Asymptotic Properties of Global Fields

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We study sequences of global fields, both number and function, with growing discriminant (respectively, genus), generalizing the Odlyzko–Serre bounds and the Brauer–Siegel theorem. This leads to asymptotic bounds on the ratio $\log hR/\log\sqrt{|D|}$ valid without the standard assumption $n/\log\sqrt{|D|}\to 0$, thus including, in particular, the case of unramified towers. Then we produce examples of class field towers, showing that this assumption is indeed necessary for the Brauer–Siegel theorem to hold.

Let K be a global field, i.e., either a number field of degree n and absolute value of the disriminant D with r_1 real and r_2 pairs of complex embeddings, or a function field over $\mathbb{F}_r$; in the function field case we set $r_1=r_2=0$ and $g=genus(K)$; in the number field case $g=\log\sqrt{|D|}$. By h we denote the class-number of K (which equals the number of $\mathbb{F}_r$-rational points on the Jacobian of K in the function field case); R denotes the regulator of K in the number field case and equals 1 in the function field case. We write log for $\log_e$ in the number field case, and for $\log_r$ in the function field case over $\mathbb{F}_r$; the Euler constant is denoted by γ. For a prime power q let $N_q(K):=|\{v\in P(K): Norm(v)=q\}|$, where $P(K)$ is the set of non-archimedean places of K.

We call a sequence $\{K_i\}$ of global fields a *family* if $g(K_i)\to\infty$ for $i\to\infty$, in the function field case r being fixed. We call a family $\{K_i\}$ *asymptotically exact* if and only if there exist all the limits (for any prime power q)

$$\phi_q := \lim_{i\to\infty}\frac{N_q(K_i)}{g(K_i)}\ ,\quad \phi_{\mathbb{R}} := \lim_{i\to\infty}\frac{r_1(K_i)}{g(K_i)}\ ,\quad \phi_{\mathbb{C}} := \lim_{i\to\infty}\frac{r_2(K_i)}{g(K_i)}\ .$$

A simple diagonal argument proves that every family $\{K_i\}$ contains an asymptotically exact subfamily. It is also easy to prove that any infinite tower $K_0\subset K_1\subset K_2\subset\ldots$ is an asymptotically exact family.

To simplify the exposition, in what follows we asume the Generalized Riemann Hypothesis. Without it, the results for number fields become weaker.

Theorem 1 (Basic Inequality). *For an asymptotically exact family of global fields one has*

$$\sum_q \frac{\phi_q\log q}{\sqrt{q}-1}+\phi_{\mathbb{R}}(\log 2\sqrt{2\pi}+\frac{\pi}{4}+\frac{\gamma}{2})+\phi_{\mathbb{C}}(\log 8\pi+\gamma)\le 1,$$

the sum being taken over all prime powers q.

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In the number field case this result generalizes the Odlyzko–Serre inequalities on the discriminant. In the function field case it generalizes the Drinfeld–Vlăduţ theorem. Then we prove

Theorem 2 (Generalized Brauer–Siegel Theorem). *For an asymptotically exact family of global fields one has*

$$\lim_{i\to\infty}\frac{\log h(K_i)R(K_i)}{g(K_i)} = 1+\sum_q \phi_q \log\frac{q}{q-1} - \phi_{\mathbb{R}}\log 2 - \phi_{\mathbb{C}}\log 2\pi\ ,$$

the sum being taken over all prime powers q.

Next question is that of the possible asymptotic behaviour of the Brauer–Siegel ratio $\log hR/g$. For the number field case we have

Theorem 3 (Bounds). *For any family of number fields*

$$BS_{low}\le \liminf_{i\to\infty}\frac{\log h(K_i)R(K_i)}{g(K_i)} \le \limsup_{i\to\infty}\frac{\log h(K_i)R(K_i)}{g(K_i)} \le BS_{up}\quad ,$$

where

$$BS_{low} = 1-\frac{\log 2\pi}{\gamma+\log 8\pi}\approx 0.5165...,$$

and

$$BS_{up} = 1+\frac{\log\frac{3}{2}+\log\frac{5}{4}+\log\frac{7}{6}}{\frac{\gamma}{2}+\frac{\pi}{4}+\log 2\sqrt{2\pi}+\frac{\log 2}{\sqrt{2}-1}+\frac{\log 3}{\sqrt{3}-1}+\frac{\log 5}{\sqrt{5}-1}+\frac{\log 7}{\sqrt{7}-1}}\approx 1.0938\ldots .$$

Thus the possible values for the ratio lie in the interval $(0.5165..., 1.0938...)$.

For the function field case we have

$$1\le \liminf_{i\to\infty}\frac{\log_r h(K_i)}{g(K_i)} \le \limsup_{i\to\infty}\frac{\log_r h(K_i)}{g(K_i)} \le 1+(\sqrt{r}-1)\log_r\frac{r}{r-1}\quad .$$

Examples. Having in mind the classical value 1 of the Brauer–Siegel theorem itself, it is natural to ask whether in our more general setting there exist examples when the ratio differs from 1. They do exist.

We prove that a class field tower over $\mathbb{Q}(\cos\frac{2\pi}{11},\sqrt{2},\sqrt{-23})$ has the Brauer–Siegel ratio somewhere in the interval $(0.5939..., 0.6025...)$. Moreover, there exists a class field tower over $\mathbb{Q}(\sqrt{11\cdot13\cdot17\cdot19\cdot23\cdot29\cdot31\cdot37\cdot41\cdot43\cdot47\cdot53\cdot59\cdot61\cdot67})$ in which nine prime ideals lying over 2, 3, 5, 7 and 71 split completely. This tower has the Brauer–Siegel ratio in the interval $(1.0602..., 1.0798...)$.

This shows that the condition $n/\log|D|\to 0$ in the Brauer–Siegel theorem is indeed indispensable for the ratio to be 1. Corresponding examples for the function field case are provided by modular curves.

To understand these results we introduce a limit zeta-function, roughly, the limit of g-th roots of classical zeta-functions. Then we prove the Asymptotic Explicit Formula relating the limit distribution of zeroes of (usual) zeta-functions in an asymtotically exact family with the parameters $\{\phi_q, \phi_{\mathbb{R}}, \phi_{\mathbb{C}}\}$.

M.A. Tsfasman, S.G. Vlăduţ. *Asymptotic properties of zeta-functions.* J. Math. Sciences, 1997, v.84, n.5, pp.1445-1467.

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