Eigen-value Functional Optimization in Mathematical Physics and Stability Problems

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The stability of a stationary solution of different dynamic models of perfect media (e.g., of atmosphere, ocean, plasma) may be investigated by the Lyapunov method only, since the non-positive real parts of the eigen-values of the linearized operator are not guarantee the true non-linear stability. The variational approach as well as the numerical methods help us to solve some problems of the mathematical physics.

For instance, the first integral

$$\mathbf{I}_f = \int f(\Delta\Psi - k^2\Psi)d^2x, \quad \mathbf{E} = \frac{1}{2}\int (\nabla\Psi)^2 + k^2\Psi^2 d^2x$$

can be used as a Lyapunov functionals for the quasigeostrophic barotropic model of atmosphere (Ψ is a stream function, f is an arbitrary function); the first integrals

$$\mathbf{I}_f = \int \Phi f\left(\frac{\Omega}{\Phi}\right) d^2x, \; \mathbf{E} = \frac{1}{2}\int \Phi\left(\Phi + u^2 + v^2\right) d^2x,$$

$$\mathbf{M}_z = \int \Phi\left[uy - vx + \frac{l}{2}\left(x^2 + y^2\right)\right] d^2x$$

can be used for the shallow-water model, where Φ is the water surface's height, $< u, \, v >$ is velocity, $\Omega = \partial_y u - \partial_x v + l$, l is the Coriolis parameter.

As about perfect MHD model, the special gauge of the vector-potential $\vec{A}$ should be choosen:

$$\partial_t \vec{A} = \left[\vec{v}, \, \vec{B}\right] - \nabla\left(\vec{A}, \, \vec{v}\right)$$

because, in the case we obtain a family of first integrals (like helicity):

$$\int \rho f\left(h/\rho\right) d^3x, \text{ where } h = \left(\vec{A}, \, \vec{B}\right),$$

ρ is density.

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The functionals $\mathbf{K}_F + \mathbf{E}$ (where $\mathbf{E}$ is energy) can be used as Lyapunov functionals to obtain stable solutions for perfect plasma.

On the other side the methods may be useful for estimations of instability.

A particle stability is determined by the minimal eigen-value of the linear Schrödinger equation with a potential, i.e. the degree of the stability depends on the potential. The optimization problem with respect to the potential $U(|\vec{x}|)$ in $\mathbf{R}^n$ under the restriction $\int U^k \, d\vec{x} = const$ is reduced to the non-linear Schrödinger equation. It was solved analytically at $n = 1$ and numerically at $n = 2, 3$. The similar problem of the minimal eigen-value optimization with respect to magnetic field was considered at $n = 2$.

The plane-parallel current in a channel may be unstable if there are inflection points of the basic velocity profile $V(z)$. The maximal increment of small perturbations is determined by the profile. The maximal unstable profile under the restriction $\int_0^L V'^2 \, dz = const$ is computed. Thus the most exact evaluation of instability is obtained.

The similar approaches are useful also in models of atmosphere, ocean, plasma etc.

References

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