The Whitehead link, the Hudson-Habegger invariant and classification of embeddings $S^1 \times S^3 \to \mathbb{R}^7$

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We shall give a short survey on classification of links, knotted tori and embeddings of 4-manifolds into R^m for $m \geq 7$. Since 1970 until now a complete classification of embeddings of closed connected 4-manifolds into R^7 was known only for simply-connected 4-manifolds. A particular case of our main result is that

$$\mathrm{Emb}^7_{PL}(S^1 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$$

and there is an exact sequence

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{12} \to \mathrm{Emb}^7_{DIFF}(S^1 \times S^3) \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0.$$

We denote CAT = DIFF or PL (we omit CAT if a statement holds in both categories). Let $\mathrm{Emb}^m_{CAT}(N)$ be the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT isotopy.

We show how our results appear in the general embedding theory. Suppose that N is a closed connected CAT n-manifold (in this and the next paragraph). Then $|\mathrm{Emb}^m(N)| = 1$ for $m \geq 2n+1$. In the theory of classification of embeddings into Euclidean spaces for $m \leq 2n$, as well as in other branches of mathematics, the approach using configuration spaces, or 'complements to diagonals' proved to be very fruitful [Vas92]. This approach gives the *Haefliger-Wu invariant* defined as follows. Let $\widetilde{N} = \{(x,y) \in N \times N \mid x \neq y\}$ be the *deleted product* of N, i.e. the configuration space of pairs of distinct points of N. Let $\mathbb{Z}_2$ act on $\widetilde{N}$ and on S^{m-1} by exchanging factors and antipodes, respectively. For an embedding $f : N \to \mathbb{R}^m$ define a map

$$\widetilde{f} : \widetilde{N} \to S^{m-1} \qquad \text{by} \qquad \widetilde{f}(x,y) = \frac{fx - fy}{|fx - fy|}.$$

The equivariant homotopy class $\alpha(f)$ of $\widetilde{f}$ in the set $\pi^{m-1}_{eq}(\widetilde{N})$ of equivariant maps $\widetilde{N} \to S^{m-1}$ is clearly an isotopy invariant. Thus the *Haefliger-Wu (deleted product) invariant* is defined as

$$\alpha = \alpha^m_{CAT}(N) : \mathrm{Emb}^m_{CAT}(N) \to \pi^{m-1}_{eq}(\widetilde{N}).$$

Classical results of van Kampen, Shapiro, Wu, Haefliger and Weber [Vas92, ReSk99] reduce the problems of existence and classification of embeddings into $\mathbb{R}^m$ of an n-polyhedron or a closed smooth n-manifold N to equivariant homotopy problems involving the deleted product of N. Precisely, the Haefliger-Weber theorem states that $\alpha^m(N)$ *is bijective for* $2m \geq 3n+4$ *and a polyhedron (or a smooth n-manifold) N.* This results requires the metastable dimension assumption $2m \geq 3n+4$, which can be weakened to $2m \geq 3n+3-d$ for d-connected closed PL manifold N and $m \geq n+3$ [Sko97, Sko01]. For links and knotted tori this approach gives many specific results which will be listed in the talk.

Classical and modern examples of Haefliger, Zeeman, Hsiang-Levine-Szarba, Levine, Boechat-Haefliger, Freedman-Krushkal-Teichner, Segal-Spież and the author [ReSk99, Sko01] show the incompleteness of the deleted product invariant without the above dimension assumptions. Most of these examples also show that all the other invariants derived from configuration spaces are in general incomplete. Some of these examples were subsequently generalized to obtain a complete classification of embeddings.

In this talk we consider the case when the above dimension assumption $2m \geq 3n+3-d$ is just not fulfilled, i. e. when $2m = 3n+2-d$. For this case the above generalized configuration spaces invariants give no more information with respect to the deleted product invariant [Sko01]. However, the proof of [Sko01] suggests the definition of an invariant, required to classification of embeddings in this case. Analogous invariant in other situations was studied by Haefliger (implicitly), Hudson, Habegger, Fenn-Rolfsen, Kirk, Koschorke, Vassiliev and, possibly, others. Using our new invariant we show that the dimension restriction (or the connectedness assumption) is not only necessary, but also *sharp* in the above improvement of the Haefliger-Weber theorem [Sko01]. We also extend this example to obtain a PL isotopy classification for the case $2m = 3n+2-d$.

Theorem 1.1. *(a)* $\alpha^{6k+1}_{PL}(S^1 \times S^{4k-1})$ *is not injective for each* k.

(b) $\mathrm{Emb}^{6k+1}_{PL}(S^1 \times S^{4k-1}) \cong \pi^S_{2k-2} \oplus \pi^S_{2k-1} \oplus \mathbb{Z}$.

(c) For any d*-connected PL* $(2l+d)$*-manifold* N *the map* $\alpha^{3l+d+1}_{PL}(N)$ *is a surjection, each of whose preimages contains at most* $\#H_{d+1}(N, \mathbb{Z}_{(l)})$ *elements.*

Representatives of isotopy classes of embeddings $S^1 \times S^{4k-1} \to \mathbb{R}^{6k+1}$ are explicitly constructed (using, in particular, the higher-dimensional Borromean rings and Whitehead link).

References

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