Decomposable skew-symmetric functions

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A (real) function of n (real) variables is said to be *skew-symmetric*, if it changes sign whenever any two variables are interchanged:

$$f(x_1, ..., x_i, ..., x_j, ..., x_n) = -f(x_1, ..., x_j, ..., x_i, ...x_n).$$

A skew-symmetric function $f(x_1, ..., x_n)$ is *decomposable*, if there exist functions of one variable f_1, ..., f_n such that

$$f(x_1, ..., x_n) = \det \|f_i(x_j)\|_{i,j=1}^n.$$

Problem 1. Find a criterion that a given skew-symmetric function $f(x_1, ..., x_n)$ be decomposable.

Theorem 1. In the class of analytic functions (or in any ring of functions without zero divisors) a skew-symmetric function $f(x_1, ..., x_n)$ is decomposable if and only if it satisfies the identity

$$f(x_1, x_2, ...)f(x_3, x_4, ...) - f(x_1, x_3, ...)f(x_2, x_4, ...) + f(x_1, x_4, ...)f(x_2, x_3, ...) = 0, \qquad (1)$$

where the dots mean one and the same set of $(n-2)$ variables.

Besides the above notion of (completely) decomposable, one can introduce the notion of partially decomposable skew-symmetric functions. If $\lambda = (\lambda_1, ..., \lambda_k)$ is a partition of n, then by a λ-decomposable skew-symmetric function of n variables we understand the complete antisymmetrization of the product of k arbitrary functions of λ_1, ..., λ_k variables. The partition $(1, 1, ..., 1)$ gives completely decomposable functions, while the partition (n) yields the class of all skew-symmetric functions in n variables.

Problem 2. For a given λ, find a criterion of λ-decomposability.

For more general classes of functions, the assertion of the above theorem is true only one-way. However, equation 1 is meaningful by itself (it comes from one construction of weight systems in the theory of finite type knot invariants), and the problem, inverse to Problem 1 from the point of view of Theorem 1, presents certain interest.

Problem 3. Find a complete description of the solutions to equation 1 for non-analytic functions f. For example, it is interesting to study functions with finite support, or whose support is an algebraic variety in $\mathbb{R}^n$ and the restriction of the function to its support is algebraic.