Research statement of Mikhail Bondarko for 2011

My research (starting from 2006) is dedicated to various motivic categories and related aspects of homological algebra.

I am currently finishing my paper [Bon11] on mixed motivic sheaves (this is the category of mixed motives over a base whose existence was conjectured by A. Beilinson) and weights for them. I prove: if certain 'standard' conjectures on motives over algebraically closed fields hold, then over any 'reasonable' base scheme S there exists a *motivic* t-structure for the triangulated category $DM^c(S)$ (of S-motives; it was introduced in [CiD09]). If S is an equicharacteristic scheme, then the heart of this t-structure (the category of *mixed motivic sheaves* over S) is endowed with a *weight filtration* with semi-simple factors. I also prove a certain 'motivic decomposition theorem' (assuming the conjectures mentioned) and characterize semi-simple motivic sheaves over S in terms of those over its residue fields. Note here: the standard conjectures mentioned are certainly very hard; yet it is nice to know that passing to relative motives in this matter conceals no additional difficulties. This group of results is much stronger than all of the previous achievments in this direction; the theory of weight structures (and the corresponding spectral sequences, as introduced in [Bon10a]) was a very important tool in their proof.

An important part of my reasoning is the degeneration of *Chow-weight spectral sequences* for 'perverse étale homology'. I prove this fact unconditionally; this statement also yields the existence of the *Chow-weight filtration* for such (co)homology that is strictly restricted by ('motivic') morphisms (by the results of [Bon10c]). Using this, I prove that w_{Chow} (the Chow weight structure for S-motives intoduced in [Bon10b]) is *transversal* to the motivic t-structure (modulo the standard conjectures mentioned), and prove several new properties of mixed motivic sheaves.

Next I would like to study some cases where my results on mixed motivic sheaves could be applied without relying on any conjectures. Moreover, the results mentioned mostly use certain 'axiomatics' of relative motives; so it seems possible to apply them to other 'motive-like' categories. In particular, I plan to study relative 1-motives. Even more 'simple' motivic category is the category DAT of Artin-Tate motives (defined by J. Wildeshaus). I plan to study (relative) Artin-Tate motives with various coefficients rings, and introduce a series of weight structures for them.

I am going to extend the results of [Bon10b] and find more applications for them. The (existence of the) Chow weight structure (and the corresponding filtrations and spectral sequences) for relative motives should yield new information on intersection cohomology, and its functoriality. In [Bon10b] it was proved that $K_0(DM^c(S)) \cong K_0(Chow(S))$; a certain 'motivic Euler characteristic' (with multiplicative structure) for S-schemes was defined. This could possibly improve our understanding of motivic integration. Note here: motivic integration usually studies the so-called jets (i.e. for a variety V/k one considers the projective system $V(k[t]/(t^n))$ for $n > 0$); so one naturally obtains certain comotives (as introduced in [Bon10d]). This refines the 'usual' motivic integration invariant (as introduced by Kontsevitch).

I would like to introduce a certain Gersten weight structure for relative (co)motives (joining the results of [Bon10d] with those of [Bon10b]). This would yield motivic functoriality of the coniveau filtration and spectral sequences for cohomology theories that factorize through S-motives. Also it seems possible to obtain some results related with the (very hard) Gersten conjecture.

I also plan to improve my results on weight structures for localizations of triangulated categories. An easy consequence, I will deduce the existence of the *Chow weight structure* for the category $DM^{bir}(S)$ of *birational motives* (over any base scheme S), and calculate the heart of this weight structure. As noted in [Bon10d], in the case when the base is a perfect field k, this weight structure coincides (up to the corresponding shifts) with the ones 'induced' by the Chow and the Gersten weight structures (for Voevodsky's category DM_{gm}^{eff}) on 'slices' i.e. on the localizations $DM_{gm}^{eff}(i)/DM_{gm}^{eff}(i+1) \cong DM^{bir}(k)$ (for any $i \geq 0$). These observations should yield a way to relate coniveau and Chow-weight filtrations for cohomology; this could be important for several long-standing 'motivic' problems (the Hodge and Tate conjectures). Besides, I hope to relate this matter with the famous Beilinson-Soule vanishing conjecture.

Also, I am going to define *higher truncation* functors t_N (for $N \geq 0$) for triangulated categories endowed with weight structures; those should be exact and conservative (in the 'bounded' case). Note that in [Bon10a] only a rough version of t_0 (the *weight complex functor*) was constructed for the general case; the target of this functor is a certain factor of $K(\underline{Hw})$ (the homotopy category of complexes over the heart of the weight structure w) that is not triangulated. On the other hand, for triangulated categories that possess a differential graded enhancement all the t_N were constructed in [Bon09a]. The main problem with this construction is that it heavily depends on the choice of an enhancement; hence this method cannot be applied to 'non-algebraic' tringulated categories, whereas for algebraic ones it hardly could be applied to the study of the functoriality of t_N. Now, I found a way to define t_N in a way that does not depend on any choices (to this end their targets are described as categories of certain sequences of 'weight pieces' of objects of the original category). The functoriality of this approach allows to prove (in particular) that the singular realization of Voevodsky's motives DM_{gm} factorizes through t_1 (hence, the singular realization only 'detects' the information that is contained in the target of t_1; the latter is much simpler then DM_{gm} since t_1 'kills higher motivic cohomology groups' in a certain sense).

Lastly, I have an idea whose successful implementation could be very useful for the theory of motives. One of the main ingredients of the existing attempts to construct the theory of mixed motives (as conjectured by Beilinson) is the so-called Beilinson-Murre filtration for the Chow groups (of smooth projective varieties); this filtration should come from the motivic t-structure truncations of the motives of these varieties. Now, the i-th Chow group of a (smooth projective) P over a perfect field is given by $Chow^i(P) = \mathrm{Hom}_{DM_{gm}}(\mathcal{M}(P), \mathbb{Z}(i)[2i])$ (for $\mathbb{Z}(i)[2i]$ being the corresponding Tate motives). Hence the yoga of adjacent structures yields that it is (probably) possible to define the Beilinson-Murre filtration for $Chow^i(P)$ using *weight truncations* of $\mathbb{Z}(i)[2i]$ with respect to a certain weight structure (that is *right adjacent* to the mixed motivic t-structure; possibly one should construct some new 'motivic' category in order to define this weight structure, or maybe Poincare duality would help here). So, it suffices to calculate a single series of 'truncations' for $\mathbb{Z}(i)[2i]$ in order to calculate the filtrations in question for all $Chow^i(P)$ simultaneously!

References

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