ORDER, RANK, AND CLASS OF PROJECTIVE ALGEBRAIC VARIETIES

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Matrices and their invariants, such as order, rank, determinant, play an outstanding role in mathematics. In an attempt to understand these notions geometrically and to generalize them to arbitrary projective varieties, one arrives at several possible definitions. We study the corresponding invariants, prove an inequality between them, classify the varieties for which they coincide and discuss some open problems.

To be a little more precise, nonzero square matrices of order $r+1$ (modulo multiplication by a nonzero constant) are represnted by points of the projective space $\mathbb{P}^{(r+1)^2-1}$. Geometrically, we consider the subvariety of matrices of rank one which is just the Segre variety $S = \mathbb{P}^r \times \mathbb{P}^r \subset \mathbb{P}^{(r+1)^2-1}$. Then matrices whose rank does not exceed m correspond to the points of S^m (the m-th join of S with itself), where S^m is swept out by the $(m-1)$-dimensional linear subspaces spanned by collections of m points of S. The order $r+1$ of matrix is equal to the smallest number k such that $S^k = \mathbb{P}^{(r+1)^2-1}$. Furthermore, the last nontrivial join S^r is actually a hypersurface whose equation (modulo multiplication by a nonzero constant) is the determinant and whose degree is also equal to the order of matrix.

Alternatively, in the dual space $\mathbb{P}^{(r+1)^2-1\,*}$ one can consider the dual variety S^* consisting of the points corresponding to hyperplanes tangent to S (i.e. containing a tangent space to S). Then it is not hard to check that S^* is a hypersurface defined (again modulo multiplication by a nonzero constant) by vanishing of determinant, so that $\deg S^* = r+1$.

Thus one has definitions of order and determinant in terms of projective geometry of S and S^*, and one would like to see what kind of invariants these definitions yield for an arbitrary variety $X^n \subset \mathbb{P}^N$. In the talk we describe some progress in this direction.

Finding relations between the above invariants is closely connected with the very old classical problem of bounding from below the class d^* of a projective variety $X^n \subset \mathbb{P}^N$ (by definition, $d^* = \deg X^*$) and describing varieties of small class. In the talk we explain what should be understood by "small", prove a (sharp) bound for class and classify varieties of the smallest possible class.

We also discuss another conjectural bound $d^* \geq 2\frac{N+1}{n+2}$, give a "proof" of this bound based on computation of Hessians, and explain why this "proof" is wrong. The error going back to Hesse leads to a still unsolved problem of classifying forms with vanishing Hessian. While this problem seems difficult, we formulate an easier elementary problem on the Hessian rank of polynomial matrices whose solution would yield the above bound.