Research Statement of A. Skopenkov

Knotting of manifolds below the metastable dimension

Introduction.

According to C. Zeeman [Z], the classical problems of topology are the following.

• Homeomorphism Problem: Classify n-manifolds.

• Embedding Problem: Find the least dimension m such that given space embeds into m-dimensional Euclidean space $\mathbb{R}^m$.

• Knotting Problem: Classify embeddings of a given space into $\mathbb{R}^m$ up to isotopy.

These problems have played an outstanding role in the development of topology. Various methods for the investigation of the *Knotting Problem* (in higher dimensions) were created by such classical figures as G. Alexander, L.S. Pontryagin, R. Thom, H. Whitney, V.A. Rokhlin, W.S. Massey, R. Penrose, J.H.C. Whitehead, C. Zeeman, W. Browder, J. Levine, S.P. Novikov, A. Haefliger, M. Hirsch, J.F.P. Hudson, M. Irwin and others. For recent surveys see [RS99, S08, HCEC]; whenever possible we refer to these surveys not to original papers.

The Knotting Problem is known to be difficult. There are only a few cases in which there are complete readily calculable classifications. (However, in these cases the statements, as opposed to the proofs, are simple and accessible to non-specialists.) For the best known specific case of codimension 2 embeddings a complete classification is neither known nor expected.

The Knotting Problem is most interesting for manifolds of dimension at most 4, because embeddings of such manifolds often appear in other branches of mathematics. For example, certain constant energy manifolds of Hamiltonian systems with two degrees of freedom are 3-dimensional manifolds, algebraically embedded into $\mathbb{R}^6$; configuration spaces of generic pentagonal planar linkages are 2-dimensional manifolds algebraically embedded in $\mathbb{R}^6$.

I work in the smooth category unless PL (piecewise linear) category us explicitly mentioned.

Classical results of Wu, Haefliger, Hirsch (1960-s) on embeddings of n-dimensional manifolds into $\mathbb{R}^m$ have the *metastable dimension restriction*

$$2m > 3n + 3.$$

In particular, in low dimensions Haefliger and Hirsch classified embeddings of 3-dimensional manifolds into $\mathbb{R}^m$ for $m \geq 7$, and of 4-dimensional manifolds into $\mathbb{R}^m$ for $m \geq 8$.

The main intention of this research proposal is to classify embeddings for

$$2m \leq 3n + 3$$

and for *closed connected* manifolds. For N not a homology sphere until 2005 no classification was known, in spite of the existence of interesting partial results, results in the PL category and approaches of Browder-Wall [B68, W99, §11, CRS04] and Goodwillie-Weiss [GW99].

Let $E^m(N)$ ($E^m_{PL}(N)$) be the set of smooth (PL) embeddings $N \to \mathbb{R}^m$ up to smooth (PL) isotopy.

Embeddings of 4-manifolds.

For $m > n + 2$ there is the 'connected sum' group structure on $E^m(S^n)$ and the *'connected sum' action* $\#$ of $E^m(S^n)$ on the set $E^m(S^n)$ for a closed connected orientable n-manifold N. Embeddings $S^n \to S^m$ for $m > n + 2$ were classified by A. Haefliger in 1960s. In particular, he proved that $E^7(S^4) \cong \mathbb{Z}_{12}$.

The quotient set of the action $\#$ was known for some cases including the case $m = 6 = 2n$ and the case N simply-connected, $m = 7 = 2n - 1$ [H69, BH70, V77, F94], cf. [Y84, S10', T]. There remained to find the orbits of $\#$. For N not a homology sphere until 2005 no description of the orbits was known.

A description of orbits for these cases appeared in [S08', S10, CS11]. This yielded a classification of embeddings for 3-manifolds in $\mathbb{R}^6$ and for simply-connected 4-manifolds in $\mathbb{R}^7$. Cf. [M].

D. Crowley and I plan to obtain a classification of embeddings of non-simply-connected 4-manifolds in $\mathbb{R}^7$, both in the smooth and in the PL category. This is a significant new step because as opposed to the situation in [S08', S10, CS11], all known invariants defined on the quotient set of # (or on the set of PL embeddings) are incomplete. *My contribution to our joint plan* would be a description of the quotient set of # and the 'lower estimation' in the description of the orbits of #.

The main methods to be used are

• Kreck's modification of surgery [K99] (an exposition of how to apply this to embeddings could be found in [CS11, §2]), and

• parametric connected sum of embeddings $S^1 \times S^3 \to \mathbb{R}^7$ and $N \to \mathbb{R}^7$ [S07, S10, PCS].

The main problems are that

• surgery obstructions are hard to express in calculable terms and

• the parametric connected sum is not well-defined for 4-manifolds in $\mathbb{R}^7$.

Let me present a precise statement for $N = S^1 \times S^3$.

Conjecture 1. *There is a commutative diagram of surjections*

$$\begin{array}{ccc} E^7(S^4) \oplus (\mathbb{Z} \oplus \mathbb{Z}) & \xrightarrow{\#\oplus\tau} & E^7(S^1 \times S^3) \\ \downarrow{\scriptstyle 0\oplus(\mathrm{id}\oplus\rho_6)} & & \downarrow{\scriptstyle \mathrm{forg}} \\ \mathbb{Z} \oplus \mathbb{Z}_6 & \xrightarrow{\tau_{PL}} & E^7_{PL}(S^1 \times S^3) \end{array} \quad \textit{such that} \quad \# \textit{ is free},$$

$$\tau(b,c) = \tau(b',c') \quad \textit{if and only if} \quad b = b' \quad \textit{and} \quad c \equiv c' \mod 2b$$

$$\textit{and} \quad \tau_{PL}(b,c) = \tau_{PL}(b',c') \quad \textit{if and only if} \quad b = b' \quad \textit{and} \quad c \equiv c' \mod 2GCD(b,3).$$

Here forg *is the forgetful map and τ, τ_{PL} are explicitly defined in the paper.*

Concerning partial results toward Conjecture 1 see my report on achievements in 2009-2010.

Knotted tori.

Many interesting examples of embeddings are embeddings $S^p \times S^{n-p} \to \mathbb{R}^m$, i.e. knotted tori [KT]. A classification of knotted tori is a natural next step (after the link theory and the classification of embeddings of highly-connected manifolds) towards classification of embeddings of arbitrary manifolds. Since the general Knotting Problem is very hard, it is very interesting to solve it for the important particular case of knotted tori. Recent classification results for knotted tori [S06, CRS07, CRS08] give some insight or even precise information concerning arbitrary manifolds (cf. [S07, S10, PCS]) and reveal new interesting relations to algebraic topology.

Assume that $1 \le p \le n/2$. A classification for $m \ge 2n-p+1$ and $2m \ge 3n+4$ was obtained by Haefliger-Hirsch in 1963, and for $m \ge n+p+3$ and $2m \ge 3n+4-p$ in [S97, S02]. These are results in the 'metastable' dimension range where the Haefliger-Wu invariant is bijective. In [S06] the simplest case $2m = 3n+3-p$ outside the 'metastable' dimension range was solved. However, it was not clear what happens when $2m < 3n+3-p$ and especially for $2m < 3n+4-2p$. *I plan to obtain a classification of embeddings $S^p \times S^q \to S^m$ for $m \ge n+p+3$ and $2m < 3n+3-p$.* More precisely, *I plan to prove that Theorem 2 below holds for $2m \ge 3n+4-2p$ and to obtain its generalization to $2m < 3n+4-2p$.*

Denote by $V_{k,l}$ the Stiefel manifold of l-frames in $\mathbb{R}^k$. Consider the 'connected sum' group structure on $E^m(S^q)$ and an 'S^p-parametric connected sum' group structure on $E^m(S^p \times S^q)$. They are well-defined for $m \ge 2p+q+3$ [H66, S06].

Theorem 2. [S97, S02, S06] *For $1 \le p \le n/2$, $m \ge n+p+3$ and $2m \ge 3n+3-p$*

$$E^m(S^p \times S^{n-p}) \cong \pi_{n-p}(V_{m-n+p,p+1}) \oplus E^m(S^n).$$

Cf. Conjecture 1 above. A description of $E^m(S^p \times S^{n-p}) \otimes \mathbb{Q}$ was obtained for $m \ge n+p+3$ in [CRS07, CRS08, S]. However, looking at the exact sequences there it is non-trivial even to guess a simple description as above. Thus one would need new geometric arguments.

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