Lecture 1 part 1: Kissing numbers and spherical codes

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History: Thomas Harriot (1585)

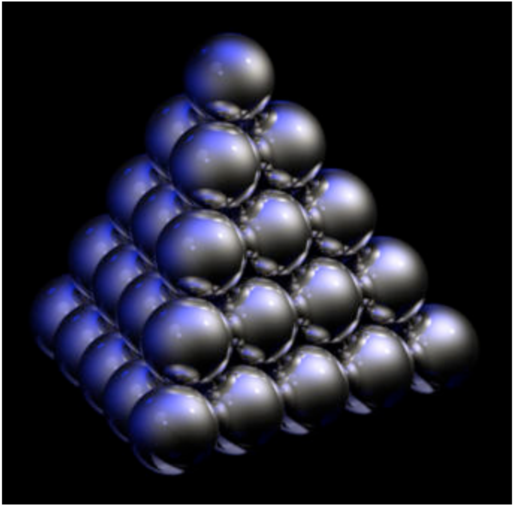
Sir Walter Raleigh's problem:

To develop a formula that would allow to know how many cannonballs can be in a given stack simply by looking at the shape of the pile.

Harriot discovered that for sufficiently large pile the highest density gives the so-called *face centered cubic* (FCC) packing. For the FCC packing the density is:

$$\frac{\pi}{3\sqrt{2}} \approx 0.74048$$

Face Centered Cubic (FCC) packing



History: Johannes Kepler (1611)

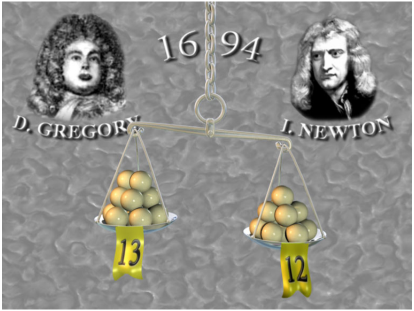
J. Kepler. The Six-Cornered Snowflake, 1611
In this little booklet Kepler examined several questions:
– Why honeycomb are formed as hexagon?
– Why the seeds of pomegranates are shaped as dodecahedra?
– Why the petals of flowers are most often grouped in fives?
– Why snowflakes are shaped as they are?

The Kepler Conjecture (1611):
The highest density of a packing of 3*-space by equal spheres* $= 0.74048\ldots$

Hilbert's Problem 18:3 (1900):
"How can one arrange an infinite number of equal solids, of given form, most densely in space, e.g., spheres with given radii... How can one fit them together in a manner such that the ratio of the filled space to the unfilled space be as great as possible?"

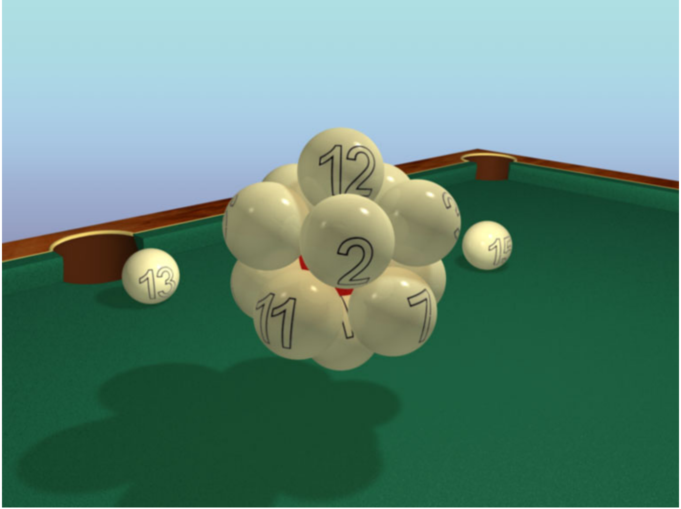
History: Gregory vs. Newton (1694)

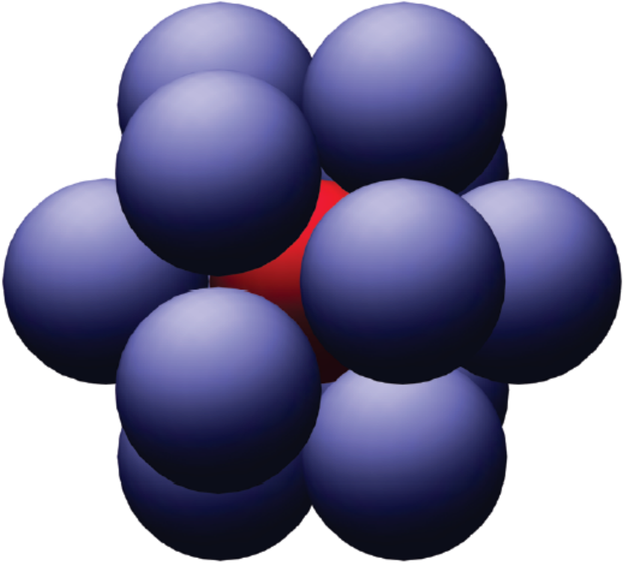
On May 4, 1694 David Gregory paid a visit to Cambridge for several days nonstop discussions about scientific matters with the leading scientist of the day Isaac Newton. Gregory making notices of everything that great master uttered. One of the points discussed, *number* 13, in Gregory's memorandum was 13 spheres problem.
Newton: $k(3) = 12$ vs.
Gregory: $k(3) = 13$ (The main Gregory argument was: area of the unit sphere $\approx 14.9\times$ area of a spherical cap of radius 30°.)



The Newton – Gregory problem = The thirteen spheres problem

The most symmetrical configuration, 12 billiard balls around another, is achieved if the 12 balls are placed at positions corresponding to the vertices of a regular icosahedron.





(Graphics: Detlev Stalling, ZIB Berlin)

History: Lattice sphere packing

Carl Friedrich Gauss (1831): *The FCC packing is the unique densest lattice sphere packing for dimension three.*

Hérmit (1850,1874); **Lebesgue** (1856); **Selling** (1874); **Minkowski** (1883), . . . , **Mahler** (1992).

Korkine & Zolotareff: $n = 4$ (1872), $n = 5$ (1877).

Blichfeldt (1925, 1929, 1935): $n = 6, 7, 8$.

Cohn & Kumar (2009): $n = 24$.

History: Schütte & v. d. Waerden (1953)

Reinhold Hoppe thought he had solved the thirteen spheres problem in 1874. However, there was a mistake — an analysis of this mistake was published by **Thomas Hales**: *The status of the Kepler conjecture*, Mathematical Intelligencer, 16 (1994), 47-58.
Finally, the thirteen spheres problem was solved by **Kurt Schütte** and **Baartel Leendert van der Waerden** in 1953. They had proved:

$$k(3) = 12.$$

It's not the end of the story about 13 spheres ...

John Leech(1956) : two-page sketch of a proof $k(3) = 12$.

. . . It also misses one of the old chapters, about the "problem of the thirteen spheres," whose turned out to need details that we couldn't complete in a way that would make it brief and elegant.

Proofs from THE BOOK, M. Aigner, G. Ziegler, 2^{nd} edition.

W. –Y. Hsiang (2001);

H. Maehara (2001, 2007);

K. Böröczky (2003);

K. Anstreicher (2004);

M. (2006)

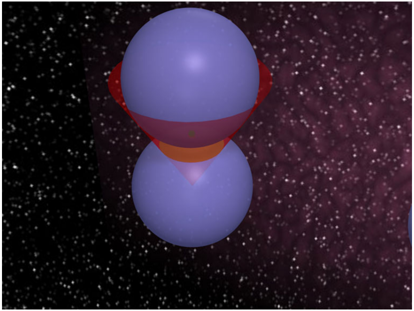
History: Coxeter (1963)

Coxeter proposed upper bounds on $k(n)$ in 1963 for $n = 4, 5, 6, 7$, and 8 these bounds were 26, 48, 85, 146, and 244, respectively.

Coxeter's bounds are based on the conjecture that equal size spherical caps on a sphere can be packed no denser than packing where the Delaunay triangulation with vertices at the centers of caps consists of regular simplices. This conjecture has been proved by **Böröczky** in 1978.

Packing by spherical caps

If unit spheres kiss the unit sphere S, then the set of kissing points is the arrangement on S such that the angular distance between any two points is at least 60°. Thus, the kissing number is the maximal number of nonoverlapping spherical caps of radius 30° on S.



Delsarte's method

Ph. **Delsarte** (1972); **V. M. Sidelnikov** (1974)
Delsarte, **Goethals** and **Seidel** (1975, 1977)

Theorem (Delsarte et al)

If

$$f(t) = \sum_{k=0}^{d} c_k G_k^{(n)}(t)$$

is nonnegative combination of Gegenbauer polynomials, with $c_k \geq 0$ *and* $c_0 > 0$*, and if* $f(t) \leq 0$ *holds for all* $t \in [-1, \frac{1}{2}]$*, then the kissing number in* n *dimensions is bounded by*

$$k(n) \leq \frac{f(1)}{c_0}$$

K(8)=240; k(24)=196560

G.A. Kabatiansky and **V.I. Levenshtein** (1978):

$$2^{0.2075n(1+o(1))} \leq k(n) \leq 2^{0.401n(1+o(1))}$$

In 1979: **V. I. Levenshtein** and independently **A. Odlyzko** and **N.J.A. Sloane** using Delsarte's method have proved that $k(8) = 240$, and $k(24) = 196560$.

Odlyzko & Sloane: upper bounds on $k(n)$ for $n = 4, 5, 6, 7$, and 8 are 25, 46, 82, 140, and 240, respectively.

Kissing numbers: 1979 – 2003

1993: **W.-Y. Hsiang** claims a proof of $k(4) = 24$ (as well as a proof of Kepler's conjecture). His work has not received yet a positive peer review.

1999: **V.V. Arestov** and **A.G. Babenko** proved that the bound $k(4) < 26$ cannot be improved using Delsarte's method.

2003: [**O.M.**] $k(4) = 24$.

History: Hales (1998)

In 1998, **Thomas Callister Hales,** following the approach suggested by **László Fejes Tóth** in 1953, announced a proof of the Kepler conjecture. Hales' proof is a proof by exhaustion involving checking of many individual cases using complex computer calculations. On 10 August 2014 Hales announced the completion of a formal proof using automated proof checking, removing any doubt.

History: Viazovska (2016)

In 2016, **Maryna Viazovska** announced a proof that the E_8 lattice provides the optimal packing in eight-dimensional space, and soon afterwards she and a group of collaborators (**Cohn, Kumar, Miller, Radchenko**) announced a similar proof that the Leech lattice is optimal in 24 dimensions.

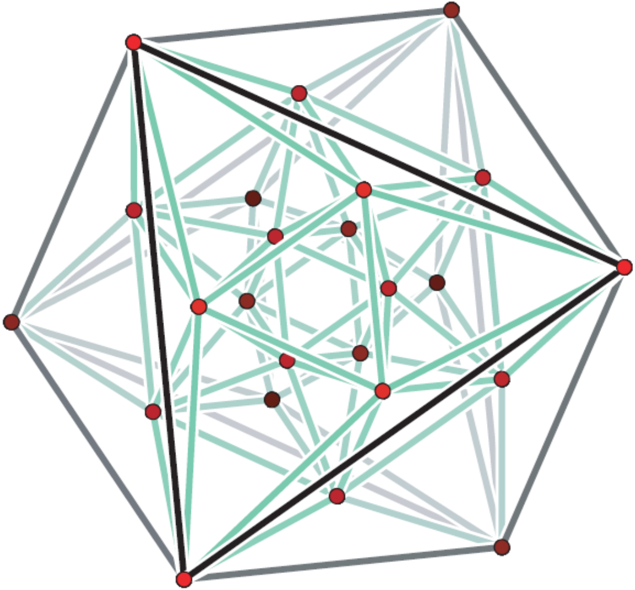
Status 2019: Kissing numbers

The only exact values of kissing numbers known:

n	lattice	regular polytope
$k(1) = 2$	A_1	
$k(2) = 6$	A_2	hexagon
$k(3) = 12$	H_3	icosahedron
$k(4) = 24$	$?D_4$	?24-cell
$k(8) = 240$	E_8	
$k(24) = 196560$	Λ_{24}	

The 24–cell

$n = 4$: There are 24 vectors with two zero components and two components equal to ± 1; they all have length $\sqrt{2}$ and a minimum distance of $\sqrt{2}$. Properly rescaled (that is, multiplied by $\sqrt{2}$), they yield the centers for a kissing configuration of unit spheres and imply that $k(4) \geq 24$. The convex hull of the 24 points yields a famous 4-dimensional regular polytope, the "24-cell", discovered in 1842 by Ludwig Schläfli. Its facets are 24 regular octahedra.



(Graphics: Michael Joswig/polymake [13])

The kissing problem in three dimensions

$$f_3(t) = 30.3875t^9 - 64.35t^7 + 45.8325t^5 + 8.575t^4 - 8.3t^3 - 2.13t^2 + 0.1t - 0.005$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be unit vectors in $\mathbb{R}^3$ (i.e. points on the unit sphere S^2). Then

$$S(P) = \sum_{k,\ell} f_3(p_k \cdot p_\ell) \geq m^2.$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be a kissing arrangement on the unit sphere S^2 (i.e. $p_k \cdot p_\ell \leq \frac{1}{2}$). Then

$$S(P) = \sum_{k,\ell} f_3(p_k \cdot p_\ell) < 13m.$$

Theorem

$$k(3) = 12.$$

Proof.

Suppose P is a kissing arrangement on S^2 with $m = k(3)$. Then P satisfies the assumptions in Lemmas 1, 2. Therefore, $13m > S(P) \geq m^2$. From this $m < 13$ follows, i.e. $m \leq 12$. From the other side: $k(3) \geq 12$, showing that $m = k(3) = 12$. □

The kissing problem in four dimensions

$$f_4(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.38t^4 - 9.83t^3 - 4.12t^2 + 0.434t - 0.016$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be unit vectors in $\mathbb{R}^4$ (i.e. points on the unit sphere S^3). Then

$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) \geq m^2.$$

Lemma

Let $P = \{p_1, \ldots, p_m\}$ be a kissing arrangement on the unit sphere S^3 (i.e. $p_k \cdot p_\ell \leq \frac{1}{2}$). Then

$$S(P) = \sum_{k,\ell} f_4(p_k \cdot p_\ell) < 25m.$$

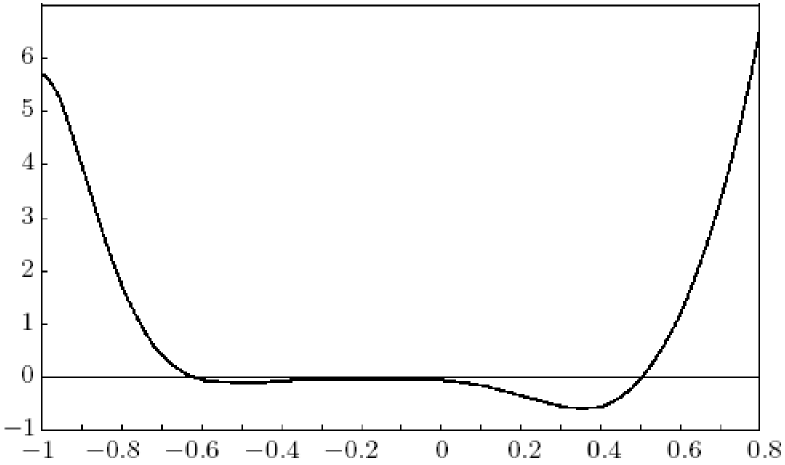
Theorem

$$k(4) = 24.$$

Proof.

Suppose P is a kissing arrangement on S^3 with $m = k(4)$. Then P satisfies the assumptions in Lemmas 3, 4. Therefore, $25m > S(P) \geqslant m^2$. From this $m < 25$ follows, i.e. $m \leqslant 24$. From the other side: $k(4) \geqslant 24$, showing that $m = k(4) = 24$. $\square$

The graph of the function $y = f_4(t)$



dim=4: uniqueness of the maximal kissing arrangement

LP bound [Odlyzko & Sloane; Arestov & Babenko] = 25.558...

M. (2003): $k(4) < 24.865$

C. Bachoc & F. Vallentin (2008): $S_7(4) = 24.5797...$

H. D. Mittelmann & F. Vallentin (2010)
$S_{11}(4) = 24.10550859...$
$S_{12}(4) = 24.09098111...$
$S_{13}(4) = 24.07519774...$
$S_{14}(4) = 24.06628391...$

F.C. Machado & F.M. de Oliveira Filho (2017, 2019+)
$S_{15}(4) = 24.062758...$
$S_{16}(4) = 24.056903...$

Perhaps, it is possible to combine the SDP and irreducible graphs to prove the uniqueness.

Towards a proof of the uniqueness conjecture

E. Bannai and N.J.A. Sloane: *Uniqueness of certain spherical codes.* Canadian J. Math. 33, 437–449 (1981)

The uniqueness conjecture. *In dimension 4 the maximal kissing arrangement is the 24–cell.*

O.R. Musin. *An extension the semidefinite programming bound for spherical codes,* arXiv:1903.05767

I think that above theorems and analysis of the distance distributions will help to prove of the uniqueness conjecture in 4 dimensions.

LP and SDP bounds

$$N \le \frac{f(1)}{f_0}$$

$$N \le \frac{f(1) + \hat{h}(n, T, f)}{f_0}$$

$$N^2 \le \frac{F(1,1,1) + 3(N-1)B}{f_0}$$

$$N^2 \le \frac{F(1,1,1) + 3(N-1)\,B + 3N\,\hat{h}(n, T, g)}{f_0}$$