The Hurwitz problem and the geometry of spaces of meromorphic functions

S. K. Lando

A ramified covering $f : C \to S^2$ of the 2-sphere by a genus g two-dimensional surface C determines a sequence of permutations $\sigma_1, \dots, \sigma_c$ acting on a nondegenerate fiber of f. These permutations correspond to the monodromy of the covering along simple loops around the ramification points $t_1, \dots, t_c \in S^2$. The cyclic structures of these permutations form the *passport* of the covering. The number of ramified coverings with given passport is finite. The *Hurwitz problem* consists in enumeration of ramified coverings with given passport. Hundred years ago Hurwitz gave an elegant answer to his problem for a special case, where the covering surface is also the sphere and all ramification points but one are simple. Recently, the Hurwitz problem attracted a considerable interest due to important applications in physics.

The efforts of a number of research groups all over the world led to a remarkable progress in solving the Hurwitz problem during the last decade. Main achievements result from understanding the connection between the enumeration and the geometry of spaces of meromorphic functions on complex curves. These spaces have, in fact, a lot of in common with moduli spaces of complex curves with marked points. The geometry of the last spaces was, and still is, an object of a thorough attack of many researchers, because of its crucial role in modern physics and topology. Among the most prominent results in this direction is Kontsevich's proof of the Witten conjecture concerning the intersection theory on the moduli spaces.

It will be shown in the talk how the Witten conjecture is related to the Hurwitz problem. This connection led already to a number of important explicit calculations. The very recent result of Okounkov and Pandharipande consists in a new proof of the Witten conjecture based on this relationship and known properties of the ramified coverings enumeration.