Astroidal geometry and hyperbolic polynomials

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An astroid is a hypocycloid with four cusps. It is a remarkable curve possessing many geometric properties. Here is an example of such a property. Consider the family of all straight lines tangent to the astroid (these lines will be called *rays*, and the astroid itself will be called the *caustic* for the system of rays). Trajectories orthogonal to the rays are called *fronts*. It happens that all fronts of an astroid are algebraic curves, and one of the fronts is an astroid itself, which is two times less than the original astroid; this fact is not obvious.

One can define caustics and fronts for any periodic function. It happens that the caustic of arbitrary periodic function has at least four cusps, and if there are precisely four cusps (as in the case of an astroid), then they necessarily form a parallelogram. These results generalize theorems of Sturm and Hurwitz concerning zeroes of Fourier series.

The geometry of astroids can be applied to the study of topological properties of hyperbolic polynomials. A homogeneous polynomial $f(x, y)$ is called *hyperbolic* if its second differential is a quadratic form of hyperbolic signature $(+, -)$ at each point (x, y) of the plane except of the origin. The harmonic polynomial $r^4 \cos(4\phi)$, where $x = r\cos\phi$, $y = r\sin\phi$, is an example of a hyperbolic polynomial. Astroidal geometry implies

Theorem 1. *The space of hyperbolic homogeneous polynomials of degree* 4 *is connected.*

Corollary. *Topological invariants of the field of crosses determined by the zeroes of the second differentials of a homogeneous hyperbolic polynomial of degree* 4 *do not depend of the coefficients of the polynomial.*

Theorem 2. *Hyperbolic homogeneous polynomials of degree* 6 *determine at least two topologically distinct fields of crosses of zeroes of the second differentials.*

Corollary. *The number of connected components of the space of homogeneous hyperbolic polynomials of degree* 6 *is at least two.*

I think that the number of connected components grows as the degree of the polynomials grows (conjecturally, as a linear function in the degree, but this conjecture is not proved yet).

Astroidal geometry also implies the existence of "almost hyperbolic" polynomials (whose second differential has no points of *elliptic* signature $(+, +)$ or $(-, -)$) that do not belong to the boundary of the space of hyperbolic polynomials (that is, they do not become hyperbolic after a small modification of coefficients). An example is given by the polynomial $x^4 - y^4$. It cannot be hyperbolized because the index of its field of crosses is wrong (it does not coincide with that for hyperbolic polynomials of degree 4).

The theory of hyperbolicity and astroidal geometry are closely related to inequalities between derivatives of smooth functions on the circle and, further, to the optimal control theory. On this way we find qualitative and quantitative difference between the cases of smooth functions (analysis) and trigonometric polynomials of given degree (algebraic geometry): they lead to different inequalities for topological invariants and for the numbers of connected components.

Astroidal geometry corresponds to the 3-parameter family of trigonometric polynomials of degree 4 on the circle,

$$F(\phi) = C + A\cos(2\phi) + B\sin(2\phi) + \cos(4\phi).$$

The *caustic* of a periodic function is the set of those of its linear combinations with the sine and the cosine functions that are not Morse functions. Astroid is the caustic of the function $\cos(2\phi)$.

The hyperbolic theory seems to be an instructive example (and, probably, one of standard normal forms) of a much more general symplectic and contact theory of singularities of Hessians, and of even more general qualitative theory of differential equations of degree greater than one (two, in the case of Hessians).

The talk will discuss both the results already obtained and semiexperimental conjectures, as well as further theoretical perspectives.