Capacities in spectral theory of magnetic Schrödinger operators

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This is a joint work with Vladimir Kondratiev (Moscow State University) and Vladimir Maz'ya (University of Linköping, Sweden).

It is well known that for the Schrödinger operator $-\Delta + V(x)$ in $L^2(\mathbb{R}^n)$ with a real-valued potential V the spectrum is discrete if we assume that $V(x) \to +\infty$ as $|x| \to \infty$. This was established by H.Weyl (1910) in case $n = 1$, and by K.Friedrichs (1934) for arbitrary n. A.Molchanov (1953) found a necessary and sufficient condition which should be imposed on V for the spectrum to be discrete in case when it is a priori assumed that $V \geq 0$ (or, more generally, if V is semi-bounded below). This condition is formulated by use of Wiener capacity. V.Mazya (1985) found a similar necessary and sufficient condition for the strict positivity of the Schrödinger operator with $V \geq 0$.

We consider the magnetic Schrödinger operator in $L^2(\mathbb{R}^n)$ which has the form

$$H_{A,V} = -\sum_{j=1}^{n} \left(\frac{\partial}{\partial x^j} + iA_j \right)^2 + V,$$

where A_j, V are real-valued functions, $A_j \in C^1(\mathbb{R}^n)$, $V \in L^2_{loc}(\mathbb{R}^n)$, $V \geq 0$. It becomes the usual Schrödinger operator (as above) if the vector potential A vanishes identically. We establish necessary and sufficient conditions for the spectrum of this operator to be discrete or to be separated from 0. Both conditions are formulated in terms of Wiener capacity as well as energy μ of the magnetic field in cubes Q_d of edge size $d > 0$ which form a tiling of $\mathbb{R}^n$. This energy is defined by the formula

$$\mu = \mu(Q_d) = \inf \left\{ \int_{Q_d} \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x^j} + iA_j u \right|^2 dx \;\middle|\; u \in C^\infty(Q_d),\ \int_{Q_d} |u|^2 dx = 1 \right\}.$$

Let us define the Molchanov functional

$$M_c(Q_d; V) = \inf_F \left\{ \int_{Q_d \setminus F} V(x)dx \;\middle|\; \mathrm{cap}\,(F) \leq c\ \mathrm{cap}\,(Q_d) \right\},$$

where cap means the Wiener capacity, F is a compact subset in Q_d, $c > 0$.

For simplicity of formulations we will only consider the case $n \geq 3$.

Theorem 1. *There exists a constant $c = c_n > 0$ (depending only on n) such that the following condition is necessary and sufficient for the discreteness of the spectrum of the operator $H_{A,V}$: for any $d > 0$*

$$\mu(Q_d) + d^{-n} M_{cf(\mu)}(Q_d; V) \to +\infty \quad as \quad Q_d \to \infty,$$

where

$$f(\mu) = (\mu d^2 + 1)^{2-n}, \quad \mu = \mu(Q_d).$$

For $A \equiv 0$ this becomes A. Molchanov's theorem (1953).

Theorem 2. *Let us assume that $V \geq 0$. Then the operator $H_{a,V}$ is strictly positive if and only if there exist $c > 0$, $d_1 > 0$ and $d > 0$ that for any cube $Q_d \subset \mathbb{R}^n$*

$$\mu(Q_d) + d^{-n} M_c(Q_d; V) \geq \frac{1}{d_1^2}.$$

For $A \equiv 0$ this result is due to V. Maz'ya (1985).

Simpler and more explicit necessary conditions and (separately) sufficient conditions for the discreteness of spectrum and strict positivity can be deduced from Theorems 1 and 2.

Theorem 1 holds also on Riemannian manifolds of bounded geometry where the magnetic Schrödinger operator should be defined as $H_{A,V} = d_A^* d_A + V$, where $d_A = d + iA$ acts from scalar functions to 1-forms, and the vector potential A is a 1-form on the manifold.