REPORT ON THE YOUNG RUSSIAN MATHEMATICS FELLOWSHIP 2021

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Obtained results

The following results were obtained. The (Morse) index of an important non-orientable free boundary minimal surface, the critical Möbius band, in the 4-dimensional Euclidean found by Fraser and Schoen ball was computed. The machinery developed in order to get this result also enables us to reprove a similar result on the index of another important orientable free boundary minimal surface, the critical catenoid, in the 3-dimensional Euclidean ball also found by Fraser and Schoen. Also some nontrivial inequalities on the index of a free boundary minimal surface in the ball were obtained. Moreover, the index and the nullity of new immersed free boundary minimal surfaces in the 4D-ball studied recently by Fraser and Sargent were estimated from below, which is the first result on the index and nullity of these surfaces. Also, it was shown that the flat equatorial disk is the only free boundary minimal surface in the ball with zero quartic Hopf differential. It was also conjectured that the critical Möbius band is the only free boundary minimal Möbius band in the Euclidean 4D-ball.

Papers

[1] On the index of the critical Möbius band in $\mathbb{B}^4$
arXiv:2112.04883

In this paper we prove that the Morse index of the critical Möbius band in the $4-$dimensional Euclidean ball $\mathbb{B}^4$ equals 5. It is conjectured that this is the only embedded non-orientable free boundary minimal surface of index 5 in $\mathbb{B}^4$. One of the ingredients in the proof is a comparison theorem between the spectral index of the Steklov problem and the energy index. The latter also enables us to give another proof of the well-known result that the index of the critical catenoid in $\mathbb{B}^3$ equals 4.

Scientific conferences and seminar talks

[1] Conference "Young Researchers in Spectral Geometry", Montreal, September, 13

Talk "On the index of the critical Moebius band in the 4-ball"

[2] Moscow, November, 5

Talk "On free boundary minimal submanifolds" at "Laboratory of algebraic geometry: weekly seminar" (Higher School of Economics)

[3] Moscow, October, 2

Talk "On the index of the critical Moebius band in the 4-ball-I" at "Seminar on Spectral Geometry" (Independent University of Moscow and Interdisciplinary Scientific Center J.-V. Poncelet, ISCP, UMI 2615)

[4] Moscow, October, 9

Talk "On the index of the critical Moebius band in the 4-ball-II" at "Seminar on Spectral Geometry" (Independent University of Moscow and Interdisciplinary Scientific Center J.-V. Poncelet, ISCP, UMI 2615)

Teaching

[1] Analytic geometry, People's Friendship University of Russia, I year students, February-April 2021, 2 hours per week.

Program.

I. Vector algebra. Vectors. Scalar, vector and mixed products of vectors. Length of a vector. Angle between two vectors. Equations of a line and a plane. Distance between a point and a line and a point and a plane. Euclidean space. Hypersurfaces.

II. Ellips, hyperbola and parabola. Metric classification of the curves of second order. Surfaces of second order and their classification.

[2] Linear Algebra. Higher School of Economics, I year students, January-May 2021, 4 hours per week.

Program

I. Introduction: sets, algebra of sets, maps between sets, injections, surjections, bijections, complex numbers, trigonimetric form of a complex number, De Moivre's formula, Fundamental Theorem of Algebra, quantifiers.

II. Systems of linear algebraic equations (SLAE): equivalent systems, consistent and inconsistent systems, geometric sense of a SLAE of two equations with two variables and of two equations with three variables, matrices and the matrix form of SLAE, vector-columns and vector-rows, vectors in $\mathbb{R}^n$, linear combinations of vectors, linear dependence and linear independence of vectors, rank and free variables, Kronecker-Capelli-Rouché Theorem, Gauss eliminations method, space of solutions of a SLAE.

III. Matrices and determinants: algebraic operation with matrices, matrix algebra, matrix equations, matrix exponential, invertible matrices, inverse matrix, Inverse Matrix Theorem, Gauss-Jordan algorithm of finding the inverse matrix, trace of a matrix, determinant of a matrix and its properties, methods and strategies of finding the determinant of a matrix, method of computation of the inverse matrix via the adjoint matrix, Cramer's rule, kernel and image of a matrix.

IV. Vector spaces and linear maps: vector space $\mathbb{R}^n$, abstract vector spaces, linear dependence and linear independence of vectors in abstract vector spaces, vector subspaces of a vector space, Theorem about a Vector Subspace of a Vector Space, vector spaces generated by vectors, basis and coordinates, dimension of a vector space, Rank Theorem, linear maps between vector spaces, kernel and image of a linear map, canonical matrix of a linear map, transformation matrix, isomorphisms of vector spaces, Theorem about isomorphic finite dimensional vector spaces.

V. Scalar products: scalar product of vectors in $\mathbb{R}^n$, abstract scalar products in vector spaces, length of a vector, distance between two vectors, Cauchy-Bunyakovski-Schwarz inequality, triangle inequality, orthogonal vectors, orthogonal and orthonormal basis, the orthogonal complement of vector subspace of a vector space with a scalar product, direct sum of vector spaces, orthogonal projection formula, Gram-Schmidt orthogonalization.

VI. Diagonalization and quadratic forms: eigenvectors and eigenvalues of a matrix, eigenspace, characteristic equation, Cayley-Hamilton theorem, similar matrices, diagonalizable matrices, Theorem about a Diagonalizable Matrix, Jordan's normal form of a matrix, symmetric matrices and their orthodiagonalizability, Spectral Theorem, bilinear and quadratic forms, definite and indefinite forms, canonical form of a quadratic form, Sylvester's criterion, Sylvester's law of inertia, Lagrange's method.

[3] Introduction to Modern Topology. Higher School of Economics, I-IV year students, January-May 2021, 4 hours per week.

Program

I. Intuitive topology.

II. Topology of subsets of $\mathbb{R}^n$.

III. Abstract topological spaces.

IV. Graphs.

V. Surfaces and their classification.

VI. Homotopy. Brower's fixed-point theorem.

VII. Vector fields.

VIII. Coverings and the fundamental group.

IX. The Kakutani fixed-point theorem.

[4] Non-Euclidean Geometry. Math in Moscow, I year students, February-May 2021, 2 hours per week.

Program

I. Toy geometries.

II. Abstract groups and group representations.

III. Finite subgroups of $SO(3)$ and the Platonic bodies.

IV. Discrete subgroups of the isometry group of the plane and tilings.

V. Reflection groups and Coxeter geometries.

VI. Spherical geometry.

VII. The Poicaré disk model of the hyperbolic geometry.

VIII. The Poicaré half plane model.

IX. The Cayley-Klein model.

X. Hyperbolic trigonimetry.

XI. Projective geometry.

XII. Finite geometries.

XIII. The hierarchy of geometries.

[5] Introduction to Geometric Analysis-II. Independent University of Moscow, III-V year students, February-May 2021, 2 hours per week.

Program

I. Basic facts of Riemannian geometry and partial differential equations: Riemann tensor, Ricci tensor, Weyl tensor, scalar curvature, mean curvature, injectivity radius, conformal metrics, change of curvature tensors under the conformal deformation of the metric, PDE on Riemannian manifolds, weak and strong solutions, Green's function, Sobolev spaces on Riemannian manifolds and their embeddings, the critical case in the Sobolev Embedding Theorem.

II. Positive Mass Theorem: Einstein's equations, variation of the scalar curvature, Hilbert-Einstein functional, Jacobi operator, nonpositive scalar curvature manifolds vs positive scalar curvature manifolds, minimal submanifolds and their stability, asymptotically flat manifolds, Arnowitt-Deser-Misner mass, the proof of the Positive Mass Theorem by Schoen-Yau.

III. Yamabe Problem: Yamabe equation and the critical case in the Sobolev Embedding Theorem, variational formulation of the Yamabe problem, Yamabe invariant, solution of the Yamabe problem in 2D and the Uniformization theorem, solution of the Yamabe problem in higher dimension by Aubin and Schoen.

[6] Calculus-II. Higher School of Economics, II year students, September-December 2021, 2 hours per week.

Program

I. Infinite series. Finite sums and products. Harmonic numbers. Convergent and divergent series. Examples of series: telescoping series, geometric series, decimal fractions, p-series, alternating series. Necessary condition for convergence. Integral test. Tail of a series. Series of Nonnegative Terms. Convergence Tests. Series of nonnegative terms. Comparison test. Limit comparison test. Ratio and Root tests, their relationship. Rate of convergence. Gauss test. Alternating series. Absolute and conditional convergence. Cauchy criterion. Alternating series test. Dirichlet/Abel tests. Absolute and conditional convergence. Sine and cosine sums. Conditionally convergent alternating and trigonometric series. Products of series. Infinite Products. Rearrangement of series, Cauchy's and Riemann's theorems. Product of series, Cauchy products. Convergence and divergence of infinite products, reduction to series. Wallis product. Stirling's formula.

II. Double integrals. Riemann sums. Double integrals over rectangles. Lower and upper Darboux sums. Darboux criterion. Properties of double integrals. Fubini's theorem, reduction to iterated integrals. Double integrals over general regions. Change of variables in double integrals. Polar coordinate system. Triple integrals. Applications of double and triple integrals. Fubini's theorem, reduction to iterated integrals. Change of variables in triple integrals. Cylindrical and spherical coordinate systems. Calculating areas of domains, volumes of solids, areas of surfaces. Improper integrals. Multiple integrals. Exhaustions. Improper double integrals.

III. Uniform convergence. Sums of functions. Pointwise and uniform convergence of functional sequences and series, their relationship. Cauchy criterion for the uniform convergence. Tests for the uniform convergence of series: alternating series test, Weierstrass M-test, Dirichlet/Abel tests. Interchange of limits, continuity of a limit function. Term-by-term integration and differentiation of uniformly convergent series. Riemann zeta function.Power series. Examples of power series. Radius and interval of convergence of power series. Cauchy-Hadamard formula. Uniform convergence of power series. Term-by-term differentiation and integration of power series. Abel's theorem. Products of power series. Uniqueness of power series expansion. Taylor series of common functions. Binomial series. Analytic functions. Complex power series. Euler's formula.Fourier series. Trigonometric series. Fourier coefficients and Fourier series. Parseval's identity. Piecewise functions. Riemann-Lebesgue lemma. Dirichlet kernel. Pointwise convergence of the Fourier series of a 2π-periodic piecewise continuously differentiable function. Applications of Fourier series.

[7] Advanced Riemannian Geometry. Higher School of Economics, III-V year students, September-December 2021, 2 hours per week.

Program

I. Theory of geodesics in Riemannian manifolds.

II. Complete Riemannian manifolds.

III. Riemannian manifolds of nonpositive curvature.

IV. Comparison theorems.

V. Quarter-pinched theorem.

[8] Theory of Minimal Submanifolds-I. Independent University of Moscow, III-V year students, September-December 2021, 2 hours per week.

Program

I. Basic facts of Riemannian geometry and partial differential equations: Riemann tensor, Ricci tensor, scalar curvature, mean curvature, Gauss equation, injectivity radius, conformal metrics, change of curvature tensors under the conformal deformation of the metric, PDE on Riemannian manifolds, weak and strong solutions, Green's function, Sobolev spaces on Riemannian manifolds and their embeddings, the critical case in the Sobolev Embedding Theorem, elliptic regularity, Harnack's inequality, spectrum of a Riemannian manifold.

II. Foundation of the theory of minimal submanifolds: volume functional vs energy functional, harmonic maps, minimal submanifolds of the Euclidean space and the standard spheres, minimal graphs, Gauss map, Bernstein theorem, Plateau problem.

III. Stability theory of minimal submanifolds: second variation of the volume functional, Jacobi operator, Morse index of a minimal submanifold, Fisher-Colbri theorem about the index of a minimal hypersurface, Schoen-Yau theorem about the stable minimal hypersurface, Fisher-Colbri-Schoen theorem, Barbosa-do Carmo theorem about stable domains, minimal submanifolds with ends, Cheng-Tysk theorem, stables cones and their applications.

IV. Minimal submanifolds of higher codimension: Kähler geometry, Virtinger's inequality, special lagrangian submanifolds, Bernstein theorem in higher codimension, harmonic maps in grassmanians, calibrations.