SYMMETRIC POWERS AS ALGEBRAIC VARIETY

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The talk is devoted to my joint recent results with Elmer G. Rees (Edinburgh University).

Given a commutative algebra A over the complex numbers $\mathbb{C}$, we define certain algebraic subsets Φ_n of the space of all linear maps $Hom(A, \mathbb{C})$. The space Φ_1 is the set of ring homomorphisms. The elements of Φ_n are characterised by algebraic equations similar to those introduced by Frobenius in his definition of n-characters of finite groups. We prove that in the case when $A = \mathbb{C}(X)$, the ring of complex valued continuous functions on a compact Hausdorff space X, the variety Φ_n is canonically homeomorphic to the symmetric power $Sym^n(X) = X \times \cdots \times X/S_n$. The case $n = 1$ is the Gelfand correspondence. Another important application of this theory is the case $X = \mathbb{C}^m$ and $A = \mathbb{C}[u_1, \ldots, u_m]$, the ring of polynomial functions on X. We prove that Φ_n is the symmetric power $Sym^n(\mathbb{C}^m)$. The case $n = 1$ here is classical as well.

The description of Φ_n immediately produces an embedding into the finite dimensional subspace of linear functionals vanishing on monomials of degree $> n$. The algebraic equations defining Φ_n in this space are then exactly equations are give the relations satisfied in the ring of multisymmetric polynomials. The corresponding syzygies have been studied for more than a century and only particular examples were obtained through difficult calculations. The problem of describing the symmetric powers of $\mathbb{C}^m$ (and also algebraic subvarieties in $\mathbb{C}^m$) as algebraic varieties arises in different aspects of classical invariant theory, algebraic geometry and the theory of Abelian functions. Besides discussion these questions, we will also consider modern applications in the theory of integrable systems.