Four dimensional geometry of circles

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September 18, 2001

The classical Beltrami theorem says that any metric whose geodesics are straight lines has constant curvature. But what can we say about geometries whose geodesics are circles (to be more precise — parts of circles; by a circle we always mean a circle or a line)? In dimensions 2 and 3 the answer is the same: these are classical geometries — Euclidean, spherical and hyperbolic [1, 2]. But in dimension 4 this is not true. The simplest counterexample is the Fubini-Study metric restricted to an affine chart of $\mathbb{CP}^2$.

Note that a family of geodesics satisfies the following nice property: at each point there is a germ of diffeomorphism that rectifies all the geodesics, i.e., sends them to straight lines. This diffeomorphism is just the inverse of the exponential mapping. Hence the following question arises. Suppose that we have a collection of circles passing through one point, say 0. What are geometric conditions for this collection to be rectifiable by a germ of diffeomorphism?

A collection of curves in $\mathbb{R}^n$ passing through 0 is said to be a *simple bundle of curves* if no two of them are tangent at 0. A simple bundle of curves is called *rectifiable* if there exists a germ of diffeomorphism in a neighborhood of the origin that sends all curves from this bundle to straight lines. A. G. Khovanskii proved in [1] that a rectifiable simple bundle of more than 6 circles in the plane necessarily pass through some point different from the origin. F. A. Izadi [2] generalized Khovanskii's arguments to dimension 3. A rectifiable simple bundle of circles at 0 in $\mathbb{R}^3$ containing sufficiently many circles in general position must pass through some other common point.

In dimension 4 this is not true. The simplest counterexample is a family of circles that are obtained from straight lines by some complex projective transformation (with respect to some identification $\mathbb{R}^4 = \mathbb{C}^2$ such that the multiplication by i is an orthogonal operator).

It turns out that in dimension 4 there is a huge family of transformations that take all lines to circles. To construct such a family, fix a quaternionic structure on $\mathbb{R}^4$ compatible with the Euclidean structure. If A and B are some general affine maps, then the map $A^{-1}B$ takes lines to circles (the multiplication and the inverse are in the sense of quaternions). Such maps will be called (left) *quaternionic fractional transformations*. Right quaternionic fractional transformations AB^{-1} also round lines (i.e. take them to circles). Any real projective, complex projective or quaternionic projective transformation is quaternionic fractional.

We prove that a rectifiable simple bundle of circles containing sufficiently many circles in general position is the image of a bundle of lines under some left or right quaternionic fractional transformation.

In arbitrary dimension, we have a purely algebraic description of rectifiable simple bundles of circles. So the analytic problem of classification of such bundles is reduced to an algebraic problem (more precisely, to a problem in complex algebraic geometry).

It is an interesting open question — what are metrics in an open subset of $\mathbb{R}^4$ whose geodesics are circles? We will give a partial answer. Fix a complex structure on $\mathbb{R}^4$ (compatible with the standard inner product) and confine ourselves to consideration of Kähler metrics only. We will classify all families of circles that can be geodesics with respect to some Kähler metric. These families correspond to the Euclidean geometry, the Fubini-Study metric on the complex projective plane and the Bergman metric on the complex hyperbolic plane (the latter two geometries can be viewed as complexifications of the spherical and hyperbolic geometries respectively).

References

[1] Khovanskii A. G., *Rectification of circles*, Sib. Mat. Zh., **21** (1980), 221–226

[2] Izadi F. A., *Rectification of circles, spheres, and classical geometries*, PhD thesis, University of Toronto, (2001)