi(n)\int_0^\infty y^{s/2}e^{-\pi n^2 y/m}\,\frac{dy}{y},
\end{aligned}$$

where $y = mt/n^2$. Interchanging the sum and integral,

$$\Lambda(s,\chi) = \int_0^\infty \sum_{n\geq 1}\chi(n)y^{s/2}e^{-\pi n^2 y/m}\,\frac{dy}{y} = \int_0^\infty \left(\sum_{n\geq 1}\chi(n)e^{-\pi n^2 y/m}\right)y^{s/2}\,\frac{dy}{y}. \tag{4.2}$$

Looking at the series in (4.2), set

$$h(y,\chi) = \sum_{n\geq 1} \chi(n)e^{-\pi n^2 y/m} = e^{-\pi y/m} + \chi(2)e^{-4\pi y/m} + \chi(3)e^{-9\pi y/m} + \cdots$$

for $y > 0$, so

$$\Lambda(s,\chi) = \int_0^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y} \tag{4.3}$$

and for large y, $\boxed{h(y,\chi) \approx e^{-\pi y/m}}$. Define a related series over all integers:

$$\theta(y,\chi) = \sum_{n\in\mathbf{Z}} \chi(n)e^{-\pi n^2 y/m} = \sum_{n\geq 1}(\chi(n)+\chi(-n))e^{-\pi n^2 y/m} = \sum_{n\geq 1} 2\chi(n)e^{-\pi n^2 y/m},$$

where the last equation uses the fact that χ is even (if χ were odd then $\chi(n)+\chi(-n) = 0$, so this definition of $\theta(y,\chi)$ would be 0). Unlike $\theta(y)$, the constant term of $\theta(y,\chi)$ is 0 since $\chi(0) = 0$. We have

$$\theta(y,\chi) = 2e^{-\pi y/m} + 2\chi(2)e^{-4\pi y/m} + 2\chi(3)e^{-9\pi y/m} + \cdots = 2h(y,\chi). \tag{4.4}$$

When $\boxed{\chi \bmod m \text{ is odd}}$ and $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
\Lambda(s,\chi) &= m^{(s+1)/2}\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) \\
&= \sum_{n\geq 1} m^{(s+1)/2}\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)\frac{\chi(n)}{n^s} \\
&= \sum_{n\geq 1}\chi(n)\int_0^\infty m^{(s+1)/2}\frac{t^{(s+1)/2}}{n^s}e^{-\pi t}\,\frac{dt}{t} \quad \text{by (4.1)} \\
&= \sum_{n\geq 1} n\chi(n)\int_0^\infty \frac{m^{(s+1)/2}t^{(s+1)/2}}{n^{s+1}}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1} n\chi(n)\int_0^\infty \frac{m^{(s+1)/2}t^{(s+1)/2}}{(n^2)^{(s+1)/2}}e^{-\pi t}\,\frac{dt}{t} \\
&= \sum_{n\geq 1} n\chi(n)\int_0^\infty y^{(s+1)/2}e^{-\pi n^2 y/m}\,\frac{dy}{y},
\end{aligned}$$

where $y = mt/n^2$. Interchanging the sum and integral,

$$\Lambda(s,\chi) = \int_0^\infty \sum_{n\geq 1} n\chi(n)y^{\frac{s+1}{2}}e^{-\pi n^2 y/m}\,\frac{dy}{y} = \int_0^\infty \left(\sum_{n\geq 1} n\chi(n)e^{-\pi n^2 y/m}\right)y^{\frac{s+1}{2}}\,\frac{dy}{y}. \tag{4.5}$$

Looking at the series in (4.5), set

$$h(y,\chi) = \sum_{n\geq 1} n\chi(n)e^{-\pi n^2 y/m} = e^{-\pi y/m} + 2\chi(2)e^{-4\pi y/m} + 3\chi(3)e^{-9\pi y/m} + \cdots$$

for $y > 0$, so

$$\Lambda(s,\chi) = \int_0^\infty h(y,\chi)y^{(s+1)/2}\,\frac{dy}{y} \tag{4.6}$$

and for large y, $\boxed{h(y,\chi) \approx e^{-\pi y/m}}$. Define a related series over all integers:

$$\theta(y,\chi) = \sum_{n\in\mathbf{Z}} n\chi(n)e^{-\pi n^2 y/m} = \sum_{n\geq 1}(n\chi(n) + (-n)\chi(-n))e^{-\pi n^2 y/m} = \sum_{n\geq 1} 2n\chi(n)e^{-\pi n^2 y/m},$$

where the last equation uses the fact that χ is odd (if χ were even then $n\chi(n)+(-n)\chi(-n) = 0$ so this definition of $\theta(y,\chi)$ would be 0). We have

$$\theta(y,\chi) = 2e^{-\pi y/m} + 4\chi(2)e^{-4\pi y/m} + 6\chi(3)e^{-9\pi y/m} + \cdots = 2h(y,\chi) \tag{4.7}$$

for $y > 0$. As in the case of even characters, $\theta(y,\chi)$ has constant term 0.

In (3.2) we gave an important formula that connects $\theta(y)$ and $\theta(1/y)$. This formula has a generalization to $\theta(y,\chi)$, depending on if χ is even or odd: there is a complex number w_χ with absolute value 1 such that

$$\boxed{\theta\left(\frac{1}{y},\chi\right) = w_\chi\sqrt{y}\,\theta(y,\overline{\chi}) \text{ if } \chi \text{ is even}} \tag{4.8}$$

and

$$\boxed{\theta\left(\frac{1}{y},\chi\right) = w_\chi y^{3/2}\theta(y,\overline{\chi}) \text{ if } \chi \text{ is odd}}. \tag{4.9}$$

The proof of these formulas involves Fourier analysis, like (3.2) does, and we omit the details.[5] Notice these formulas use $\overline{\chi}$ on the right side.

Without giving an exact formula for w_χ,[6] we can read off one property of these numbers of absolute value 1:

$$\overline{w_\chi} = w_{\overline{\chi}}. \tag{4.10}$$

For example, by the definition of $\theta(y,\chi)$ whether χ is even or odd, we have $\overline{\theta(y,\chi)} = \theta(y,\overline{\chi})$, so by applying complex conjugation to both sides of (4.8) or by replacing χ with $\overline{\chi}$ everywhere in (4.8) we get two formulas when χ is even:

$$\theta(1/y,\overline{\chi}) = \overline{w_\chi}\sqrt{y}\,\theta(y,\chi) \quad\text{and}\quad \theta(1/y,\overline{\chi}) = w_{\overline{\chi}}\sqrt{y}\,\theta(y,\chi).$$

From this we get (4.10) since all terms besides $\overline{w_\chi}$ and $w_{\overline{\chi}}$ on both sides of each formula are equal and $\theta(y,\chi)$ is not identically zero. The proof of (4.10) for odd χ is similar, using (4.9) instead of (4.8).

Since $h(y,\chi) = \frac{1}{2}\theta(y,\chi)$ whether χ is even or odd (see (4.4) and (4.7)), formulas (4.8) and (4.9) turn into formulas using $h(y,\chi)$ by dividing both sides of (4.8) and (4.9) by 2:

$$h\left(\frac{1}{y},\chi\right) = w_\chi\sqrt{y}\,h(y,\overline{\chi}) \text{ if } \chi \text{ is even} \tag{4.11}$$

and

$$h\left(\frac{1}{y},\chi\right) = w_\chi y^{3/2}h(y,\overline{\chi}) \text{ if } \chi \text{ is odd.} \tag{4.12}$$

We will use these to get a formula for $\Lambda(s,\chi)$ that makes sense at all $s \in \mathbf{C}$.

[5]This is *exactly* the place in the proof where we need χ to be "primitive". For nonprimitive characters, (4.8) and (4.9) are actually not true.

[6]For primitive χ, $w_\chi = \sum_{a \bmod m}\chi(a)e^{2\pi i a/m}/\sqrt{m}$ if χ is even, and $w_\chi = \sum_{a \bmod m}\chi(a)e^{2\pi i a/m}/(i\sqrt{m})$ if χ is odd.

Case 1: χ is even. Start from (4.3) and break up the integral $\int_0^\infty$ as $\int_0^1 + \int_1^\infty$:

$$\begin{aligned}\Lambda(s,\chi) &= \int_0^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y}\\ &= \int_0^1 h(y,\chi)y^{s/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y}.\end{aligned}$$

The second integral makes sense for all $s \in \mathbf{C}$ since the lower bound of integration is positive and since $h(y,\chi) \approx e^{-\pi y/m}$ for large y. Make the change of variables $y \mapsto 1/y$ in the first integral: $d(1/y)/(1/y) = -dy/y$, so

$$\begin{aligned}\Lambda(s,\chi) &= \int_1^\infty h\left(1/y,\chi\right)y^{-s/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y}\\ &= \int_1^\infty w_\chi\sqrt{y}h(y,\overline{\chi})y^{-s/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y} \quad \text{by (4.11)}\\ &= \int_1^\infty w_\chi h(y,\overline{\chi})y^{(1-s)/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi)y^{s/2}\,\frac{dy}{y}.\end{aligned}$$

Now the first integral makes sense for all $s \in \mathbf{C}$, so this provides a method of extending $\Lambda(s,\chi)$ to all $s \in \mathbf{C}$ (finite values everywhere). Combining the integrals into a single integral over $(1,\infty)$,

$$\Lambda(s,\chi) = \int_1^\infty \left(h(y,\chi)y^{s/2} + w_\chi h(y,\overline{\chi})y^{(1-s)/2}\right)\frac{dy}{y}. \tag{4.13}$$

To prove the functional equation $\Lambda(s,\chi) = w_\chi\Lambda(1-s,\overline{\chi})$, replace χ with $\overline{\chi}$ and s with $1-s$ in (4.13) to get

$$\Lambda(1-s,\overline{\chi}) = \int_1^\infty \left(h(y,\overline{\chi})y^{(1-s)/2} + w_{\overline{\chi}}h(y,\chi)y^{s/2}\right)\frac{dy}{y}$$

By (4.10), $w_\chi w_{\overline{\chi}} = w_\chi\overline{w_\chi} = |w_\chi|^2 = 1$, so

$$w_\chi\Lambda(1-s,\overline{\chi}) = \int_1^\infty \left(w_\chi h(y,\overline{\chi})y^{(1-s)/2} + h(y,\chi)y^{s/2}\right)\frac{dy}{y} = \Lambda(s,\chi)$$

by (4.13). This completes the proof of Theorem 4.6 when χ is even.[7]

Case 2: χ is odd. By (4.6),

$$\begin{aligned}\Lambda(s,\chi) &= \int_0^\infty h(y,\chi)y^{(s+1)/2}\,\frac{dy}{y}\\ &= \int_0^1 h(y,\chi)y^{(s+1)/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi)y^{(s+1)/2}\,\frac{dy}{y}.\end{aligned}$$

The second integral makes sense for all $s \in \mathbf{C}$ since the lower bound of integration is positive and since $h(y,\chi) \approx e^{-\pi y/m}$ for large y. By the change of variables $y \mapsto 1/y$ in the first

[7]Strictly speaking, this proves the theorem when χ is even and "primitive."

integral,

$$\begin{aligned}\Lambda(s,\chi) &= \int_1^\infty h\left(1/y,\chi\right) y^{-(s+1)/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi) y^{(s+1)/2}\,\frac{dy}{y} \\ &= \int_1^\infty w_\chi y^{3/2} h(y,\overline{\chi}) y^{-(s+1)/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi) y^{(s+1)/2}\,\frac{dy}{y} \quad \text{by (4.12)} \\ &= \int_1^\infty w_\chi h(y,\overline{\chi}) y^{(1-s+1)/2}\,\frac{dy}{y} + \int_1^\infty h(y,\chi) y^{(s+1)/2}\,\frac{dy}{y}.\end{aligned}$$

These integrals each make sense for all $s \in \mathbf{C}$, so this provides a method of extending $\Lambda(s,\chi)$ to all $s \in \mathbf{C}$ (finite values everywhere, as before). Combining the integrals into a single integral over $(1,\infty)$,

$$\Lambda(s,\chi) = \int_1^\infty \left(h(y,\chi) y^{(s+1)/2} + w_\chi h(y,\overline{\chi}) y^{(1-s+1)/2}\right) \frac{dy}{y}. \tag{4.14}$$

To prove the functional equation $\Lambda(s,\chi) = w_\chi \Lambda(1-s,\overline{\chi})$, replace χ with $\overline{\chi}$ and s with $1-s$ in (4.14) to get

$$\Lambda(1-s,\overline{\chi}) = \int_1^\infty \left(h(y,\overline{\chi}) y^{(1-s+1)/2} + w_{\overline{\chi}} h(y,\chi) y^{(s+1)/2}\right) \frac{dy}{y}$$

By (4.10), $w_\chi w_{\overline{\chi}} = w_\chi \overline{w_\chi} = |w_\chi|^2 = 1$, so

$$w_\chi \Lambda(1-s,\overline{\chi}) = \int_1^\infty \left(w_\chi h(y,\overline{\chi}) y^{(1-s+1)/2} + h(y,\chi) y^{(s+1)/2}\right) \frac{dy}{y} = \Lambda(s,\chi)$$

by (4.14). This completes the proof of Theorem 4.6 when χ is odd.[8] □

Remark 4.7. While $\theta(y)$ has a nonzero constant term (at $n = 0$), the other functions $\theta(y,\chi)$ for nontrivial χ have constant term 0, and this is the reason why $Z(s)$ has value ∞ at $s = 0$ and 1 while $\Lambda(s,\chi)$ is finite everywhere.

Corollary 4.8. *For* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(s) < 0$*,* $\Lambda(s,\chi) \neq 0$*.*

Proof. First we will treat the case $\mathrm{Re}(s) > 1$ and then we will use the functional equation for $\Lambda(s,\chi)$ to treat the case $\mathrm{Re}(s) < 0$.

If $\mathrm{Re}(s) > 1$, $\Lambda(s,\chi)$ is $m^{s/2}\pi^{-s/2}\Gamma(s/2)L(s,\chi)$ or $m^{(s+1)/2}\pi^{-(s+1)/2}\Gamma((s+1)/2)L(s,\chi)$ depending on whether χ is even or odd, and all the factors are nonzero: $L(s,\chi) \neq 0$ by the Euler product, $\Gamma(s/2)$ and $\Gamma((s+1)/2)$ are not 0 because the Γ-function is finite and nonzero on $\mathrm{Re}(s) > 0$, and exponential functions are nonzero everywhere. The exact same reasoning shows $\Lambda(s,\overline{\chi}) \neq 0$ when $\mathrm{Re}(s) > 1$.

If $\mathrm{Re}(s) < 0$ then $\mathrm{Re}(1-s) > 1$ and $\Lambda(s,\chi) = w_\chi \Lambda(1-s,\overline{\chi})$ by the functional equation, so the right side is not 0 by what we showed about the completed L-function of $\overline{\chi}$. □

Corollary 4.9. *For nontrivial* χ*,* $L(s,\chi)$ *extends from* $\mathrm{Re}(s) > 1$ *to all* $s \in \mathbf{C}$*. If* χ *is even then* $L(s,\chi) = 0$ *when* s *is* 0 *or a negative even integer, and if* χ *is odd then* $L(s,\chi) = 0$ *when* s *is a negative odd integer.*

Notice the contrast with $\zeta(s)$: $\zeta(0) = -1/2 \neq 0$ while the corollary is saying $L(0,\chi) = 0$ if χ is an even *nontrivial* character.[9]

[8] Strictly speaking, this proves the theorem when χ is odd and "primitive."

[9] When a character is not primitive, Corollary 4.9 is still true for its L-function.

Proof. For all $s \in \mathbf{C}$, the definition of $\Lambda(s, \chi)$ motivates us to define

$$L(s,\chi) = \frac{m^{-s/2}\pi^{s/2}\Lambda(s,\chi)}{\Gamma(s/2)} \tag{4.15}$$

if χ is even and

$$L(s,\chi) = \frac{m^{-(s+1)/2}\pi^{(s+1)/2}\Lambda(s,\chi)}{\Gamma((s+1)/2)} \tag{4.16}$$

if χ is odd. These are consistent with the original definition of $L(s, \chi)$ when $\mathrm{Re}(s) > 1$.

The function $\Lambda(s, \chi)$ makes sense everywhere (finite values) on $\mathbf{C}$, while $\Gamma(s/2) = \infty$ for $s \in \{0, -2, -4, \dots\}$ and $\Gamma((s + 1)/2) = 0$ for $s \in \{-1, -3, -5, \dots\}$, so the above definition of $L(s, \chi)$ makes sense everywhere except perhaps at 0 and negative even integers if χ is even, and except for negative odd integers if χ is odd. What happens in these cases?

If χ is even and $s \in \{-2, -4, -6, \dots\}$ then $\Lambda(s, \chi) \neq 0$ by Corollary 4.8 while $\Gamma(s/2) = \infty$, so it is natural to interpret (4.15) as saying $L(s, \chi) = 0$. If χ is odd and $s \in \{-1, -3, -5, \dots\}$ then $\Lambda(s, \chi) \neq 0$ by Corollary 4.8 while $\Gamma((s+1)/2) = \infty$, so it is natural to interpret (4.16) as saying $L(s, \chi) = 0$.

What happens to (4.15) when $s = 0$? The numerator is $\Lambda(0, \chi)$ and the denominator is $\Gamma(0) = \infty$. Using the functional equation, $\Lambda(0, \chi) = w_\chi \Lambda(1, \overline{\chi})$, and a hard theorem in analytic number theory says the completed L-function of a nontrivial Dirichlet character is in $\mathbf{C}^\times$ at $s = 1$.[10] Therefore $\Lambda(0, \chi) \in \mathbf{C}^\times$, so $L(0, \chi) = 0$ using (4.15). $\square$

From now on, we consider $L(s, \chi)$ for nontrivial χ as a function on $\mathbf{C}$ that is finite everywhere.

Remark 4.10. It is possible to write the functional equation $\Lambda(s, \chi) = w_\chi \Lambda(1 - s, \overline{\chi})$ as an ugly functional equation for $L(s, \chi)$:

$$L(s,\chi) = w_\chi m^{1/2-s}\frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)L(1-s,\overline{\chi})$$

for even χ and

$$L(s,\chi) = w_\chi m^{1/2-s}\frac{(2\pi)^s}{\pi}\cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)L(1-s,\overline{\chi})$$

for odd χ.[11]

The zeros of $L(s, \chi)$ at integers ≤ 0 are called *trivial* and all other zeros of $L(s, \chi)$ are called *nontrivial*.[12]

Generalized Riemann Hypothesis: *Every nontrivial zero of* $L(s, \chi)$ *whose real part is strictly between* 0 *and* 1 *has real part* $\frac{1}{2}$.

Example 4.11. The first few nontrivial zeros of $L(s, \chi_4)$ with positive imaginary part have the form $1/2 + it$ for the following approximate values of t:

$$6.0209, \quad 10.2437, \quad 12.5880.$$

[10] This is a contrast to the case of the completed zeta-function, where $Z(1) = \infty$.

[11] Such functional equations are only true for primitive χ.

[12] There may be additional zeros on the imaginary axis if χ is not primitive and these are considered trivial zeros also.

Example 4.12. The first few nontrivial zeros of $L(s, \chi_5)$ with positive imaginary part have the form $1/2 + it$ for the following approximate values of t:

$$6.1835, \quad 8.4572, \quad 12.6749.$$

For every nontrivial Dirichlet character χ, there are infinitely many nontrivial zeros of $L(s, \chi)$. While it is not hard to show $\zeta(s) \neq 0$ on the interval $(0, 1)$, without the Generalized Riemann Hypothesis there is no proof that $L(s, \chi) \neq 0$ on the interval $(0, 1)$ for all χ, except possibly at $s = 1/2$. It is conjectured that $L(1/2, \chi) \neq 0$ for all Dirichlet characters χ.

5. Analytic functions

We created extensions of $Z(s)$ and $\Lambda(s, \chi)$ from $\mathrm{Re}(s) > 1$ to $\mathbf{C}$ (allowing that $Z(0) = \infty$ and $Z(1) = \infty$), from which we got extensions of $\zeta(s)$ and $L(s, \chi)$ from $\mathrm{Re}(s) > 1$ to $\mathbf{C}$ (allowing that $\zeta(1) = \infty$). In what sense can we say these extensions are unique without referring to explicit formulas? To answer this we will use the language of analytic functions.

Definition 5.1. A *domain* in $\mathbf{C}$ is a connected open subset.

Intuitively, this means a domain is in one piece (connected) and when any point in the domain is moved by a small amount it remains in the domain.

Examples of domains in $\mathbf{C}$ include open discs and open half-planes such as $\{s : \mathrm{Re}(s) > 1\}$. Domains do not include any point on their boundary.

Definition 5.2. A function $f \colon \Omega \to \mathbf{C}$ on a domain Ω is called *analytic* if it can be written as a power series near each point of Ω: for every $a \in \Omega$ we can write

$$f(s) = \sum_{n \geq 0} c_n (s - a)^n$$

for all s close to a.

Example 5.3. The function e^s is defined on $\mathbf{C}$ and is analytic there: for each $a \in \mathbf{C}$ and $s \in \mathbf{C}$,

$$e^s = e^a e^{s-a} = e^a \sum_{n \geq 0} \frac{(s-a)^n}{n!} = \sum_{n \geq 0} e^a \frac{(s-a)^n}{n!}.$$

Example 5.4. Every rational function is analytic on the domain that is $\mathbf{C}$ without the finitely many points where the denominator is 0. For instance, $1/(s^2 + 1)$ is analytic on $\mathbf{C} - \{\pm i\}$.

Example 5.5. For $s \in \mathbf{C}$, $1/n^s$ is analytic since $1/n^s = n^{-s} = e^{-s \ln n}$, and this can be written as a power series in $s - a$ for all $a \in \mathbf{C}$ by the same ideas as in Example 5.3.

Example 5.6. The infinite series $\zeta(s) = \sum_{n \geq 1} 1/n^s$ and $L(s, \chi) = \sum_{n \geq 1} \chi(n)/n^s$ are both analytic on $\mathrm{Re}(s) > 1$.

Example 5.7. The function $\Gamma(s)$ is analytic on $\mathbf{C} - \{0, -1, -2, -3, \ldots\}$.

Example 5.8. The absolute value function $f(s) = |s|$ is *not* analytic on any domain in $\mathbf{C}$.

Analytic functions have a "unique extension" property, as described in the following theorem that is proved in courses on complex analysis.

Theorem 5.9. *If $f \colon \Omega \to \mathbf{C}$ is analytic on a domain Ω and Ω' is a larger domain, then there is at most one extension of f to an analytic function on Ω'.*

This theorem is *not* saying an analytic function on a domain can be extended to an analytic function on every larger domain (try to extend $1/s$ from $\mathbf{C} - \{0\}$ to $\mathbf{C}$), but only that if this can be done then any two ways of doing this must lead to the same result. There is nothing like Theorem 5.9 for continuous functions being extended to continuous functions on a larger domain.

It can be shown from the integral formulas for $Z(s)$ and $\Lambda(s, \chi)$ that these functions are analytic on $\mathbf{C} - \{0, 1\}$ and on $\mathbf{C}$, and then that $\zeta(s)$ and $L(s, \chi)$ are analytic on $\mathbf{C} - \{1\}$ and on $\mathbf{C}$, so by Theorem 5.9 the extensions we have found of these functions from the half-plane $\mathrm{Re}(s) > 1$ are the only possible analytic extensions. In this sense we have found the "right" extensions of $\zeta(s)$ and $L(s, \chi)$ to $\mathbf{C}$.

6. Contour integrals

We now turn to a very important tool for studying analytic functions: complex contour integration. It will not at first look like this has anything to do with describing $\zeta(s)$ or $L(s, \chi)$, or using the Riemann Hypothesis, so some patience will be needed.

Definition 6.1. If $\gamma\colon [a, b] \to \mathbf{C}$ is a differentiable path[13] and f is a continuous complex-valued function on the image of γ, then we define the complex contour integral of f along γ to be

$$\int_\gamma f(s)\,ds := \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{6.1}$$

If γ is only piecewise differentiable then we define $\int_\gamma f(s)\,ds$ to be the sum of the contour integrals of f on each piecewise differentiable piece of γ.

The right side of (6.1) can be calculated from an antiderivative (indefinite integral) of the integrand: if $g'(t) = f(\gamma(t))\gamma'(t)$ then $\int_\gamma f(s)\,ds = g(b) - g(a)$. This will be used below in Example 6.2.

Our definition of contour integrals looks similar to line integrals of real-valued functions in $\mathbf{R}^2$, but there is a crucial difference: the multiplication $f(\gamma(t))\gamma'(t)$ is a product of complex numbers, not an inner product of vectors in $\mathbf{R}^2$. The value of $\int_\gamma f(s)\,ds$ is a complex number, not (usually) a real number. Its value does *not* have a direct geometric interpretation as an area, volume, density, *etc.*, but we will see that the machinery of contour integration is very powerful.

The integral $\int_\gamma f(s)\,ds$ can be defined as a limit of Riemann sums, but for the sake of being efficient we have given the more direct definition above instead of the more conceptual definition using limits.

Example 6.2. Let's compute $\displaystyle\int_{C_R^+} s^n\,ds$ where C_R^+ is a circle centered at the origin of radius $R > 0$, going once around the origin counterclockwise, The "+" in C_R^+ refers to this "positive" orientation on the circle. This path around the circle can be described by $\gamma\colon [0, 2\pi] \to \mathbf{C}$ where

$$\gamma(t) = Re^{it},$$

[13] This means that if we write γ in terms of its real and imaginary parts, say $\gamma(t) = u(t) + iv(t)$, then the component functions $u(t)$ and $v(t)$ are differentiable.

so $\gamma'(t) = iRe^{it}$ and then

$$\int_{C_R^+} s^n\,ds = \int_\gamma s^n\,ds = \int_0^{2\pi} (Re^{it})^n(iRe^{it})\,dt = iR^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt.$$

This last integral can be computed by finding an antiderivative of $e^{i(n+1)t}$. Separate cases are needed when $n \neq -1$ (so $n+1 \neq 0$) and when $n = -1$ (so $n+1 = 0$).

Case 1: $n \neq -1$. An antiderivative of $e^{i(n+1)t}$ is $e^{i(n+1)t}/(i(n+1))$, so

$$\int_0^{2\pi} e^{i(n+1)t}\,dt = \left.\frac{e^{i(n+1)t}}{i(n+1)}\right|_0^{2\pi} = \frac{e^{2\pi i(n+1)} - e^0}{i(n+1)} = \frac{1-1}{i(n+1)} = 0,$$

so $\int_{C_R^+} s^n\,ds = 0$.

Case 2: $n = -1$. Here $e^{i(n+1)t} = 1$, so an antiderivative is t and

$$\int_0^{2\pi} e^{i(n+1)t}\,dt = \left. t \right|_0^{2\pi} = 2\pi - 0 = 2\pi,$$

so

$$\int_{C_R^+} \frac{1}{s}\,ds = iR^{-1+1}\int_0^{2\pi} 1\,dt = 2\pi i.$$

Putting these two cases together,

$$\int_{C_R^+} s^n\,ds = \begin{cases} 0, & \text{if } n \neq -1,\\ 2\pi i, & \text{if } n = -1.\end{cases}$$

It is intriguing that the answer does *not* depend on R: for all circles centered at the origin, with a fixed choice of n, we get the same result!

In light of the values of this integral, it is convenient to divide by $2\pi i$ to make the two values 0 and 1:

$$\frac{1}{2\pi i}\int_{C_R^+} s^n\,ds = \begin{cases} 0, & \text{if } n \neq -1,\\ 1 & \text{if } n = -1.\end{cases}$$

Remark 6.3. If the path around the circle goes once around in the clockwise direction (negative orientation), using path $\gamma(t) = Re^{-it}$ for $0 \le t \le 2\pi$, then the contour integral would be

$$\int_{C_R^-} s^n\,ds = -iR^{n+1}\int_0^{2\pi} e^{-i(n+1)t}\,dt = \begin{cases} 0, & \text{if } n \neq -1,\\ 2\pi(-i), & \text{if } n = -1,\end{cases}$$

so

$$\frac{1}{2\pi i}\int_{C_R^-} s^n\,ds = \begin{cases} 0, & \text{if } n \neq -1,\\ -1, & \text{if } n = -1,\end{cases}$$

This illustrates how a contour integral depends on how γ traces out a path: if the image of γ traces out a path in the reverse direction or multiple times then $\int_\gamma f(s)\,ds$ changes. We will not focus on this aspect, which would require using the concept of a winding number.

We can now integrate a power series $\sum_{n\ge 0} c_n s^n$ that converges on the circle C_R^+, assuming we can interchange the order of summation and integration:

$$\int_{C_R^+} \sum_{n\ge 0} c_n s^n\,ds = \sum_{n\ge 0} c_n \int_{C_R^+} s^n\,ds = 0. \tag{6.2}$$

That does not look very interesting. Add to the power series a finite number of negative powers of s (which all make sense on C_R^+, since 0 is not on the circle) and then we find

$$\int_{C_R^+} \sum_{n \geq -1} c_n s^n \, ds = c_{-1} \int_{C_R^+} \frac{1}{s} \, ds + \sum_{n \geq 0} c_n \int_{C_R^+} s^n \, ds = 2\pi i c_{-1}$$

and

$$\int_{C_R^+} \sum_{n \geq -2} c_n s^n \, ds = c_{-2} \int_{C_R^+} \frac{1}{s^2} \, ds + c_{-1} \int_{C_R^+} \frac{1}{s} \, ds + \sum_{n \geq 0} c_n \int_{C_R^+} s^n \, ds = 2\pi i c_{-1},$$

and more generally for any $N < 0$,

$$\int_{C_R^+} \sum_{n \geq N} c_n s^n \, ds = 2\pi i c_{-1}. \tag{6.3}$$

This last formula is also valid if $N \geq 0$, since in that case the coefficient c_{-1} of $1/s$ is 0 and the integral is also 0 by (6.2).

Dividing by $2\pi i$ in (6.3),

$$\frac{1}{2\pi i} \int_{C_R^+} \sum_{n \geq N} c_n s^n \, ds = c_{-1}.$$

This calculation shows that the coefficient c_{-1} of $1/s$, unlike every other coefficient in $\sum_{n \geq N} c_n s^n$, can be detected by contour integration. This will lead to a very powerful result in complex analysis called the *residue theorem*.