Hilbert's Sixteenth and Related Problems in Dynamics, Geometry and Analysis

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Yulij Sergeevich Ilyashenko

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Multipliers of homoclinic orbits

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It is supposed to describe in this talk new differential invariants of dynamical systems, the multipliers of homoclinic orbits.

It is one of the most basic results in smooth dynamical systems that a transversal crossing of the stable and unstable manifolds of a hyperbolic fixed point produces a hyperbolic "horseshoe" and associated chaotic dynamics. This phenomenon which was discovered by Poincare has been discussed in many places. In this treatement the role of the eigenvalues of the linearization of the map at the fixed point is well understood. These eigenvalues control the rates of decrease of the size of the blocks which form the hyperbolic set. These rates in turn influence the numerical invariants such as Hausdorff dimension, box dimension and thickness of the hyperbolic sets. However, in previous works there always appear certain "constants" which remain unexplained. In this talk we show that these constants are in fact determined by two numerical invariants of the homoclinic orbit. They are indeed invariants with respect to smooth conjugacies. We call these invariants the stable and unstable multipliers of the homoclinic. We show that these multipliers appear in the asymptotic formulae for the thickness and Hausdorff dimension of the invariant sets in the vicinity of the homoclinic as the size of the vicinity tends to 0.

Geometrically, the multipliers we introduced capture the expansion and contraction effects which cannot be attributed to the local ones.To obtain these multipliers we essentially cancel out the effects of linear parts. A nice feature of te multipliers is that they may be calculateed by straightforward computations along the homoclinic orbit. Similar quantities can be introduced for heteroclinic transversal orbits and for heteroclinic contours.

All results of the talk were obtained together with Todd Young.

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Kepler's cubes and finite Lobachevsky geometry

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Dedicated to Yu.S.Iliashenko

Kepler had inscribed 5 cubes into the dodecaedron in his "Harmonia Mundi" (to describe the planets orbits greate axis length distribution).

The 12 edges of a Kepler's cube are some diagonals of the 12 faces of the dodecaedron (one diagonal on each face).

This cubes are related to some 5 Hamilton subgroups of the finite modular group $G = SL(2, \mathbb{Z}_5)$ (of second order matrices of determinant 1, whose elements are residues modulo 5).

The vertices ot the Kepler's cubes are the elements of order 3 in G_5, being also the vertices of a dodecaedron, forming the mod 5 version of the Lobachevsky plane.

Trying to extend this theory, developed in [1], to the more general mod p version of Lobachevsky geometry and to finite modular groups G_p was led to the following theory (p being a prime number).

Definition 1. *The Lobachevsky plane is the group of the line affine transformations, $u \to au + b$ (equipped with the left-invariant metric), verifying the orientation positivity condition, $a = c^2$, $c \neq 0$.*

This definition provides the usual Lobachevsky geomemtry of the half-plane $a > 0$ in the real line case. For the finite field $\mathbb{Z}_p$ case one should replace the positivity condition by its p-version: a should be *a nontrivial quadratic residue*, everything else remaining similar to the ordinary (real geometry) case.

The replacement of the calculus inequalities by the quadratic residues condition might be useful in other instances, as I had used it for the comlexification goal in real algebraic geometry [2].

Let G be any finite group.

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Definition 2. *The squaring graph of G has as its its vertices the elements of G and its (oriented) edges lead from each element g to g^2.*

Theorem 1. *The* squaring graph *of any finite group consists of connected components, each of these components being one cycle, equipped homogeneously by isomorphical trees, leading to its point (the identity element being a cycle of length 1 and its component being a rooted tree).*

Example For G_5 the graph is

$$T_{1,1,30} \bigsqcup (10A_2) \bigsqcup (6A_4),$$

where A_n denotes a cycle of length n equipped with trees of 1 edge at each point of the cycle, $T_{1,q_1,\dots}$ being the rooted tree with q_r vertices at the floor number r.

Definition 3. *A Hamilton group is a group, isomorphic to the group of the 8 quaternionic units,* $\{\ \pm 1,\ \pm i,\ \pm j,\ \pm k\ \}$.

Theorem 2. *The above tree T contains 5 Hamilton subgroups. Each of the Kepler's cubes in G_5 consists of those elements of order 3 in G_5, which preserve one of the 5 Hamilton subgroups, while acting on G_5 by conjugation.*

The conjugations, defined by this subgroup's elements, preserve the chosen cube.

Remark Denoting by A one of the vertices of the cube, one might attribute to the other vertices the notations A^ω, $\omega \in \{\pm 1, \pm i, \pm j, \pm k\}$.

Trying to extend the Hamilton subgroups and the Kepler's cubes to $G_p = SL(2, \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z})$, I had obtained the following results.

Theorem 3. *A squarings chain of length r, $\{A_j : 0 \le j \le r, A_{j+1} = A_j^2, A_r = 1, A_{r-1} \ne 1\}$ exists in G_p if and only if $p \equiv \pm 1 (mod\ 2^r)$.*

Example The maximal length $r = 2$ for $p = 5$ or 11, $r = 4$ for $p = 17$ or 47. This theory is related to the matrix version of the small Fermat (and Euler) theorem: $tr(A^n) \equiv tr(A^{n-\varphi(n)})$ for $n = p^a$ (and for any order matrices). Here φ denotes the Euler function: $\varphi(n)$ is the number of the residues, relatively prime to n.

Theorem 4. *The elements A of order 3 ($A^3 = 1$, $A \ne 1$) in G_p do form 2 equally large conjugation classes, each of $p(p \pm 1)/2$ elements (provided that $p = 3c + 1$), each class forming the set of points either of a* mod p-*Lobachevsky plane, or of its de Sitter world.*

The *de Sitter world* is the continuation of the Klein model of Lobachevsky plane model, habitating inside a disc of projective plane. The continuation

represents the complementary Möbius band of the disc, as it is explained in [3].

Example For $p = 7$ the Lobachevsky geometry of the Riemannian surface, associated to the squarings graph of group G_7 is described in [1]: this regular polyhedron of genus 3 consists of 24 heptagonal faces, being the $p = 7$ brother of the $p = 5$ dodecahedron.

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Statistical properties of the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations

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An induced map of an interval exchange transformation on a smaller interval is again an interval exchange transformation, and the smaller interval can be chosen in such a way that one obtains an exchange of the same number of intervals. One therefore obtains a renormalization map on the space of interval exchange transformations, called the Rauzy-Veech-Zorich induction. Like the Gauss map on the unit interval, the Rauzy-Veech-Zorich induction map has a finite absolutely continuous invariant ergodic measure (a theorem of Anton Zorich); the square of the map is exact with respect to that measure. Estimates on the decay of correlations are obtained for the Rauzy-Veech-Zorich map with the methods of Bunimovich–Sinai and of Lai-Sang Young.

Higher order Poincaré-Pontryagin functions and iterated path integrals

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Let $f, P, Q \in \mathbb{R}[x, y]$ be real polynomials in two variables. How many limit cycles the perturbed foliation

$$df + \varepsilon(Pdx + Qdy) = 0 \qquad (1)$$

can have? This problem is usually refered to as the weakened 16th Hilbert problem (see Hilbert [6], Arnold [1, p.313]).

Suppose that the foliation defined on the real plane by $\{df = 0\}$ has a period annulus $\mathcal{A} = \cup_{t\in\Delta}\gamma(t)$ formed by a continuous family of periodic orbits $\gamma(t) \subset f^{-1}(t)$, where $\Delta \subset \mathbb{R}$ is an open interval. Take a segment σ, transversal to each orbit in $\mathcal{A}$ and parameterized by $t = f|_\sigma$. The first return map $\mathcal{P}(t, \varepsilon)$ associated to this period annulus and to (1) is analytic in t, ε and can be expressed as

$$\mathcal{P}(t, \varepsilon) = t + \varepsilon^k M_k(t) + \varepsilon^{k+1} M_{k+1}(t) + \dots \qquad (2)$$

where $M_k(t) \not\equiv 0$ is the kth order Poincaré-Pontryagin function. The maximal number of the zeros of $M_k(t)$ on Δ provides an upper bound for the number of the limit cycles bifurcating from the annulus $\mathcal{A}$.

In the case $k = 1$ the Poincaré-Pontryagin function is an Abelian integral depending on a parameter $M_1(t) = \int_{\gamma(t)} Pdx + Qdy$, see Pontryagin [7]. The higher order Poincaré-Pontryagin functions are computed according to the following Françoise's recursion formula [4].

Let $M_1(t) = \cdots = M_{k-1}(t) \equiv 0$. Then

$$M_k(t) = \int_{\gamma(t)} \Omega_k$$

where

$$\Omega_1 = Pdx + Qdy, \Omega_m = r_{m-1}(Pdx + Qdy), 2 \le m \le k$$

and the functions r_i are determined succesively from the (non-unique) representation $\Omega_i = dR_i + r_i df$.

As the functions or differential forms involved are analytic, then this computation can be carried in a complex domain. From now on we consider (1) as a perturbed complex foliation in $\mathbb{C}^2$. Let $l(t) \in f^{-1}(t)$, $t \in \Delta$ be a continuous family of closed loops on the affine complex algebraic curve $f^{-1}(t) \subset \mathbb{C}^2$, where $\Delta \subset \mathbb{C}$ is a suitable open disc. The holonomy (or monodromy) map associated to the family $l(t)$ is denoted $P(t, \varepsilon)$ and has the representation (2). The first non-zero Poincaré-Pontryagin function $M_k(t)$, defined by (2) depends only on the *free* homotopy class $\gamma(t)$ of $l(t)$ [5].

For one-forms $f_1(t)dt, f_2(t)dt, \dots, f_r(t)dt$ on $\mathbb{R}$ define inductively

$$\int_0^1 f_1(t)dt f_2(t)dt \dots f_r(t)dt = \int_0^1 (\int_0^t f_1(\tau)d\tau \dots f_{r-1}(\tau)d\tau) f_r(t). \qquad (3)$$

Let S be a Riemann surface and $\omega_1, \omega_2, \dots, \omega_k$ be holomorphic one-forms. For every smooth path $l : [0,1] \to S$ we define the *iterated path integral* [2, 3]

$$\int_l \omega_1\omega_2 \dots \omega_k = \int_0^1 f_1(t)dt f_2(t)dt \dots f_r(t)dt \qquad (4)$$

where $l^*\omega_i = f_i(t)dt$. This does not depend on the parametrization of l.

Every non-constant polynomial f defines a smooth fibration

$$f : \mathbb{C}^2 \backslash f^{-1}(S) \to \mathbb{C} \backslash S$$

where S is a finite set. Let ω be a polynomial one-form in $\mathbb{C}^2$. It defines a geometric section of the flat cohomology bundle associated to the above fibration. Denote by $\omega' = \frac{d\omega}{df}$ the covariant derivative of this section and consider the free associative differential ring $(R, \frac{d}{df})$ with integer coefficients generated by ω. The elements of R are integer linear combinations of words $\omega^{(i_1)}\omega^{(i_2)} \dots \omega^{(i_s)}$. Consider finally the linear differential operator $L = \omega \frac{d}{df} : R \to R$. We have for instance $L\omega = \omega\omega'$, $L^2\omega = \omega(\omega')^2 + \omega^2\omega''$ etc. Note that $L^{k-1}\omega$ is a linear combinations of words of lenght k. After this preparation we may reformulate the Françoise's recursion formulae as follows

Theorem 1. *The k-th order Poincaré-Pontryagin function M_k defined by (2) is given by an iterated path integral of lenght k*

$$M_k(t) = (-1)^{k-1} \int_{\gamma(t)} L^{k-1}\omega \ .$$

This allows to prove on its hand

Theorem 2. *The kth order Poincaré-Pontryagin function M_k satisfies an equation of Fuchs type.*

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Internal 1:3 resonance revisited

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For a two parameter unfolding

$$\dot{z} = e^{i\varphi}z + (e^{i\varphi} - 1)\bar{z} - \mu z^2\bar{z}, \quad z = x + iy, \bar{z} = x - iy,$$

being a principal approximating family for the study of a system near a periodic orbit with the multiplier $\exp[\pm 2\pi i/3]$, we present an explicit formula for a bifurcation curve in the parameter plane (μ, φ) corresponding to the the existence of heteroclinic contour. We also prove the uniqueness of a limit cycle without appealing to any small parameter methods.

Yu.S.Ilyashenko's contributions to the theory of holomorphic foliations and uniformization

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An approach to the Hilbert 16-th problem initiated by Petrovskii and Landis and continued by Yu.S.Ilyashenko was to study the foliation by complex phase curves of a polynomial vector field. In 1957 Petrovskii and Landis had made an attempt to prove that the number of limit cycles of a polynomial vector field is bounded by a cubic polynomial in the degree. They have shown that it is no greater than the maximal number (called Petrovskii-Landis genus) of homologically-independent cycles whose projections to a given complex line are homotopic to disjoint circles. Their key statement said that the latter genus is uniformly bounded by a function in the degree.

It was found that Petrovskii-Landis arguments do not give a proof. In one of his first papers [7] Yu.S.Ilyashenko had proved that the latter key statement was false: the Petrovskii-Landis genus of a vector field of a given degree can be arbitrarily large, and the number of complex limit cycles can be countable.

While developing a complex-analytic approach to the Hilbert 16-th problem, Yu.S.Ilyashenko had obtained distinguished results in the theory of holomorphic foliations. In late 60-ths - beginning of 70-ths he investigated generic properties of polynomial vector fields in complex plane. He proved in [10] that a typical polynomial vector field has at most countable number of homologically independent complex limit cycles and is rigid with respect to the orbital equivalence. In the same paper he had proved density of complex phase curves for certain class of polynomial vector fields, which generalized a previous result by Khudai-Verenov [6]. Density of phase curves of a generic complex polynomial vector field was proved by his student B.Müller [15] in dimension two and recently by F.Loray and J.Rebelo [14] in any dimension. In his joint work with A.S.Pyartli [11] Ilyashenko proved that the monodromy group at infinity of a generic polynomial vector field is free.

In late 60-ths Ilyashenko initiated the study of uniformization of foliation by complex phase curves. The classical Poincaré-Köbe uniformization theorem says that each simply-connected Riemann surface is conformally equivalent to either $\overline{\mathbb{C}}$, or $\mathbb{C}$, or the unit disc. The uniformization of arbitrary (not necessarily simply-connected) Riemann surface is its parametrisation by the canonical coordinate of its universal cover (which has one of the three previous types). As it was proved by the speaker [3], [4], Lins Neto [13], and in a particular case by Candel and Gomez-Mont [2], for a generic polynomial vector field all the universal covers of the complex phase curves are discs.

In 1972 Ilyashenko had proved that the union of universal covers of leaves intersecting a cross-section (taken with marked points on the cross-section) is a Stein manifold [8], [12].

The classical Bers' simultaneous uniformization theorem [1] says that for any holomorphic fibration by compact Riemann surfaces the fibers admit a uniformization holomorphic in the parameter of the transversal cross-section. In early 70-ths Ilyashenko had obtained an important result in the theory of Kleinian groups [9] that extended Bers' theorem to holomorphic families (parametrized by a disc) of compact surfaces with a unique singular fiber having only nondegenerate double singular points. (For general algebraic families of compact Riemann surfaces with singularities the previous statement is false [5].)

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How often a diffeomorphism has infinitely many sinks?

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This question is known as R.Thom's Conjecture [T]. S.Newhouse disproved this conjecture [N1], [N2]. It turns out that in the space of C^r smooth diffeomorphisms $\mathrm{Diff}^r(M)$ of a compact surface M there is an open set U such that a Baire generic diffeomorphism $f \in U$ has infinite many coexisting sinks. But some results indicates that these diffeomorphisms have zero measure [LY].

Our (with V.Kaloshin) work in process is devoted to this phenomena. Infinite number of sinks for two dimensional diffeomorphisms appears near homoclinic tangencies. Let V be a small neighborhood of a homoclinic contour. A sink is *V-localized* if its trajectory is inside of V. It has *a complexity s* if it turns around V exactly s times. Preliminary results are the following.

Theorem 1. *For any integer s there exists a neighborhood $V = V(s)$ of a homoclinic contour such that a prevalent unfolding of a diffeomorphism has only finite number of V-localized sinks of complexity s.*

It is not trivial to define a notion of prevalence, see [K], [HSY].

Theorem 2. *Given $\alpha > 0$ and an integer s, there exists a neighborhood $V = V(s, \alpha)$ such that a prevalent unfolding f of the diffeomorphism as above has the following property. For some constant $C > 0$ and any V-localized periodic point p of complexity not greater than s, $f^n(p) = p$,*

$$|Tr D f^n(p)| > C\mu^{(1-\alpha)n}.$$

The last Theorem is in some sense parallel to results of Palis and Takens [PT] and Palis and Yoccoz [PY]. They prove stronger hyperbolic properties in more restrictive case.

The approach we use involves Newton Interpolation Polynomials and Discretization method, derived by V.Kaloshin in his previous works [K1], [KH1], [KH2].

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Multiple Time Scale Systems: an Update

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The theory of multiple time scale dynamical systems can be traced at least as far back as the work of van der Pol during the 1920's. The subject has undergone extensive mathematical development since that time, from three distinct approaches.

- Asymptotic analysis of these systems was developed systematically in Russia, with Pontryagin an early leader,
- Nonstandard analysis of these systems was developed systematically in France with Georges Reeb in Strasbourg as a visionary leader.
- Geometric methods were introduced into the subject in the United States and Russia, with the work of Fenichel in the 1970's an early examplar.

The subject remains in an unsatisfactory state with these different languages that are difficult to translate into one another and with the inherent complexity of the phenomena the subject investigates. Periodic efforts to review the subject and organize the major results have been made. Arnold, Afrajmovich, Ilyashenko and Shil'nikov wrote a geometric overview in a paper that appeared in English in the Springer-Verlag Encyclopaedia of Mathematical Sciences. This lecture will survey some of the developments that have occurred since this work, with a focus on some of my own recent contributions. In particular, it will discuss computational aspects of multiple time scale systems, using the forced van der Pol system as an example. The origins of the hyperbolic theory of dynamical systems can be traced to the work of Cartwright and Littlewood on this example, and we shall illustrate how modern methods and computations clarify the phenomena they studied.

Instabilities of totally elliptic points of symplectic maps in dimension four.

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We show that generically certain totally elliptic points of symplectic maps in dimension four are unstable. Proof is based on variational principle developed by J.Mather. This is a joint work with J.Mather and E. Valdinoci.

Normal forms, geometric structures and rigidity

Anatole Katok

We discuss how local and semi-local normal form theory for smooth maps, differential equations and group actions can be used for proving global differentiable classification in various cases, including various rigidity properties. Here is a list of topis and key words:

- Resonanace and subresonanace relations.
- Non-stationary normal forms for contractions and applications to rigidity.
- Critical regularity of invariant geometric structures for hyperbolic dynamical systems: nonresonanace and resonance cases.
- Anosov cocycle and its generalizations.
- Invariant connections and higher–order structures.

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Circle diffeomorphisms with breaks.

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An example of non-coincidence of minimal and statistical attractors

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1. Introduction. There are many different definitions of what one should call an attracting set for a dynamical system, such as the maximal attractor, the nonwandering set, the Birkhoff center, the Milnor's attractor, the statistical attractor and the minimal attractor. The latter notion was introduced by A.Gorodetski and Yu.Ilyashenko, see [1].

The aim of this work is to present an example of a dynamical system for which the two last definitions give different attracting sets (an example which have been found only short time ago).

2. Definitions. Let us consider a smooth dynamical system (X, g^t), and let μ_L be the Lebesgue measure on X (normalized to $\mu_L(X) = 1$).

Definition 1. *An open set U is called Stat-inessential, if for μ_L–almost every point $x \in X$*

$$\frac{1}{T}\left|\{t | 0 \le t \le T, \quad g^t(x) \in U\}\right| \to 0, \qquad \textit{as } T \to \infty,$$

where $|Z|$ denotes the Lebesgue measure of the set $Z \subset \mathbb{R}$.

Definition 2. *An open set U is called Min-inessential, if*

$$\frac{1}{T}\int_0^T (g^t_*\mu_L)(U)\, dt \to 0, \qquad \textit{as } T \to \infty.$$

Definition 3. *A statistical (resp. minimal) attractor is defined as a complement to the union of all Stat-inessential (resp. Min-inessential) open sets. We denote it as A_{Min} (resp. A_{Stat}).*

3. Main result. Consider a planar vector field v. Suppose that it has two singular points, a saddlenode A and a saddle B. Also, suppose that an outgoing separatrix of A coincides with an incoming separatrix of B and that an outgoing separatrix of B coincides with an incoming separatrix of A (see fig. 11.1):

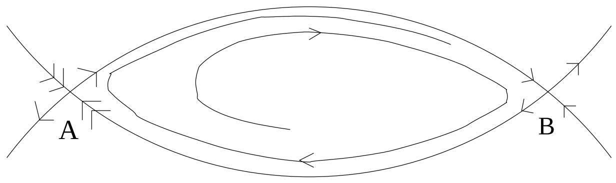

Figure 12.1: Modified Bowen's example

Let γ_{AB} (resp. γ_{BA}) be the separatrix, going from A to B (resp. from B to A), let $\Gamma = \{A, B\} \cup \gamma_{AB} \cup \gamma_{BA}$, and let M be a domain, bounded by Γ. Then there exists a small "inner neighborhood" V of the loop Γ (an intersection of some neighborhood of Γ with M) such that every trajectory with initial conditions in V stays in V forever and tends to Γ as time tends to infinity.

Let us consider the dynamical system (X, g^t), where $X = \overline{V}$, and $g^t = g^t_v$ is the flow, generated by v. We will call this system "modified Bowen's example" in an analogy with the Bowen's example (which is constructed in the same way, but both singular points are saddles).

Theorem 1. *For the modified Bowen's example, $A_{Min} = \{A\}$, $A_{Stat} = \{A, B\}$.*

4. Sketch of the proof. It is clear that every point $x \in X$ passes almost all the time in arbitrarily small neighborhoods of A and B, thus, $A_{Min} \subset A_{Stat} \subset \{A, B\}$. Consider for a point $x \in X$ two sequences T^A_n and T^B_n of time intervals, passed in some (fixed) neighborhoods of A and B respectively. For simplicity, we restrict to the case where the saddle can be written as $\dot{x} = x$, $\dot{y} = -y$, the saddlenode as $\dot{x} = x^2$, $\dot{y} = -y$ (the general case is similar, but a bit more difficult technically). Direct computation gives the following (asymptotic) behavior of this sequence:

$$T^B_n \sim T^A_n, \quad T^A_{n+1} \sim e^{T^B_n}.$$

So, for each individual point, there exists a sequence of times T_n which tends to infinity, such that for every T_n the total amounts of time, passed near A and near B, are almost equal. Thus, $A_{Stat} = \{A, B\}$.

On the other hand, it can be shown that for any two points $x, y \in V$ either after some moment θ they will never appear near B simultaneously, or they belong to the same trajectory. It can be seen from comparing the sequences

$$T, T, e^T, e^T, e^{e^T}, e^{e^T}, \ldots$$

and

$$T', T', e^{T'}, e^{T'}, e^{e^{T'}}, e^{e^{T'}}, \ldots$$

of the times, passed near A and B. Thus, the Lebesgue measure will concentrate (as time goes to infinity) near the point A:

$$g^t_* \mu_L \to \delta_A, \quad \text{as } t \to \infty.$$

Thus, the minimal attractor is $A_{Min} = \{A\}$.

5. Remarks.

Remark 1. *The modified Bowen's example is an example of codimension 3 in planar vector fields.*

The definitions of inessentiality may be rewritten in a following way. Consider for an open set U the family $\chi^U_T(x)$ of time-averages of the characteristic function χ^U of U:

$$\chi^U_T(x) = \frac{1}{T} \int_0^T \chi^U(g^t(x))dt.$$

Remark 2. *The set U is Stat-inessential (resp. Min-inessential) if and only if χ^U_T tends to 0 in measure (resp. almost surely) w.r.t. μ_L.*

Remark 3. *Considering family of functions χ^U_T for $U = U_\varepsilon(B)$, we obtain a dynamical realization of the Riesz's example.*

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Integrable linearizable and normalizable saddles in quadratic systems in $\mathbb{C}^2$

P. Mardešić

We present some results from a joint work with C. Christopher and C. Rousseau.

We study for the family of polynomial vector fields given by

$$\dot{x} = x + \sum_{i+j=2} c_{ij} x^i y^j$$
$$\dot{y} = -\lambda y + \sum_{i+j=2} d_{ij} x^i y^j, \quad \lambda > 0, c_{ij}, d_{ij} \in \mathbb{C}$$

A vector field is linearizable if by an analytic change of coordinates (without multiplication by a function) it can be transformed to a linear vector field. A vector field is integrable if it can be tranformed to a multiple of a linear vector field. A resonant vector field i.e. $\lambda \in \mathbb{Q}^+$ is normalizable (orbitally normalizable) if by an analytic change of coordinates it can be transformed to a vector field containing only resonant monomials (up to a multiplicative factor).

In the parameter space $\mathbb{R}^+ \times \mathbb{C}^6$, denote $\mathcal{L}$, $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{ON}$ the sets of the parameter values corresponding to linearizable, integrable, normalizable and orbitally normalizable vector fields and for fixed $\lambda \in \mathbb{R}^+$ let $\mathcal{L}_\lambda$, $\mathcal{I}_\lambda$, $\mathcal{N}_\lambda$ and $\mathcal{ON}_\lambda$ denote their restriction given by the prescribed value of λ.

The following inclusions are obvious

$$\begin{array}{ccc} \mathcal{L} & \subset & \mathcal{I} \\ \cap & & \cap \\ \mathcal{N} & \subset & \mathcal{ON} \end{array}$$

and analogously for their restrictions to λ fixed.

What can be said about the sets $\mathcal{L}$, $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{ON}$ and about their restrictions for λ fixed?

The dependence of these sets on λ is higly discontinuous. For instance if λ is a Bruno number, then $\mathcal{L}_\lambda = \mathbb{C}^6$, whereas if λ is rational, then the sets $\mathcal{L}_\lambda$, $\mathcal{I}_\lambda$ are algebraic sets. For λ irrational non-Bruno number these sets are very complicated. We study two mechanismes for linearizability which work well in the family: the Darboux method and the blow down to a node or saddle-node. In particular these two methods suffice for proving linearizability of all strata in $\mathcal{L}_2$.

Our calculations showed some strange-looking features: holes in the linearizability strata and their accumulation, strata stopping for some value of λ etc. We give a geometric explanation of some of these phenomena in terms of the unfolding of Ecalle-Voronin moduli.

We study also the relationship between integrability and linearizability. We show that for $\lambda \in \mathbb{R}^+$, $\mathcal{I}_\lambda = \mathcal{L}_\lambda$ if and only if λ is not a Cremer number.

We consider more generally analytic saddles. We prove the equivalence between the following three conditions:

- linearizability of an integrable saddle
- relative exactness of a certain time form η
- vanishing of the integrals of the form η along all asymptotic cycles in the leaves of the saddle.

Analogous results for saddle-nodes have been obtained by L. Teyssier.

We discuss also some generalizations.

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On the analytic solvability of the center-focus problem

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The singular point of the vector field in the plane is called *monodromic*, if the Poincare first return function (monodromy transformation) is defined for it. Tthe monodromic singular point of the analytuc vector field can be either a center or a focus. The set M of all the monodromic germs can be represented as an union of semialgebraic sets : $M = \cup M_\alpha$. All the germs from one M_α have the same process of desingularization connected with the Newton diagrams. The following theorem is proved

Theorem 1. *The center-focus problem is analytically solvable in each class* M_α.

Topology of plane real algebraic curves: some solved and open problems

S. Orevkov

How the branches of a plane real algebraic curve can be arranged on the projective plane? This is the first half of the first part of Hilbert's 16th problem (the second half is about spatial surfaces). This question just indicated a large area for investigation. However several concrete questions and conjectures served as repers in this area. For example, in the 16th problem, Hilbert himself posed an explicite question: can a 6th degree curve consist of 11 ovals, none of them being inside another (answered by Petrovski) and more generally, what arrangements of ovals are possible for a 6th degree curve (answered by Gudkov). Other questions and conjectures which motivated the development were formulated by Ragsdale, Gudkov, Arnold, Rohlin, Viro, Kharlamov et als. For instance, the famous Gudkov's conjecture about the congruence modulo 8 maybe caused the Arnold-Rohlin revolution in the area. In my talk I am going to survey the matter of art for the old problems and to formulate some new (or old but forgotten) problems which seem to me important.

Rigidity theorems for generic holomorphic germs of dicritic foliations and vector fields in $(\mathbb{C}^2, 0)$

L. Ortiz-Bobadilla

It is well known, that the analytic and formal classification of holomorphic vector fields at generic singular points coincides (Poincare). The failure of the genericity assumptions (in the case of saddle resonant and saddle node singularities, for example) leads to a relatively simple formal classification which differs from the analytic one: the analytic classification has functional moduli (Brjuno, Voronin, Grintchy, Mescheryakova, Teyssier). The same happens for orbital equivalence (analytic and formal) of holomorphic vector fields (Ilyashenko, Martinet, Ramis), and for mappings (Ecalle, Ilyashenko, Voronin). In the cases of higher codimension the remarkable presence of "rigidity" takes place: it consists on the coincidence of formal and analytic classification, while already the formal classification is, generally speaking, boundless and, as rule, has functional moduli.

The first rigidity theorem (where rigidity is understood as above) was proved for the analytic classification of nonsolvable finitely-generated groups of germs of holomorphisms on the complex line (Cerveau, Moussu, Ramis, Elizarov, Ilyashenko, Shcherbakov, Voronin). The rigidity phenomena in the classification of foliations given by holomorphic germs of vector fields, known as orbital rigidity, was found for degenerate singular points having nilpotent Jordan cell at this linearization (Cerveau, Moussu, Elizarov, Ilyashenko, Shchervakov, Voronin). A rigidity theorem for foliations in the class (next by complexity), ν_n, consisting of germs having vanishing n-jet at its singular point, was proved (for generic non dicritic germs) by S. Voronin. Later, an analogous result for the rigidity of vector fields was obtained (Ortiz, Rosales, Voronin). Finally, orbital rigidity for families of germs with fixed geometry (under weak genericity assumptions) was obtained by Le Floch and can be obtained from the results given by J.F. Mattei and E. Salem. In the present exposition a rigidity theorem (in the case of foliations and vector fields) is

obtained for generic dicritic germs in the class ν_n (joint work with S.Voronin and E. Rosales).

Algebraic Integrability of functional equations and dynamical systems with discrete time defined by birational mappings

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The talk is based on the papers [8], [9] that are a development of the algebraic-geometrical approach to integrability of functional equations (see [7]) of the form below and is devoted to an algebraic integrability (AI) of functional equations defined by birational mappings (BFEs) for functions $y(w): \quad \mathbf{C} \to \mathbf{C}^N$ in one complex variable w of the form

$$y(w+1) = F_n(y(w)), \quad y(w): \ \mathbf{C} \to \mathbf{C}^N, \quad w \in \mathbf{C}, \quad F_n \in \mathbf{Bir}(\mathbf{C}^N). \quad (1)$$

For $w = m \in \mathbf{Z}$ the BFEs above is a dynamical system with a discrete time (DDS) or cascade (see [2]). Here the map $F_n: \ y \mapsto y' = F_n(y) = \frac{f_i(y)}{f_{N+1}(y)}, i = (1, 2, \ldots, N)$, $f_i(y)$ for $\forall$ i are polynomials in y, $\deg F_n(y) = \max_{i=1}^{N+1} \{\deg(f_i(y))\} = n$, is a given birational one of the group of all automorphisms of $\mathbf{C}^N \to \mathbf{C}^N$ with coefficients from C [10]-[13]. The BFEs (1) can also be derived at author's discretization of a standard autonomous differential equation of first order with the vector field [1] connected with $F_n(y(w))$ by a corresponding manner. We obtain a general solution of these BFEs for $n = 1$ and any N but also the one for $\forall(n, N)$ at $F_n \equiv U_m \circ F_1 \circ U_m^{-1}$ where $U_m \in \mathrm{Bir}\mathrm{C}^N$.

We discuss the dynamics of the map (1) at $N = 2, \forall n$ following to [7] and such important characteristic of integrable mappings as the Arnold complexity $C^{\Phi_n}_{A;\Gamma_1\Gamma_2}(k) \sim d_{\Phi_n}(k) \stackrel{\mathrm{def}}{=} \deg \Phi_n^k$ (see Definition of the one in [3], [4] and [5], [6]). We derive for the one the linear autonomous difference

equation with constant coefficients. A spectrum of characteristic equation for this difference equation defines a polynomial or exponential growth of the Arnold complexity that is testing of algebraic or nonalgebraic integrability.

The AI of the BFEs for $n \geq 2, \quad N = 2$ is defined through an existence of the linear system of algebraic curves [12] (LSAC) $\psi_\mu(z(w))$: $\sum_{j=1}^{k+1} C_j \Psi_j(z(w) \quad (y_i(w) = z_i(w)/z_3(w), \quad z \in \mathrm{CP}^2)$ of degree μ, dimension $k \geq 1$, of genus $p \geq 0$, being invariant subspace relatively to the birational map $\Phi_n : z \mapsto z' \sim \phi(z) = z_3^n f(z_i/z_3)$. As result of this definition we derive k first integrals in extended phase space $\mathrm{C}^2 \times \mathrm{C}$ in terms of $(k+1)$ basic functions of the LSAC (in Jordan basis). These integrals are rational functions in $y(w), w, (\frac{\lambda_i}{\lambda_{k+1}})^w, i \in (1, \cdots, k)$ but λ_i are defined by the spectrum of the matrix D (in Jordan basis): $\Psi(z'(w)) \sim D \circ \Psi(z(w))$, and $z'(w) \stackrel{\mathrm{def}}{=} z(w+1) \stackrel{\mathrm{def}}{=} \Phi_n z(w)$. We established necessary and sufficient conditions of the existence of the LSAC as well as the construction of the one (LSAC) in explicit form. In [14] we made first step to the integrating of the functional equations (1) for $\forall(n, N)$.

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The cyclicity of singular loops of reversible quadratic Hamiltonian under quadratic perturbations

Robert Roussarie

As it is well-known, the principal difficulty in the study of a planar vector field family is to understand how in the family, the polycycles bifurcate into limit cycles. The reason is that a polycycle bifurcation is in general a highly non-analytic phenomenon. A case of particular importance is when one considers limit cycles created in unfolding an Hamiltonian polycycle. Introducing a small parameter ε the dual unfolding can be written

$$\nu_\lambda = dH + \varepsilon\bar{\nu}_{\bar{\lambda}} + o(\varepsilon)$$

where $\lambda = (\bar{\lambda}, \varepsilon)$. The integral function unfolding $I(h, \bar{\lambda}) = \int_{\{H=h\}} \nu_{\bar{\lambda}}$ is called Abelian integral (unfolding) of the given vector field unfolding.

Limit cycles bifurcating from an annulus filled by *regular Hamiltonian cycles,* are in general directly controlled by the zeros of the associated Abelian integral unfolding. Then, a general question is to understand in what measure the Abelian integral unfolding also control the limit cycles bifurcating from a *Hamiltonian polycycle.*

In the present talk, one restricts the investigation to quadratic polynomial vector fields. In this context, the Hamiltonian function H is a cubic polynomial and $\bar{\nu}_{\bar{\lambda}}$ is a family of quadratic 1-forms. The exact number of zeros of Abelian integrals corresponding to the integration of quadratic 1-forms along cycles of cubic Hamiltonians was recently obtained, even for the degenerate cases. These Abelian integrals have no more than 2 zeros in their whole domain of definition, which may be the union of two intervals.

One can deduce from this result the number and the bifurcation diagram of limit cycles which bifurcate from any region, union of regular annuli of Hamiltonian cycles in the space of quadratic vector fields. Moreover, one can extend this study at the whole plane, in the case of generic quadratic Hamiltonian vector fields. For such a generic Hamiltonian vector field, the boundary of region filled by the regular Hamiltonian cycles contains just non-degenerate centers or homoclinic saddle connections, and in these two cases it is known that all the bifurcating limit cycles are related to zeros of the associated Abelian integral.

For non-generic Hamiltonian vector fields, the problem seems much more difficult because one has to look at the limit cycles bifurcating from polycycles containing more than one singular point. Let us denote by Q^H and Q^R the Hamiltonian class and reversible class of quadratic integrable systems. There are several topological types for systems belonging to $Q^H \cap Q^R$. One of them is the case where the corresponding system has two heteroclinic loops with 2 saddles (2-*saddle-loops*), sharing one saddle–connection, which is a line segment (the other saddle-connections form an ellipse).

In a recent work in collaboration with Chengzhi Li, we prove that the maximal number of limit cycles which bifurcate from the union of the 2-saddle-loops with respect to quadratic perturbations is two, *in the reversible direction*. This means that every bifurcating limit cycle is related to a zero of Abelian integral. We also describe the corresponding bifurcation diagram. This diagram can not be deduced from the bifurcation diagram for Abelian integral zeros. It must be noticed that the bifurcation diagram is not completely established, up to now, outside a conic sector around the reversible direction.

On systems with homoclinic loops of saddle-focus

L. P. Shilnikov

Rectifiable pencils of conics

Vladlen Timorin

A set of curves with a common point is called *rectifiable* if there is a local diffeomorphism (defined in a neighborhood of the common point) that sends all curves from this set to straight lines. Rectifiable pencils of curves appear, for example, in Riemannian geometry — geodesics passing through a given point are rectifiable by the inverse of the exponential map.

The following is one type of questions I am particularly interested in. For some nice class of curves, determine what pencils of curves from this class are rectifiable.

Almost all rectifiable analytic pencils of conics in the plane look as follows. All conics from the pencil (except for 3 degenerate conics, which are straight lines) have 3 points of tangency with the same curve, which is the dual to a cubic.

Homoclinic bifurcations and dimension of attractors of evolutionary PDE's

Dmitry Turaev

A new method of estimating the attractor's dimension is suggested which involves analysis of homoclinic bifurcations, and which allows for expressing both upper and lower bounds for the attractor's deimension in terms of Lyapunov dimension. The method is applied for obtaining sharp estimates of the attractor's dimension for a class of damped wave equations which are beyond the reach of the classical methods. This is a joint work with S.Zelik.

Monodromy groups of degenerate singularities of holomorphic foliations

S. M. Voronin

We shall study a generalization of the usual monodromy group of holomorphic foliations. This group is obtained by glueing the monodromy groups of pasted spheres under a good blowing-up.

Center problem for Abel equation, Moments and Compositions

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Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y), \\ \dot{y} = x + G(x,y) \end{cases} \tag{1}$$

with $F(x,y)$ and $G(x,y)$ homogeneous polynomials of a given degree $d \geq 2$. This system has a center at the origin if all its trajectories around the origin are closed. A (version of) the classical Poincaré's Center-Focus problem is to find conditions on F and G necessary and sufficient for the system (1) to have a center. Closely related to this question is the second part of Hilbert's 16-th problem which asks for the maximal possible number of isolated closed trajectories (limit cycles) in the above system.

These classical problems can be naturally reformulated for the Abel differential equation

$$y' = p(x)y^2 + q(x)y^3 \tag{2}$$

This equation is said to have a *center at a pair of complex numbers* (a,b) *if* $y(a) = y(b)$ *for every solution* $y(x)$ *of (2) with the initial value* $y(a)$ *small enough.* The Center-Focus problem is to give necessary and sufficient conditions on p, q for (2) to have a center. The analogue of the Hilbert 16-th problem for Abel equation (called sometimes "Pugh problem") is to bound the number of isolated solutions $y(x)$ of (2) satisfying $y(a) = y(b)$ (the "periodic" ones).

It is a general belief that the Abel equation version of the Hilbert and the Center-Focus problems reflects the main difficulties of the original ones while suggesting serious technical simplifications. Moreover, it opens new relations with classical Analysis and Algebra. Indeed, recently center conditions for Abel equation (2) have been related to the composition factorization of $P = \int p$ and $Q = \int q$ on one side, and to the vanishing conditions for the moments

$$m_k = \int_a^b P^k(x)q(x)dx \tag{3}$$

on the other. (Finding these vanishing conditions for $m_k, k = 0, 1, \dots$ we call a "Moment problem". It turns out to be a tangential version of the Center-Focus problem).

Let us assume that the coefficients $p = P'$ and $q = Q'$ of Abel equation (2) are complex polynomials. P and Q are said to satisfy a *Polynomial Composition condition* if there exist polynomials $\tilde{P}, \tilde{Q}, W$ such that $W(a) = W(b)$ and

$$P(x) = \tilde{P}(W(x)),\ Q(x) = \tilde{Q}(W(x)). \tag{4}$$

A change of variables in (2) and (3) according to (4) shows that the Composition condition implies both the center for (2) and the vanishing of all the moments (3). We do not know any counterexample to the following

Composition Conjecture. *The Polynomial Composition condition is necessary and sufficient for Abel equation (2) to have a center.*

This conjecture has been verified for small degrees of p, q and in some special cases (M.A.M. Alwash, N.G. Lloyd, M. Briskin, J.-P. Francoise, Y.Y., C. Christopher, M. Blinov, N. Roytvarf, Yang Lijun and Tang Yun). Below we give some very recent results further supporting this conjecture.

For some time a similar conjecture for the Moment problem has been discussed. It turns out to be true in many special cases but in general the vanishing of all the moments (3) does not imply (4) (F. Pakovich).

Let us call *"definite" on a, b the polynomials P for which the vanishing of $m_k, k = 0, 1, \dots$ with any polynomial $q = Q'$ does imply Composition condition (4).*

All polynomials P up to degree 5 are definite. In the space V_l of polynomials P of a fixed degree $l \geq 6$ non-definite polynomials belong to a certain proper algebraic subset. More specifically, all indecomposable P are definite (for every $a \neq b$), as well as all P with $P'(a) \neq 0$, $P'(b) \neq 0$. Chebyshev polynomial T_6 is not definite with respect to $a = -\sqrt{3}/2$, $b = \sqrt{3}/2$. Many additional classes of definite polynomials are known (M. Briskin, J.-P. Francoise, Y.Y., C. Christopher, M. Blinov, N. Roytvarf, F. Pakovich).

The following results (Blinov, Briskin, Roytvarf, Y) support the Composition Conjecture above and illustrate the role of definite polynomials (and of the moments in general) in the Center problem for the Abel equation. In all these results the polynomial $p = P'$ is fixed while the polynomial q of a given degree d varies inside the space V_d.

Theorem 1. *Let a polynomial $p = P'$ be fixed, with P definite. Let the maximal degree d of the polynomials $q = Q'$ be fixed. There exists $\epsilon = \epsilon(p, d, a, b) > 0$ depending only on p, a, b and d such that for any q of degree d with $\| q \| \leq \epsilon$ Abel equation (2) has a center at a, b if and only if P and Q satisfy (4).*

A special case of this result for $p(x) = x$ has been obtained by Yang Lijun and Tang Yun.

For P definite *the Moment Bautin index $N = N(P, d, a, b)$ is the minimal number of the moments m_k whose vanishing implies (4) for any q of degree at most d.* This number can be explicitly computed (or bounded).

Theorem 2. *Let $p = P'$ be fixed, with P definite. The local Bautin ideal I of Abel equation (2) on a, b (generated on the ball $B_\epsilon \subset V_d$ by all the Taylor coefficients $v_k(q)$ of the Poincaré first return mapping of (2)) coincides with the ideal J generated by the moments $m_i(q), i = 0, 1, \dots, N$. The Bautin index of (2)(i.e. the minimal number of v_k generating I) is equal to N.*

This implies a bound on the number of "small" periodic solutions of (2):

Corollary 1. *There is $\delta = \delta(P, d, a, b) > 0$ such that for any q with $\| q \| \leq \epsilon$ the number of solutions y of the Abel equation (2.1) satisfying $y(a) = y(b)$ and $|y(a)| \leq \delta$ does not exceed $N(P, d, a, b)$.*

The moments $m_k(q)$ form the linear parts of the Center equations $v_k(q) = 0$, $k = 2, \dots$. As Theorems 1 and 2 show, their structure essentially determines the structure of the Center equations near the origin in V_d. It turns out that the same is true "at infinity" (i.e. in a neighborhood of the infinite hyperplane in the projectivization PV_d of the space V_d). Combining this fact with the local information above we can show in many cases that *globally the set of $q = Q'$ for which Abel equation (2) has a center consists of those q for which P, Q satisfy Composition condition (4) and possibly of a finite number of additional points q_j.*

The Bogdanov-Takens singularity

Henryk Żołądek

I will present some results about clasification of germs of vector fields in the complex plane with singular point having nilpotent linear part. Firstly, the classical Takens normal form can be obtained in an analytic way. Recall that the Takens form is not the final one, when considering it from the formal orbital equivalence point of view. I will present a complete formal orbital classification of such singualarities. These results were obtained together with Ewa Stróżyna.