KOSZUL DUALITY, ARTIN–SCHELTER GORENSTEIN PROPERTY, AND HOCHSCHILD (CO)HOMOLOGY OF THE SECOND KIND

RESEACH STATEMENT

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1. Artin–Schelter Gorenstein Coalgebras

Artin–Schelter regular algebras are a class of positively graded noncommutative, associative algebras over a field whose properties resemble those of the algebras of commutative polynomials in several homogeneous variables. Introduced in 1987 [1], they remain the popular subject of contemporary research [9, 5]. This project deals with the Artin–Schelter Gorenstein property, which is the homological part of the Artin–Schelter regularity condition.

A positively graded, locally finite-dimensional associative algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ over a field k is said to be *AS Gorenstein* if it has finite homological dimension and the bigraded vector space $\mathrm{Ext}_A(k, A)$ is one-dimensional. The thematic examples of AS Gorenstein/regular algebras are the enveloping algebras of positively graded, finite-dimensional Lie algebras.

This project aims to show that the natural generality for the AS Gorenstein property is that of ungraded (conilpotent or nonconilpotent) coalgebras. A conilpotent coalgebra C is said to be *AS Gorenstein* if it has finite homological dimension and the graded vector space $\mathrm{Ext}_C(C, k)$ is one-dimensional. The thematic examples of AS Gorenstein conilpotent coalgebras are the conilpotent coenveloping coalgebras of the Lie coalgebras dual to finite-dimensional, ungraded nilpotent Lie algebras.

The following results are expected to be obtained. More generally, a conilpotent curved DG-coalgebra C is said to be AS Gorenstein if the derived comodule-contramodule correspondence functors [13] transform the trivial C-comodule k to a homological shift of the trivial C-contramodule k [12, Appendix A] and vice versa. A conilpotent (curved DG-) coalgebra C is AS Gorenstein if and only if the graded algebra $\mathrm{Ext}_C(k, k)$ is (graded) Frobenius.

A conilpotent AS Gorenstein coalgebra C has a canonical outer authomorphism; in the case of a positively graded AS Gorenstein algebra, it becomes just a canonical automorphism (not outer). The induced autoequivalence of the derived category of finite-dimensional C-comodules coincides, up to a homological shift, with the Serre functor of this triangulated category. A generalization of these definitions and results to nonconilpotent coalgebras is also expected.

2. Hochschild (Co)homology of the Second Kind

The basics of the theory of Hochschild (co)homology of the second kind (othewise known as compact type Hochschild (co)homology [3]) for curved DG-algebras and categories were worked out in my paper with A. Polishchuk [10]. Several equivalent definitions were given, and the comparison with the conventional Hochschild (co)homology was established in many cases.

In this project I intend to define for the Hochschild (co)homology of the second kind of curved DG-algebras the rich structure that is known to exist on the Hochschild (co)homology of the first kind of DG-algebras. This includes the action of the operad of braces B_∞ [6, 7, 8] on the cohomological Hochschild complex, and the cyclic homology package as the natural complement to the definition of the Hochschild homology. The comparison (iso)morphisms constructed in [10] are supposed to preserve these structures.

Furthermore, one can define the Hochschild homology of a curved DG-coalgebra C over a field k as $HH_*(C) = \mathrm{Cotor}^{C\otimes_k C^{\mathrm{op}}}(C,C)$ and its Hochschild cohomology as $HH^*(C) = \mathrm{Ext}_{C\otimes_k C^{\mathrm{op}}}(C,C)$. I intend to define for these (co)homology the structures similar to those existing in the algebra case, and construct an isomorphism between the Hochschild (co)homology of Koszul dual CDG-algebra A and CDG-coalgebra C preserving all the structures (generalizing the results of [4, 8]).

The next problem is to define the Hochschild (co)homology of the second kind for exact CDG-categories [13, Remark 3.5]. One expects such (co)homology of the exact DG-category of finite-dimensional CDG-comodules over a CDG-coalgebra C to be isomorphic to the Hochschild (co)homology of C. Such a definition would be also helpful for the computation of the Hochschild (co)homology of the absolute derived category of coherent matrix factorizations on a singular scheme [14].

3. Absolute Quadratic Duality for Semialgebras

A *semialgebra* is an associative algebra object in the tensor category of bicomodules over a coalgebra. This is the key notion in the semi-infinite homological algebra of associative algebraic structures [12]. Unlike the relative nonhomogeneous quadratic duality developed in [12, Chapter 11], the absolute quadratic duality for semialgebras presumes performing the quadratic duality transformation along both the algebra and the coalgebra variables. Applying the absolute duality to a semialgebra, one obtains a semialgebra with the coalgebra variables corresponding to the algebra variables of the original semialgebra and vice versa.

One can easily construct the absolute duality for bigraded semialgebras that are entwined tensor products [2] of a Koszul coalgebra and a Koszul algebra. Generalizing this example, one defines an *absolutely Koszul semialgebra* as a semialgebra endowed with a grading and a filtration such that the associated bigraded semialgebra is of the above type. Using the relative quadratic duality construction twice, one can assign to an absolutely Koszul semialgebra its absolutely quadratic dual one.

Nonhomogeneous absolutely Koszul semialgebras can be defined as semialgebras with two filtrations; the absolute quadratic dual objects to these might be called "absolutely Koszul CDG-semialgebras" (cf. [11]). Defining the notion of a CDG-semialgebra is an open problem.

The goal of this line of research is to interpret the graded vector space of semi-infinite exterior forms over a Tate Lie algebra $\mathfrak{g}$ with a compact Lie subalgebra $\mathfrak{h}$ as a CDG-semialgebra over the exterior coalgebra of the vector space $\mathfrak{g}/\mathfrak{h}$. One would then assing left and right CDG-semimodules over this semialgebra to discrete $\mathfrak{g}$-modules with appropriate central charges, and the semitensor product of such CDG-semimodules would provide the complex of semi-infinite forms with coefficients in a $\mathfrak{g}$-module on the critical level. This is expected to lead to a new proof of the comparison theorem for the semi-infinite homology of Lie algebras and associative semialgebras [12, Appendix D].

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