Independent University of Moscow
Translation Series

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Arithmetical Problems

MCCME, IUM
Moscow, 2021

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Arithmetical Problems. — Moscow: MCCME: IUM, 2021. — 64 p.
ISBN 978-5-4439-4628-3

Translated into English on the initiative and with financial support of Alexander Gerko

Translated from the Russian language edition
Чулков П. В. Арифметические задачи. — М.: МЦНМО, 2019.

ISBN 978-5-4439-4628-3

Foreword

1. Arithmetical problems is the traditional name for word problems dealing with everyday situations that can be solved (without writing out equations) by *direct* reasoning based on analyzing of the specific situation of the problem (this method of solving is said to be "*arithmetical*"). Of course, most word problems can be solved "*algebraically*" (using equations), but the arithmetical method is more useful at the initial stages of learning.

In fact, the arithmetical method is better suited to the style of thinking of most 11–13 year-old students[1]: all the arguments used in it "suggest a completely visual and concrete, meaningful interpretation in the field of the quantities under consideration"[2].

This is especially important because students, as a rule, are not very well versed in the processes underlying the subject matter of word problems (motion, work, labour productivity).

Thus, the arithmetical method *expressly requires* that the student should construct a visual model, and this is important for further training: experience shows that those students who are good at solving problems arithmetically are better at writing out equations!

2. This little book contains the materials of six lessons, as well as an **Additional Problems** section.

The problems, as well as their solutions, are intended for middle school students (ages 11 to 13) and do not require any information beyond the school curriculum. Solutions of all the problems are presented, and some of the problems are accompanied by hints (for

[1]Russian 11–13 year old middle school pupils participating in math circles should be compared with middle school pupils from the better schools of the UK aged 12–14. (*Translator's note*)

[2]I. V. Arnold [3]. See also [8], [16], [17] in the list of references.

the pupil) and comments for the teacher. Each lesson is headed by an epigraph in the form of a humourous problem, whose discussion will help to cheer up the students and, perhaps, avoid some typical mistakes.

Lesson 1. First look at the arithmetical method.

Examples of problems solved by the arithmetical method are presented. In the solutions, the following verbal constructions are used: "assume that…", "if…, then…", "let…", and the solutions are written in the order of questions. It is expedient to check whether the answer satisfies the conditions of the problem by means of the simplest drawings.

Lesson 2. Percentages.

This is one of the most difficult topics of the school course, to which we must return many times. When solving problems about percentage, it is important to realize what concrete quantity is represented by 100% and to understand that, say, increasing a number by 32% means multiplying it by 1.32 and decreasing a number by 32% means multiplying it by 0.68. Let us explain the above. Let a number be given. Let's divide it into 100 equal parts. To reduce it by 32% will mean removing 32 of its parts. Then there will be 68 parts left. But the same result can be achieved by multiplying by 0.68, because there will also be only 68 parts out of 100 left!

Lesson 3. Pools, work, and all that.

Often students do not see the common mathematical basis behind different settings of such problems, because they have a poor idea of what happens in a particular process. The teacher's task is to help students to understand this.

Lesson 4. Let's look at motion!

When solving problems dealing with motion, it is useful to use drawings that record certain "key" moments of the condition of the problem.

Examples of such drawings are presented in the lesson. Such a visual presentation makes it much easier to find a solution.

Lesson 5. Distance, speed, time.

The purpose of the lesson is to practice solving motion problems, to show the relationship of problems involving motion with other types of arithmetical problems (e.g. joint work and other continuous processes), to find relations between the main parameters of motion (distance, speed, time). Among others, problems using the concept of “average speed” are considered.

Lesson 6. Downstream and upstream.

River traffic problems and other similar problems are considered. The ideas of “adding speeds (velocities)” and appropriately choosing a “reference frame” are used in their solution.

The following agreements are adopted:

1) If a body moves “with the current” (downstream), then its speed is composed of the speed of the body in standing water v and the speed of the river flow u: $w = v + u$.

2) If a body moves “against the current” (upstream), then its speed is taken equal to $w = v - u$.

3) Rafts move at the speed of the river flow.

Additional problems: These are problems intended for the independent work of students and for the organization of mathematical competitions. In addition to these problems, the teacher may need their variations, that is, problems with changed numbers and names, but mathematically equivalent (for example, Problem 8 of Lesson 6 is a variation of Problem 3 of the same lesson). As a rule, we do not write out variations explicitly, assuming that the teacher will be able to create them on his/her own.

Such problems can be found in the popular science literature.

3. **About the organization of “circles”.** One of the possible formats of a circle class is an hour and a half session with a short break between two parts:

- **browsing through new material**: problems are offered to the students one after another; after the current problem has been solved by one of the students, its collective discussion takes place;

- **work with "problems for independent solution"** is either in the form of an "oral olympiad", during which students tell the teacher their solutions orally, or as a "mathematical regatta" in which solutions must be presented in writing, and, once the solutions of the current problem have been gathered, the teacher discusses the solutions with the students.

In both cases, it is desirable to have assistants (senior school students or university students).

4. **Discussion of problems.** An important stage in working with problems is the discussion of their solutions with the participants of the circle. Here is an example of such a discussion.

Problem. On foot or by bus?

If Ann goes to school on foot and back by bus, she spends an hour and a half going there and back. If she goes both ways by bus, then the whole journey takes thirty minutes. How much time does Ann spend on the whole journey if she walks to the school and back?

Answer. 2 hours and 30 minutes.

Solution. Solution "with questions".

1) How much time does Ann spend on the journey if she goes by bus both ways? 30 minutes : 2 = 15 minutes.

2) How much time does Ann spend going from home to the school (one way) if she walks? 1 hour 30 minutes − 15 minutes = 1 hour 15 minutes.

3) How much time does Ann spend on the whole journey if she walks to the school and back? 1 hour 15 minutes × 2 = 2 hours 30 minutes.

This problem provides a good occasion to discuss what assumptions and simplifications are needed in order to translate the everyday situation into the language of mathematics and to what extent such assumptions are justified.

Below we give a sample discussion. The questions will, most likely, have to be put by the teacher. It is better not to give answers to students directly, but rather to stimulate the students to find the answers on their own.

Teacher. Let us try to identify those conditions that are not directly described in the problem, but are implicitely assumed.

Answer. We assume that going on foot or by bus takes the same amount of time in both directions.

Teacher. When is this assumption justified?

Answer. If the distance and the average speed are equal in both directions, then the time will be the same. For walking, this is usually the case when there are no additional conditions, such as an "uphill road" or a "downhill road". If we consider travel by bus, the answer may depend on whether we include the time spent, for example, waiting for the bus. In this case, it is reasonable to assume that the time spent waiting for the bus going there or going back is the same and so can be ignored.

Teacher. What else can lead to the inequality of the travel times?

Answer. Walking is not a factor if the bus stops in each direction are located near each other. Going by bus can make a difference if the routes there and back are different or if the average speed of the bus is not the same in the morning (going there) as in the afternoon (going back); for example, because the morning rush hour slows down traffic.

Teacher. What is the result?

Answer. We can assume that Ann goes along the same road in both directions and the waiting time for the bus is ignored.

In the comments for the teacher attached to the solutions to some problems, there are answers to questions that may be worth discussing with the students. The suggested guidelines are not mandatory. Teachers can make the necessary changes based on their own experience.

The author is grateful in advance for any comments and recommendations.

In conclusion, the author expresses his gratitude to A. D. Blinkov, F. A. Pchelintsev, A. V. Shapovalov, and A. V. Shevkin, who read the manuscript and whose comments contributed significantly to its improvement.

Lesson 1

A First Look at the Arithmetical Method

There are two fathers and two sons in one family.
How many different people are they?

A traditional problem.

Examples of problems solved by the arithmetical method are presented. In the solutions, the following verbal constructions are used: "assume that…", "if…, then…", "let…", and solutions are written question by question. It often helps to check whether the answer satisfies the conditions of the problem with the aid of the simplest drawings.

Problem 1. Gathering mushrooms. John picked 36 more mushrooms than his sister Mary. On the way home, Mary asked him: "Give me some mushrooms, so that I'll have as many mushrooms as you." How many mushrooms should John give her?

Answer. 18 mushrooms.

Solution. Suppose John puts his "extra" mushrooms in another basket, then both brother and sister will have the same number of mushrooms. To meet the condition of the problem, John must give his sister half of the extra mushrooms.

Problem 2. Geese and pigs in the barnyard. Peter counted the number of their heads: there were 30; then he counted the number of their legs: there were 84. How many geese and how many pigs were there in the barnyard?

Answer. 12 pigs and 18 geese.

Solution. Suppose that there are only geese in the yard. How many legs would they have? 60. Where did the 24 "extra" legs come from? They belong to the pigs, 2 legs to each. So there are 12 pigs in the yard. The rest (18) are geese.

We can argue differently: let's assume that all the pigs put their front legs on a log. It is clear that, in this case, 60 legs are on the ground, and the remaining 24 are the front legs of the pigs placed on the log. Therefore, there are 12 pigs in the yard.

Problem 3. Donkeys were grazing in a clearing. Several boys came up to them. "Let each one of us mount a donkey", suggested the eldest boy. But then two boys were left without donkeys. "Let's try to mount in twos", the eldest boy suggested again. Then two boys mounted each donkey, but this time one donkey was left without a rider. How many donkeys and how many boys were there in the clearing?

Answer. Six boys and four donkeys.

Solution. Suppose that all but two boys mounted – one to each donkey. Now let's put the boys on donkeys in twos in the following way:

1) one of the boys leaves his donkey and mounts another donkey as a second rider, so one donkey is left without a rider, as required;

2) the two boys who originally were left without a donkey join two others as second riders. Result: three donkeys are "occupied", and one is not. Hence the number of boys is 6, and of donkeys, 4.

Problem 4. How old is John? When John was asked his age, he thought about it and said: "I am three times younger than my dad, but three times older than Tommy". Then little Tommy appeared and said "Daddy is 40 years older than me". How old is John?

Answer. John is 15.

Solution. The father is three times older than John, who is three times older than Tommy, and so the father is nine times older than Tommy.

In other words, the father is older than Tommy by eight times Tommy's age, this age difference is 40. Therefore, Tommy is $40/8 = 5$ years old.

John is three times older than Tommy, so John is 15 years old.

In problems of this type, it is customary to take into account only “full” years. This is worth telling to the students.

Problem 5. An ancient problem. To buy an ice cream, Peter was short by seven pence and Mary, by one penny. Then they joined together all the money they had. But they still did not have enough to buy even one portion. How much did a portion of ice cream cost?

Answer. 7 pence.

Solution. If Peter had at least a penny, he would have given it to Mary, and they would have had enough for an ice cream (after all she was only one penny short!). Therefore, Peter had no money at all, and since he lacked seven pence for an ice cream, the ice cream cost seven pence.

Here it is desirable to talk about systems of monetary units, their changes, the dynamics of prices for ice cream, and so on.

Problem 6. The teacher posed a difficult problem during a lesson. As a result, the number of boys in the class who solved this problem was equal to the number of girls who did not solve it. Who is in the majority in the class: those who solved the problem or the girls?

Answer. Neither: their total number is the same.

Solution. Add to the boys (a persons) who solved the problem the girls who solved the problem (b persons). What do we get? All the students who solved the problem ($a + b$ persons).

Add to the girls who did *not* solve the problem (a persons), the girls who did (b persons). This time we obtain all the girls ($a + b$ persons).

We have added equal quantities, so we obtain equal quantities. Therefore, the total number of those who solved the problem is the same as the number of girls.

The solution of the problem can be neatly represented in the following table.

	Solved the problem	Didn’t solve the problem
Boys	a	
Girls	b	a

Problems for individual solution

Problem 7. Two stacks of notebooks. If you move 10 notebooks from the first stack to the second one, then the number of notebooks in the stacks will be equal. How many more notebooks are there in the first stack than in the second?

Problem 8. Chocolate bars. To buy eight chocolate bars, Susan needs 20 pence more. If she buys five chocolate bars, then she will have one pound left. How much money does Susan have?

Problem 9. Buying an album. To buy an album, Mary was short of 2 pence, Nick, of 34 pence, and Tom, of 35 pence. Then they added up their money, but it still wasn't enough to buy one album. How much does the album cost?

Problem 10. A long time ago. When the father was 27 years old, his son was 3 years old, and now he is three times younger than his father. How old is each of them now?

Problem 11. Chairs and stools. There are chairs and stools in the room. Each stool has 3 legs and each chair has 4 legs. When there are people sitting on all the chairs and stools, then the total number of "legs" in the room is 39. How many chairs and how many stools are there in this room?

Problem 12. Error when adding. When adding two integers, Nick put an extra zero at the end of one of the summands and got a total of 6641 instead of 2411. What numbers did he add up?

Answers and solutions

Problem 7. Answer. 20 more.

Solution. The first stack decreased by 10 notebooks, while the second one increased by 10 notebooks, after which the stacks became equal.

Problem 8. Answer. Susan has 3 pounds.

Solution. 1) Suppose Susan bought 5 chocolate bars. By the condition of the problem, she has 1 pound left. Let's give her 20 pence.

Now she has £ 1.2 at her disposal, with which she can buy three more chocolate bars. Therefore, a chocolate bar costs $120/3 = 40$ pence.

2) Since the purchase of five chocolates will cost 2 pounds and another one pound will remain, it follows that Susan has $2 + 1 = 3$ pounds.

Problem 9. Answer. The album costs 35 pence.

Solution. Since Nick has one penny more than Tom, he has at least one penny, and when adding his money to Mary's money, this penny will be added. But, as a result, the money for the album was not enough, that is, an amount of less than two pence was added to Mary's money; therefore, Tom did not have any money at all, that is, he was lacking the full price of the album.

Problem 10. Answer. The son is 12 years old and the father 36 years old.

Solution. The age difference between the father and the son does not vary and is equal to 24 years. Since the son is now three times younger than the father, then 24 years is twice the age of the son.

Here it is useful to make a drawing.

Problem 11. Answer. 4 chairs and 3 stools.

Solution. The number of "legs" corresponding to one chair is equal to $4+2 = 6$, so the total number of "legs" corresponding to chairs is a multiple of 6. Similarly, the total number of "legs" corresponding to stools is a multiple of $3 + 2 = 5$. To solve the problem one has to represent 39 as a sum of a multiple of 6 and a multiple of 5. The first summand (the multiple of 6) can be equal to 0, 6, 18, 24, 30, or 36. Checking out these cases one by one, it is easy to see that the second summand is a multiple of 5 only if it is is equal to $39-24 = 15$. Thus, there are 24 "chair legs" and 15 "stool legs", which corresponds to 4 chairs and 3 stools.

Problem 12. Answer. 1941 and 470.

Solution. By putting an extra zero at the end of one of the summands, Nick thereby increased that summand 10 times. Thus, that summand is 9 times less than the difference $6641 - 2411 = 4230$. Therefore, it is equal to $4230/9 = 470$.

Lesson 2

Percentages

By how many percent
is 20% more than 10%?
An absurd question

This is one of the most difficult topics of the school course, we should return to it many times. When solving problems about percentages, it is important to realize what quantity is represented by "100%" and to understand that, say, "increasing a number by 32%" means "multiplying it by 1.32" and "decreasing a number by 32%" means "multiplying it by 0.68". Let us explain the above. Suppose a number is given. Divide it into 100 equal parts. To reduce it by 32% will mean removing 32 of these parts. Then there will be 68 parts left. But the same result can be achieved by multiplying the number by 0.68, because there will also be only 68 parts out of 100 left!

Problem 1. Peter bought two books. The first book was cheaper by 75% than the second one. By how many percent is the second book more expensive than the first one?

Answer. By 300%.

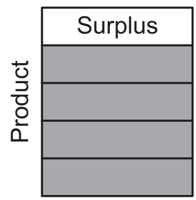
Solution. 1) In the first sentence of the condition of the problem, the price of the second book is taken as 100%. Because the price of the first book is 75% less, it follows that its price is 25% of that of the second book. Therefore, the second book is four times more expensive than the first book.

2) To answer the question of the problem, let us take the price of the *first* book as 100%. Then the price of the second book will be 400% of the price of the first book. Therefore, the price of the second book is higher by 300% than the price of the first book.

The subtlety of the solution is that two different numbers are taken equal to 100%. In the first sentence of the statement, 100% is the price of the *second* book and, in the second sentence, it is the price of the *first* book. In most cases, it is clear from the context what quantity should be taken as 100%. As a rule, this should be the quantity that we must change to get a different quantity. In our case, "the first book is cheaper than the second one by 75%"; therefore, the price of the second book should be reduced by 75%, so its price must be taken to be 100%. The question is: "By how many percent is the second book more expensive than the first one?". The required number shows by how many percent the price of the first book must be increased, so that it is precisely the price of the first book that should be taken equal to 100%. This convention is needed to unambiguously understand texts in which the notion of percentage occurs.

Problem 2. A customer's dream. The price of potatoes dropped by 20%. By how many percent more potatoes can one buy with the same amount of money?

Answer. By 25%.



The price of potatoes dropped by 20%. Therefore, we could have bought all the previously purchased potatoes by spending 80% of the money (see the figure). Let's put these potatoes in a sack. There will be 20% of the previously needed amount of money left, which means that with the remaining money we can buy a fourth of a sack of potatoes, that is, 25% more.

Problem 3. Where is it cheaper? In two nearby shops, the price of milk was the same. Then, in one shop, its price was cut by 40%, while, in the other, it was first cut by 20%, and later by 25%. In which of the shops is the milk cheaper now?

Answer. After the reductions in price, the price of milk in both shops is the same.

Solution. In the second shop, after the first price reduction, the price of milk is 80% of the original price and, after the second price cut, it will be $80\% \cdot 0.75 = 60\%$ of the original price, which is the same as the price in the first shop.

Problem 4. Garden plot. The length of a rectangular plot of land was increased by 50%, but its width was reduced by 10%. How has the area of the plot changed?

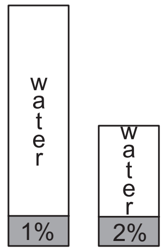
Answer. The area has increased by 35%.

Solution. As the length of the plot increases by 50%, its area is multiplied by 1.5 and as the width decreases by 10%, its area is multiplied by 0.9. Thus, the area has increased $1.5 \cdot 0.9 = 1.35$ times.

Problem 5. Dried mushrooms. The humidity of fresh mushrooms is 99% and that of dried ones is 98%. How does the mass of mushrooms change after drying?

Answer. The mass decreases by half.

Solution. Suppose that the mass of fresh mushrooms is $100m$ kg, and so the mass of dry matter in them is m kg. After drying, the mass of dry matter remains the same, but it is now 2% (one fiftieth) of the mass of the dried mushrooms (see the figure). Therefore, the mass of the dried mushrooms is $50m$ kg.



Problem 6. Two positive numbers. One positive number was increased by 1% and the other one was increased by 4%. Can the sum of these numbers increase by 3% as a result?

Answer. Yes, it can.

Solution. For example, if the first number is 100 and the second is 200, then their sum is 300. After the increase, the first number is 101, and the second is 208. Then the sum of the increased numbers is 309, which is 3% more than 300.

How can one help the students obtain this solution? We reformulate the condition of the problem as follows: Can it turn out that the sum of one percent of some positive number and four percent of another number is equal to three percent of their sum? Assume that it can.

Let's argue a little.

1) If the first summand is increased not by 1%, but, just as the second one, by 4%, then the sum will increase by 4%.

2) Hence the extra 3% of the first summand yields the extra 1% of the sum.

3) Thus, one percent of the first summand is three times less than one percent of the sum, which is possible if the first summand is three times less than the sum and two times less than the second summand.

4) We obtain an example: if the first number is, for instance, 100 (for the convenience of calculations), then the second number is 200, etc.

Problems for individual solution

Problem 7. Two students. The same scholarships for two students were increased: for the excellent student, by 100%, and, for the good student, only by 50%. At the next exam the excellent student failed one of the tests, and his scholarship was lowered to that of good student. By how many percent was his scholarship decreased?

Problem 8. Hay harvesting. Grass contains 60% of water and hay contains 20% of water. How much hay can one get from one metric ton (1000 kg) of grass?

Problem 9. A consequence of the crisis. During the last two years, our factory reduced its output by 51%, while during each of these years its output fell by an identical amount in percentage terms. By how many percent yearly?

Problem 10. A physics problem. The volume of a liquid after freezing increases by 25%. By how many percent will the volume of the frozen liquid decrease after melting?

Problem 11. After washing. After each wash, the volume of a bar of soap is reduced by 20%. After how many washes will it decrease by at least half?

Problem 12. Compare the numbers. It is known that 2% of a positive number A is greater than 3% of a positive number B. Is it true that 5% of the number A is greater than 7% of the number B?

Answers and solutions

Problem 7. Answer. By 25%.

Solution. Let the initial scholarship of each student be equal to x; then, after the increase, the scholarship of the first student became $2x$ and that of the second student, $1.5x$. The scholarship of the second student is now three quarters of that of the first student, that is, 75%.

Problem 8. Answer. 500 kg.

Solution. A ton of grass contains 400 kg of dry matter, which is 80% of the mass of hay; hence the mass of hay (100%) is 500 kg.

Problem 9. Answer. By 30%.

Solution. After two years, the output volume was 0.49 of the original volume, and this volume was the same fraction of the output volume for the previous year as that of the year before. It remains to find this fraction, knowing that its square is equal to 0.49. It follows that the fraction is 0.7. Consequently, each year the output volume decreased by 30%.

Problem 10. Answer. By 20%.

Solution. The volume of water during freezing increases by a fourth (that is, it becomes equal to five fourths of the previous volume). So, one fourth of the original volume is equal to one fifth of the "new" volume. When melting, the volume of the resulting liquid will be equal to the original volume of the liquid, that is, it will decrease by one fifth.

Problem 11. Answer. After the fourth wash.

Solution. Let the initial volume of a bar of soap be x; then, after the first wash, the volume will be equal to $0.8x$, after the second wash, to $0.64x$, after the third, to $0.512x$, and after the fourth, to $0.4096x$, which is less than one half of the original volume.

It is useful to use a little algebra here.

Problem 12. Answer. Yes, it's true.

Solution. Since 2% of the number A is greater than 3% of the number B, it follows that A is greater than B and 4% of the number A is greater than 6% of the number B (we have doubled both numbers). Besides, 1% of the number A is greater than 1% of the number B. "Adding up" the last two statements, we see that 5% of the number A is greater than 7% of the number B.

One could also use more algebra. To wit, let $A = 100x$, $B = 100y$; then 2% of the number A is $2x$, and 3% of the number B is $3y$. We see that $2x > 3y$ and $4x > 6y$. Also note that 1% of the number A is greater than 1% of the number B, whence $x > y$.

Lesson 3

Pools, Work, and all that

One egg is cooked in 4 minutes.
How long will it take
to cook three eggs?

A well-known problem

Often students do not see the common mathematical basis behind different settings of such problems, because they have a poor idea of what happens in a particular process. The task of the teacher is to help students understand this.

Problem 1. Carlson at the Kid's birthday. The Kid eats a jar of jam in 6 minutes and Carlson, twice as fast. How long will it take them to eat this jam together?

Answer. Two minutes.

Solution. Carlson eats twice as much jam per unit of time as the Kid. Therefore, together they will eat a jar of jam three times faster than the Kid alone.

In order to solve the problem, it can be reformulated as follows: "the Kid eats a jar of jam in 6 minutes. How long will it take three Kids to eat this amount of jam?" Thus, here, the rate of eating is measured in "Kids".

Problem 2. An ancient problem. A man drinks up a cask of ale in 14 days and, together with his wife, he will finish the same cask in 10 days. How many days will it take his wife alone to drink up the same cask?

Answer. 35 days.

Solution. 1) If the man drinks up a cask of ale in 14 days, then, in 70 days, he will drink five times more (5 casks).

2) If the man and his wife drink up a cask of ale in 10 days, then they will drink up seven times more (7 casks) in 70 days.

3) Therefore, in 70 days, the wife will drink up 2 casks.

Hence she will drink up one cask in 35 days.

Another solution is possible: let us divide the cask into 70 identical tumblers. Every day the husband drinks 70 : 14 = 5 tumblers and, together with his wife, 70 : 10 = 7 tumblers. Consequently, the wife drinks 2 tumblers of ale per day, so the entire cask will be finished by her in 35 days.

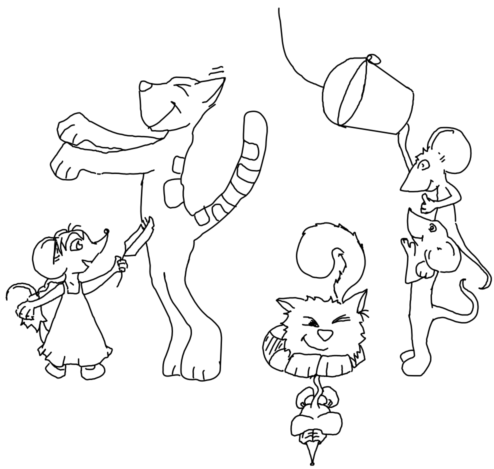
Here "proportionality" appears in the following form: during the same period of time, the man (or, for example, his wife) drinks the same amount of ale (not necessarily the same amount for the man as for his his wife).

Where did the number 70 come from? It is the least common multiple of 10 and 14.

Problem 3. Cats and mice. Five cats caught five mice in five minutes. How many cats can catch ten mice in ten minutes?

Answer. Five.

Solution. If five cats caught five mice in five minutes, then, in ten minutes, the same five cats will catch twice as many mice.



The following "problem" is worth discussing: "A plumber can fix our water tap in 20 minutes. How long will it take 150 plumbers to fix the same water tap?"

Problem 4. Beavers and a dam. Ten beavers calculated that they can build a dam in 8 days. After they had worked for two days, it turned out that, in view of an impending flood, they had to finish the work in 2 days. How many beavers do they need to call for help?

Answer. 20 beavers.

Solution. In four days, 10 beavers can do one-half of all the work; therefore, in the remaining two days, the invited beavers must do the second half of the work. Therefore, 20 more beavers are needed.

Problem 5. Firewood. In one day, two workers can either saw up three logs into chunks or chop up six such chunks into firewood for the chimney. How many such logs should they saw into chunks in a similar way in order to have time to chop up all of the resulting chunks on the same day?

Answer. Two logs.

Solution. In two days, the workers will saw up six such logs, and then chop up the resulting chunks on the next day. So, in total, this will take three days. Therefore, they will manage two logs (both sawing and splitting) in one day.

Problem 6. A classical topic. Three workers have dug a hole. They worked in turns; each worked as long as it would take the other two to dig half the hole working together. Working like this, they have dug out a certain hole. How many times faster would they have finished the job if they had worked together?

Answer. 2.5 times faster.

Solution. Assume that they worked together. How many holes will they have dug in this case? While the first worker digs his share of the "total" hole, in the same period of time the other two dig out one-half of such a hole. Then the second worker does his share, while the others will dig out another half of such a hole. Finally, the third worker will finish his share of the work and the others will dig out another half of the hole. A total of 2.5 holes will be dug out. Therefore, working together, they would finish the job 2.5 times faster.

Problems for individual solution

Problem 7. Filling the tank. If only the first tap is open, water fills the tank in three hours and if only the second tap is open, the tank is filled in 9 hours. How long will it take to fill the tank if both taps are open?

Problem 8. Two diggers digging a ditch. One of them can dig twice as much in an hour as the other one, but they are paid the same for each hour of work. What comes out cheaper: The diggers working simultaneously from opposite ends of the ditch up to the "meeting point", or each digger digging out one half the ditch?

Problem 9. Two burning ropes. Each of them burns unevenly, but burns out in exactly one hour. How can these ropes can be used to measure exactly 45 minutes?

Problem 10. A big cake. The Kid eats a big cake in 24 minutes and Carlson, in 12 minutes. The Kid started to eat first and Carlson joined him after 6 minutes. How long will it take the two of them to finish the cake? What share of the cake will the Kid get?

Problem 11. Mary, Blackie, and Whitie. When Mary had two adult cats and a puppy, named Blackie, they all ate equally and a bag of animal food lasted 6 days. Then Blackie grew up; the bag now lasted only 4 days. Then Mary got another dog, Whitie, and the bag of food now lasts only 3 days. Who eats more: each cat or the dog Whitie, and how many times more?

Problem 12. Three diggers. Three diggers contracted to dig out a pit of a certain size. If John is missing, then the other two can do the work in 30 days; if Peter does not come, then the work can be done in 15 days, but if Andrew doesn't appear, then, in 12 days. Andrew went to work alone. How many days will it take him to do the work?

Answers and solutions

Problem 7. Answer. 2 hours 15 minutes.

Solution. The first tap is equivalent to three second taps. Therefore, we can assume that 4 second taps are open. Hence the pool will fill up in 9/4 hours.

Problem 8. Answer. Simultaneous work "from opposite ends".

The "fast" digger digs faster, so he will be paid less than the "slow" one for the same work. When working "towards each other", the "fast" digger will do more than half the work (but will be paid as much as the "slow" digger), and when working in turn, the "fast" digger will only do half the work.

The problem is solved without calculations, using a qualitative estimation.

Problem 9. Solution. 1) Let's set fire to the ropes at the same time; the first one at one end and the second, at both ends. When the second rope burns out completely, exactly 30 minutes will pass.

2) At this point, let us set fire to the first rope from the other end. When that rope burns out, 15 more minutes will have passed. This makes a total of 45 minutes.

If after some time it becomes clear that the students are unable to solve the problem, it is useful to ask them the following questions: What can be done with the ropes? Why is it useless to fold a rope in several layers or cut it into pieces?

Problem 10. Answer. 12 minutes; half the cake.

Solution. The Kid managed to eat one fourth of the cake in 6 minutes and then the remaining three fourths of the cake were eaten together, with Carlson swallowing half the cake and the Kid, one fourth (Carlson eats twice as fast!).

Problem 11. Answer. Whitey eats 1.5 times more than one cat.

Solution. Let's assume that each animal ate one helping; then, in 6 days, together they would have eaten up 18 helpings, the contents of one bag. When Blackie grew up, the cats still ate 2 helpings a day, that is, in 4 days they ate 8 helpings, while the other 10 helpings were eaten by Blackie, that is, Blackie ate 2.5 helpings a day. Thus, in three days, the three of them would have eaten $2\times3+2.5\times3 = 13.5$ helpings. Therefore, Whitie eats 4.5 helpings in three days, which is 1.5 helpings per day. Thus, Whitie eats one and a half times more than one cat.

Problem 12. Answer. 120 days.

Solution. Suppose that the joint work lasts 60 days.

During that time, Peter and Andrew would dig out two pits, John and Andrew, four pits, John and Peter, five pits.

But, in total, they would not dig 11 moats, but only 5.5, because each digger's share of work must be counted once, not twice. Recall that, in 60 days John and Peter can dig out $60/12 = 5$ pits. Hence Andrew's share is one half of one pit in 60 days, or one pit in 120 days.

Lesson 4

Let's Look at Motion!

A troika of horses (three horses harnessed abreast) ran 30 km in 3 hours. In what time will one horse run 30 km?

A well-known problem

When solving problems dealing with motion, it is useful to use drawings that record certain "key" moments of the conditions of the problem.

Examples of such drawings are presented in the materials of this lesson. Such a visual presentation makes it much easier to find a solution.

Problem 1. Dragonfly and two ants. Two ants, Basil and Cyril, went on a visit to their friend, the Dragonfly. Basil crawled all the way, while Cyril rode half the way on a Caterpillar, whose speed was half the speed of crawling, and rode the rest of the way on Grasshopper, which was ten times faster than crawling. Which of the ants came first if they started at the same time and covered the same distance?

Answer. Basil came first.

Solution. At the moment when Cyril reached half way and got on Grasshopper, Basil had crawled two halves of the way and was already on the spot.

When analyzing the problem, it is useful to depict the position of the ants at the time of Cyril's "transfer" from Caterpillar to Grasshopper.

Problem 2. Caterpillar on a pole. A caterpillar crawls along a pole, climbing 5 feet upwards during the day and descending 4 feet during the night. The height of the pole is 10 feet. How long will it take the caterpillar to reach the top of the pole?

Answer. Five days and nights and one more day.

Solution. Usually students give the answer "10 days", but they are wrong.

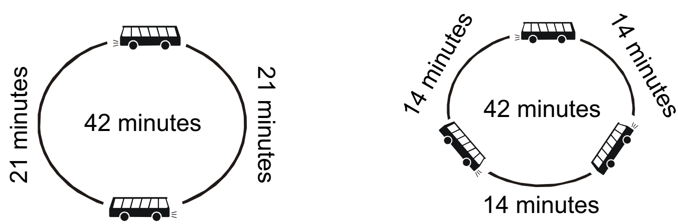
In 24 hours, the Caterpillar moves 1 foot higher; therefore, in 5 days and nights, it will move up 5 feet and, during the *sixth day*, it will climb 5 feet more and *reach* the top of the pole.

It is convenient to illustrate the problem with drawings showing the position of the caterpillar at the end of each day.

Problem 3. Circular route. Two buses run on a circular route, with a 21 minute interval between them. What will the interval between buses be if there are three buses running on this route?

Answer. 14 minutes.

Solution. Since the interval between the arrival of two buses on the route is 21 minutes (see the left-hand figure), it follows that the length of the whole route "in minutes" is 42 minutes. Therefore, when there are three buses running, the interval between successive buses is 14 minutes (see the right-hand figure).



We see that sometimes distance can be measured in minutes.

Problem 4. The road to work. An engineer usually arrives by train to the train station at 8 a.m. At 8.00 a.m. sharp, a car from the factory pulls up to the train station and takes the engineer to the factory. Once, the engineer arrived at the train station at 7 a.m and decided to walk to meet the car. When he met the car, he got in and arrived at the factory 20 minutes earlier than usual. At what time did the engineer meet the car?

Answer. At 7.50.

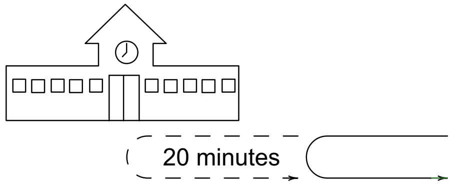
Solution. The car saved time in two ways, because:

1) it did not have to reach the train station, covering less distance from the factory;

2) it did not have to cover the full distance on the way back to

the factory.

Thus, since the car returned to the factory 20 minutes earlier than usual, it follows that, at the time of the meeting with the engineer, it only needed 10 minutes to drive up to the train station.

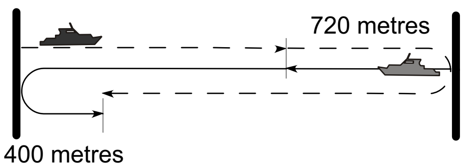


Problem 5. Encounters on the way. Two ferries depart simultaneously from the opposite banks of the river and cross the river at constant speed perpendicularly to the banks. The ferries meet each other at the distance of 720 m from the nearest bank. When they reach the shore, they immediately go back and then they meet at 400 m from the other shore. What is the width of the river?

Answer. 1760 m.

Solution. The sum of the distances that the ferries have travelled before the first encounter is equal to the width of the river and the sum of distances that they have travelled up to the second encounter is equal to the tripled width of the river. This means that the period of time up to the second encounter is three times that to the first encounter. Therefore, if, by the time of the first encounter, one of the ferries has travelled 720 m, then, by the time of the second encounter, it has travelled 2160 m, which is 400 m more than the width of the river.

All this can be seen from the drawing.

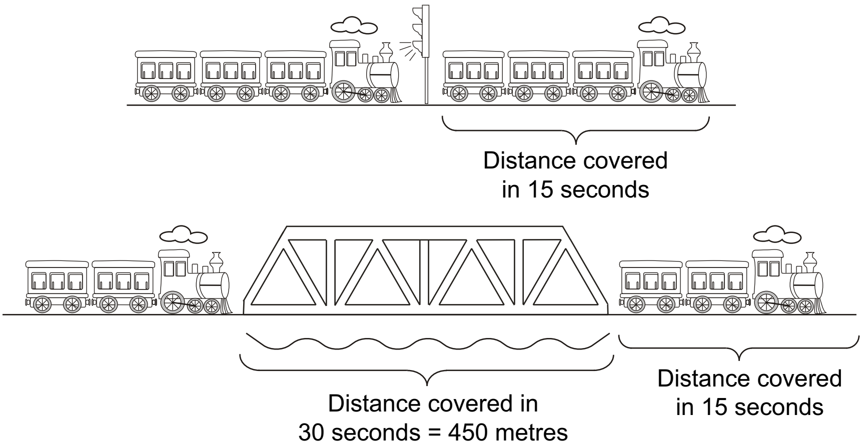


Problem 6. Train on the bridge. The train passes a bridge 450 m

long in 45 seconds and runs past a signal pole in 15 seconds. Calculate the length of the train and its speed.

Answer. 225 m, 54 km/h.

Solution. The train covers the distance equal to its length in 15 seconds (see the upper picture) and the distance greater by 450 m in 45 seconds (see the lower picture). Hence the front part of the train covers 450 m in 30 seconds, that is, the train itself is two times shorter than the bridge and so the speed of the train is 15 m/s = 54 km/h.



It is important to discuss the meaning of the assertion: "the train passes a bridge of length 450 m in 30 seconds" with the students.

Problems for Individual Solution

Problem 7. Flags from the start to the finish line. Flags are placed at the same distance from one another from the start to the finish line. A runner covers the distance from the first to the seventh flag in 7 seconds. How long will it take him to run from the first to the tenth flag?

Problem 8. Trams on a circular line. 12 trams run at the same speed and with equal intervals along a circular line. How many trams must be added so as to reduce the intervals between the trams by one fifth if they all run at the same speed?

Problem 9. Trains again. A train 180 m long runs past a singal pole in 9 seconds. How long will it take for that train to completely pass a 360 metre long bridge?

Problem 10. Two friends. Two friends, Willy and Freddy, walked towards each other at constant speeds. Willy left Williamstown at 10 a.m. and reached Fredericktown at 3 p.m. Freddy left Fredericktown at 11 a.m and reached Williamstown at 4 p.m. At what time did the friends meet?

Problem 11. The motorcyclist and the cyclist. A motorcyclist and a cyclist started at the same time on their way from point A to point B. After covering one third of the distance, the cyclist stopped and started again only when the motorcyclist had one third of the distance to B left. Having reached B, the motorcyclist made a U-turn and immediately rode back to A. Who will arrive earlier: the motorcyclist to point A or the cyclist to point B?

Problem 12. The king with his retinue travels from point A to point B at a speed of 5 km/h. Every hour, the king sends a messenger to B, the messengers run at 20 km/h. At what intervals do the messengers arrive at point B?

Answers and solutions

Problem 7. Answer. 10.5 seconds.

Solution. The athlete runs six (not seven!) gaps between the flags in 7 seconds, so 9 such gaps will require $7/6 \cdot 9 = 10.5$ seconds.

Hint. Make a drawing!

Problem 8. Answer. Three trams.

Solution. As in Problem 3, we will measure distance in time intervals. Divide the gap between the trams into five equal parts. Then the route consists of 60 such parts. If the interval between the trams is reduced by one-fifth, then the interval between the trams will be equal to four parts. Therefore, there will be 15 trams along the route.

Problem 9. Answer. In 27 seconds.

Solution. The front part of the train will pass the bridge in 18 seconds and the rear part will need 9 more seconds to clear the bridge.

Problem 10. Answer. At 1 p.m.

Solution. The boys' speeds were the same (why?) and, in an hour, each of them covered one fifth of the distance; before the encounter, each of the boys had walked two fifths of the total distance.

Consequently, Willy and Freddy met at 13 : 00.

Problem 11. Answer. The cyclist will arrive earlier.

Solution. Two key facts: 1) the cyclist covered one-third of the way in less time than it took the motorcyclist to cover two-thirds of the way; hence the cyclist's speed was more than half the speed of the motorcyclist; 2) after the cyclist set off, he had to travel two-thirds of the distance from A to B, while the motorcyclist had to travel four-thirds of the same distance.

Problem 12. Answer. Every 45 minutes.

Solution. Any messenger sent by the king will be 15 km away from him in an hour. So the distance between this messenger and the next one will be 15 km. The speed of each messenger is 20 km/h; thus, the messenger will cover 15 km in 45 minutes. Hence the messengers will arrive at B every 45 minutes.

Lesson 5

Distance, Speed, Time

How fast should a dog run,
so as not to hear the clinking of the tin can
tied to its tail?

A well known problem

The purpose of the lesson is to practice solving problems dealing with motion, to show the relationship of such problems with other types of arithmetical problems (joint work and other continuous processes), to find out the relations between the main parameters of motion (distance, speed, time). Among others, problems using the concept of “average speed” are given.

Problem 1. A bicycle tire burst when the cyclist had travelled two-thirds of the way. It took him twice as long to walk the rest of the way. How many times faster did the cyclist ride than he walked?

Answer. Four times.

Solution. The cyclist spent twice as much time walking as riding the bicycle, but, at the same time, covered twice as much distance.

Problem 2. George and Helen. George left the house 5 minutes after Helen, walking twice as fast as she did. How long will it take George to catch up with Helen?

Answer. 5 minutes.

Solution. George walks the same distance in 5 minutes as Helen in 10 minutes.

It is useful to compare this problem with the following one: “George and Helen must weed out two identical flowerbeds. George started work 5 minutes later than Helen, but he worked twice as fast as his sister, and they finished the work at the same time. How long did George work?” The undelying situation is completely different, but mathematically they are the same.

Problem 3. One hundred metres. Three runners, Andrew, Boris, and Alex compete in a 100 metre event. When Andrew reached the finish line, Boris was 10 metres behind him. When Boris reached the finish line, Alex was 10 metres behind Boris. How many metres was Alex behind Andrew at the moment when Andrew reached the finish line?

Answer. 19 metres.

Solution. When Andrew had run 100 metres, Boris was 10 metres behind him, that is, he had covered 90 metres. Hence his speed is 0.9 of Andrew's speed. Similarly, Alex's speed is 0.9 of Boris' speed, or 0.81 of Andrew's speed. Consequently, when Andrew was at the finish line, Alex had run 81 metres.

It is useful to compare this problem with the following one: "A merchant sold 10% of the available apples before lunch and 10% of the remainder after lunch. How many apples were sold? Express the answer in percentages".

Problem 4. Average speed, what is it? A person walked for a certain time at the speed of 4 km/h, and then, for twice as much time, at the speed of 7 km/h. What was his average speed of motion?

Answer. 6 km/h.

Solution. Suppose that "for a certain time" is t hours; then, in total, this person walked $4t + 7 \cdot 2t = 18t$ km in $3t$ hours, and so his average speed was 6 km/h.

It is necessary that students realise that "average speed" is not, generally speaking, the arithmetic average (mean) of several speeds. In the solution of this and other problems in the framework of arithmetical methods, it is useful to use algebraic notation.

Problem 5. If a cyclist rides at 10 km/h, he will be one hour late. If he rides at 15 km/h, then he will arrive one hour early. At what speed should he or she ride so as to arrive on time?

Answer. 12 km/h.

Solution. Suppose that there are two cyclists and their speeds are 10 km/h and 15 km/h, respectively. If the first cyclist leaves two hours earlier than the second one, then they would arrive at the same time. In our case, the second cyclist gave the first one a 20 km head start. The second cyclist can make up for this handicap in exactly 4 hours. Therefore, in order for the second cyclist to be at the final destination at the same time as the first one, he or she must travel 60 km. It only remains to determine the speed of a cyclist travelling 60 km in 5 hours.

Problem 6. Two trains move towards each other on parallel tracks at the same speed of 60 km/h. A passenger in the second train noticed that the first train ran past him in six seconds. What is the length of the first train?

Answer. 200 metres.

Solution. Imagine that the second train is standing and the first train is moving at twice the speed. The speed of the first train relative to the passenger of the second train, the speed of the first train is 120 km/h, which is 100/3 m/s.

Therefore, the length of the first train is $100/3 \cdot 6 = 200$ m.

Here, in fact, a convenient reference frame was chosen. This will be useful later in physics lessons.

Problems for individual solution

Problem 7. A car runs at 60 miles per hour. How should one increase its speed to save one minute per kilometre?

Problem 8. A lion cub decided to ride a tortoise, but needs to catch up with the tortoise first. What is the distanc the lion cub will have to run before he can catch up with the tortoise if his speed is 10 times greater than that of the tortoise, and the tortoise is 180 metres away from the cub?

Problem 9. Traffic police post. On the highway, a column of cars of length 300 metres is travelling at 60 km/h. Passing by the traffic police post, the cars slow down to 40 km/h and then proceed at that speed. What will be the length of the column when all the cars have passed the traffic police post?

Problem 10. Basil and Peter, having quarrelled, ran away from each other at the same speed in opposite directions. Five minutes later, Basil changed his mind, turned back and, with increased speed, ran to catch up with Peter. By how many times did Basil increase his speed if he caught up with Peter five minutes after he turned back?

Problem 11. A hurrying tourist. After completing half of the route, the tourist increased his speed by 25% and arrived at his destination half an hour ahead of time. How long did it take the tourist to complete the route?

Problem 12. A cyclist left point *A* on the way to point *B*. Simultaneously, a pedestrian started from point B to point A to meet the cyclist. After their meeting, the cyclist turned back, and the pedestrian continued on his way. It is known that the cyclist returned to point A 30 minutes before the pedestrian and his speed was five times the speed of the pedestrian. How much time did the pedestrian spend on the road from A to B?

Answers and solutions

Problem 7. Answer. This is not possible.

Solution. To save a minute per mile, one needs to cover each mile one minute faster, that is, in our case (when the speed is 1 mile per minute), in 0 minutes.

Problem 8. Answer. 200 metres.

Solution. Imagine first that the tortoise does not move; then the lion cub approaches it at a speed 10 times greater than the speed of the tortoise. But, in reality, during this time the tortoise will crawl away from the lion cub by 20 metres more, so the cub will have to run 200 metres.

Problem 9. Answer. 200 metres.

Solution. Since the initial speed of the column is 1000 m/min, it follows that the "tail" of the column will be at the traffic police post 0.3 minutes later than its "head". During that time, the "head" will cover $40/60 \cdot 0.3 = 0.2$ (km) $= 200$ (m), which is the length of the column.

Problem 10. Answer. Three times.

Let's make the following drawing: O is the place where the quarrel took place, B and P are the points where Basil and Peter were located 5 minutes after the quarrel, respectively.

In the next 5 minutes, Peter covered a distance equal to OP and arrived at point K. Therefore, Basil had to cover the distance BK, which is three times greater than PK, in the same perod of time.

Problem 11. Answer. 4 hours 30 minutes.

Solution. If the tourist had walked the entire route at the increased speed, he would have finished his trip one hour earlier. If he had walked, at the same speed, for another hour, he would have covered the "extra" 25% of the distance.

Since, in one hour of walking at increased speed, the tourist walks 25% of the distance, it follows that walking the whole distance at that speed will take 4 hours. Hence, at normal speed, the tourist will need 5 hours for the whole journey. In addition, it is known that, having increased the speed in the second half of the way, the tourist ended his journey 30 minutes earlier.

Problem 12. Answer. 45 minutes.

Solution. Let us mentally divide the segment AB into six equal parts.

1) Since the speed of the cyclist is five times the speed of the pedestrian, it follows that, at the time of their meeting, the cyclist covered five parts and the pedestrian walked one part.

2) Then the cyclist turned back and returned to point A, having covered five parts of the route and, during this period of time, the pedestrian covered only one part.

3) Therefore, the pedestrian completed the remaining four parts of the route in 30 minutes. Thus, for the entire route (six parts), the pedestrian must spend 45 minutes.

Lesson 6

Downstream and Upstream . . .

Three boatmen can transport
three bags of potatoes down the river,
one each, in one hour. In what time
will one boatman do this whole job?

An old problem

River traffic problems and other similar problems are considered. The ideas of "adding speeds (velocities)" and appropriately choosing a "reference frame" are used in their solution.

The following agreements are adopted:

1) If a body moves "with the current" (downstream), then its speed is composed of the speed of the body in standing water v and the speed of the river flow u: $w = v + u$.

2) If a body moves "against the current" (upstream), then its speed is considered equal to $w = v - u$.

3) Rafts move at the speed of the river flow.

Problem 1. Two swimmers simultaneously jumped off a raft and swam in opposite directions: one downstream and the other upstream. After 5 minutes, they simultaneously turned and swam back. Which of the swimmers will reach the raft first?

Answer. They will arrive at the same time.

Solution. Relative to the raft, each swimmer moves at a constant speed, be it downstream or upstream. Therefore, after covering some distance (from the raft) in 5 minutes, the swimmer will also need 5 minutes to return.

Problem 2. A powerboat covers 90 km downstream and 70 km upstream in the same period of time. How far will a raft travel in that same period of time?

Answer. 10 km.

Solution. First method. Assume that the powerboat and the raft began to move at the same time and in the same direction and that the boat, having covered 90 km in t hours, turns around and continues until it meets the raft. It is clear from the solution of the previous problem that this will take the same amount of time (t hours). Therefore, the boat will cover 70 km before meeting the raft.

Thus, the raft covered 20 km in twice the time ($2t$ hours); hence, in the given time (t hours), it covered 10 km.

We have shown that the difference between the speed of the boat going with the current and against it is equal to twice the speed of the current. The same result can be obtained from the speed addition rule.

Second method. Speed is the distance covered per unit of time. For the "unit of time", we take the time it takes the boat to travel 90 km downstream (or 70 km upstream). Then the speed of the river flow is 10 km per this "unit of time". Therefore, in that "unit of time", the raft will have covered 10 km.

In algebraic terms: let T be the time in question, then $90/T - 70/T = 20/T$ is twice the speed of the current, and so on.

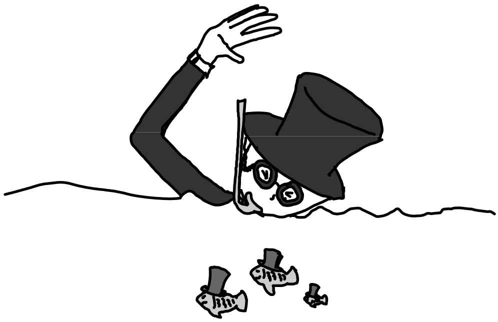
Problem 3. Down the Volga river. A river boat takes 5 days to go from Nizhny Novgorod to Astrakhan (down the Volga river), and 7 days to come back. How long will it take a raft to float down from Nizhny Novgorod to Astrakhan?

Solution. A river boat moving downstream covers one fifth of the course in a day, while moving upstream, one-seventh of the course. Therefore, the doubled speed of the current measured in parts of the course per day is $1/5 - 1/7 = 2/35$.

Thus, in a day, a raft will cover 1/35 of the course per day, and it will take it 35 days for it to float down the whole course.

One can do without fractions. Divide (mentally) the distance between the cities into 35 equal parts. Then, in a day, the river boat moving with the current covers a distance equal to seven such parts and when moving upstream, a distance equal to five parts. The difference of the speeds "downstream" and "upstream" is equal to twice the speed of the river flow. Hence, the speed of the current is one part per day. Thus, the raft will float down from Nizhny Novgorod to Astrakhan in 35 days.

Problem 4. The floating hat. While canoeing down the river, Peter lost his hat under the bridge, but continued to paddle in the same direction. After 15 minutes, he noticed the loss, returned, and found the hat floating 1 km from the bridge. What is the speed of the current of the river?



Answer. 2 km/h.

Solution. Peter recovered his hat 15 minutes after noticing its loss. Consequently, the hat floated 1 km in half an hour.

It is important to emphasize that the course of the solution is not affected by the direction in which the boat was going, be it upstream or downstream. The following analogy helps to understand this fact. Suppose that a similar situation occurred on a train: a passenger left his hat on a shelf and returned. How long will it take him to retrieve his hat? The students answer that the passenger will return in 15 minutes. How can one find out how far the train will travel during this period of time? The speed of the train must be multiplied by the time. This is a convincing analogy.

Note that in the physical language, this analogy actually means that the hat floating on the surface of the river is chosen as the origin of a coordinate system.

Problem 5. Counting escalator steps. Two people walk down, without skipping any steps, on a down going escalator. One descends faster than the other. Which of them will touch more steps and why?

Answer. The one who walks faster.

Solution. Each person sees the same number of steps at the initial moment. During the descent, some of the steps will disappear under the escalator treshold and the rest will be counted. So there will be fewer steps disappearing under the threshold for the person who descends faster.

Problem 6. A clever trick. With its tank full of fuel, a boat can cover 72 km upstream or 120 km downstream. What is the greatest distance that the boat can cover, leaving enough fuel for the return journey?

Answer. 72 km.

Solution. When going upstream, 1/72 of the tank contents is spent for each kilometre covered, and when going downstream, 1/120. Therefore, in total, 1/45 of the tank is spent for each kilometre. So the boat can cover 45 km.

This would seem to be all? But no.

One can pull the following clever trick: float 72 km downstream with the engine shut off and then return with the engine switched on.

Problems for Individual Solution

Problem 7. The escalator of the underground takes a passenger walking down on it in one minute. If the passenger walks twice as fast, she or he will descend the escalator in 45 seconds. How long will it take a standing passenger to descend the escalator?

Problem 8. Piers. A river boat covers the distance between the piers A and B downstream in 5 hours and upstream for 6 hours. How long will it take a raft to float down this same distance?

Problem 9. A raft floating down the river covers some distance in 18 hours, while a motor boat moving upstream covers this same distance in 3 hours. How long will it take the motor boat to cover this distance, moving with the current?

Problem 10. A motor boat runs downstream at the speed of 28 km/h and upstream at 20 km/h. It took 6 hours for the motor boat to move from pier A to pier B and then back to A. What is the distance between these piers?

Problem 11. Eddie and Paul walk down a descending escalator without skipping any steps. Paul manages to take two steps while Eddie takes only one. Paul, while descending, took 28 steps in all, while Eddie, only 21 steps. How many steps are there in the visible part of the escalator?

Problem 12. Peter and a bully. Peter was riding an escalator. When he was in the middle of it, a hooligan ran past him, grabbed his cap, and threw it onto the escalator moving in the opposite direction. Peter wants to get the cap back as quickly as possible. Should he run up or down?

Answers and solutions

Problem 7. Answer. 1.5 minutes.

Solution. Let's measure the speeds in "escalators per minute". In the first case, the speed is 1 and, in the second case, 4/3. The increment 1/3 is the passenger's speed with respect to the moving escalator (in the escalator's frame of reference). This means that the speed of the escalator (with repect to the walls) is $1 - 1/3$ and the time of descent of the standing passenger is $1 : 2/3 = 1.5$ min.

Problem 8. Answer. 60 hours.

Solution. In one hour, the riverboat covers 1/5 of the downstream route and 1/6 of the upstream route. The difference between 1/5 and 1/6, which is equal to 1/30 of the route, corresponds to twice the speed of the current. Therefore, in an hour, a raft will cover 1/60 of the route

Problem 9. Answer. 2 hours 15 minutes.

Solution. The speed of the motor boat going downstream differs from its upstream speed by twice the speed of the current. Therefore, in an hour, the boat goes downstream $1/9 + 1/3 = 4/9$ of the way. Thus, the boat runs the whole way downstream in 9/4 hours.

Problem 10. Answer. 70 km.

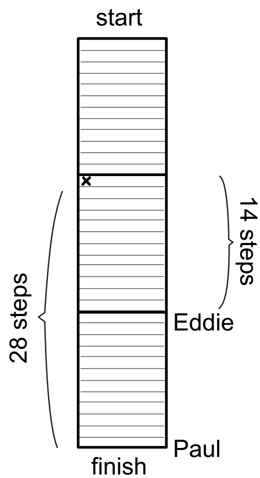
Solution. Assume that the distance between the piers is 140 km. Then the boat will cover this distance downstream in 5 hours and upstream, in 7 hours. But $5+7 = 12$ is twice as much as 6. Therefore, it is necessary to reduce the distance by half; then the travel time will also be reduced by half.

We have solved the problem by applying the so-called method of false position, or *regula falsi*, which was very popular in the Middle Ages.

Why did we choose 140? Because 140 is the least common multiple of 20 and 28. We also took into account the fact that the length of each part of the route is directly proportional to the time needed to cover it.

Problem 11. Answer. The length of the escalator is 42 steps.

Solution. Mark the escalator step that the boys stepped on at the beginning of the movement. Until Paul gets off the escalator, he will always be twice as far from the mark as Eddie. When Paul got off, Eddie had only walked 14 steps (half as many as Paul). Since 14 is 2/3 of 21, it follows that Eddie had then gone down 2/3 of the length of the escalator. Thus, at this moment, Eddie and the marked step divide the escalator into 3 equal parts (see the picture) and the total number of steps is 42.



Problem 12. The answer to the problem depends on the relation between the values of u and v: in the case when the speed u of the escalator is more than one third of Peter's speed v (speed with respect to the escalator), then it is better to run "in pursuit of the cap"; otherwise the direction of the motion does not matter.

Solution. Suppose that both escalators are joined to form a ring. Then we have a closed circular moving track, on which Peter and his hat are located. On such a track, it does not matter whether Peter runs "towards" the cap or "after" it, since his speed relative to the cap is equal to his speed v relative to the escalator.

Is it always possible to argue in this way? No, this argument is only valid if Peter has time to reach the cap before it is stopped by the escalator threshold. If he does not have that much time, then he should run "after" the cap. After all, at some point, the cap will stop and Peter's speed relative to the cap will increase by the speed of the escalator u and become equal to $v + u$.

The problem has a "hidden parameter", which was not mentioned in the condition but essentially affects the answer.

Additional Problems

What is this about: two feet sat on three,
and when four came and dragged one away,
then two legs, grabbing three, threw them
at four, so four left one?

An old problem

Problems for independent work by the students and for the organization of mathematical competitions are given. In addition to these problems, the teacher may need their modifications, that is, problems with different numbers and names, mathematically equivalent to the given ones (for example, Problem 8 of Lesson 6 is a modification of Problem 3 of the same lesson). As a rule, we do not explicitely present such modifications, assuming that the teacher wiol be able to create them independently.

Problems of the type given here can be found in the popular literature (see, for example, [1], [2–5], [6–10] in the list of references).

Problem 1. A petrol problem. Is it possible to distribute 50 gallons of petrol into three tanks so that the first tank will contain 10 gallons more than the second one and then, after pouring 26 gallons from the first tank into the third one, the third tank will contain the same amount of petrol as the second one?

Problem 2. Two sacks contain 140 pounds of flour. If we transfer 1/8 of the flour from the first sack to the second one, then both sacks will contain the same amount of flour. How much flour is there in each sack?

Problem 3. Once upon a time a king rewarded a peasant with one apple from his garden. The peasant went to the garden and saw that it was enclosed by three fences with only one gate in each fence and one watchman at each gate. The peasant came up to the first watchman, showed him the royal decree, and the watchman answered: "Go and pick the apples, but when you leave, you must give me half of the apples that you are carrying plus one more apple". The second and third watchmen told him the same thing. How many apples should the peasant pick, so that after paying off the guards, he should have precisely one apple left?

Problem 4. In search of adventure. The distance between the musketeers Athos and Aramis, riding on a straight road, is 20 leagues. In one hour, Athos covers 4 leagues and Aramis covers 5 leagues. What distance between them will there be in 1 hour?

Problem 5. What is more profitable? In which of the two cases will the depositor get more money: if the bank pays 12% of the deposit once a year or if it pays 1% every month? (The accrued income is immediately added to the amount available on the account.)

Problem 6. Seawater contains 5% of salt. How much fresh water must be added to 40 pounds of seawater so that the salt content should be reduced to 2%?

Problem 7. At an international conference, 85% of the delegates speak English and 75% speak Spanish (each of the delegates speaks at least one of these languages). What is the percentage of the delegates who speak both languages?

Problem 8. Family budget. There are four people in the family. If Mary's scholarship is doubled, then the total family income will increase by 5%. If, instead, Mum's salary is doubled, then the income will go up by 15%. If Dad's salary is doubled, then it will go up by 25%. How will the income of the whole family increase if the grandfather's pension is doubled?

Problem 9. Ali Baba and the thieves. Last year, Ali Baba was chased by 40 thieves, but they were unable to catch him. This year, the leader of the thieves promised to increase the number of thieves by at least 47%. How many thieves will be chasing Ali Baba this year?

Problem 10. Piglet was presented with several coloured balloons for his birthday, of which 45% were red. Piglet gave one blue balloon and one green one to Eeyore. Exactly half of the balloons which Piglet had after that were red. How many balloons did he receive for his birthday?

Problem 11. In a three-litre jar, there is a litre of pure alcohol and, in a five-litre jar, there is a litre of water. One is allowed to pour any amount of liquid from one vessel to another. Is it possible, as a result of a certain number of pourings, to obtain a 54% concentration solution of alcohol in the five-litre jar? Water and alcohol mix uniformly.

Problem 12. The numbers *A* and *B*. The number A is greater by 400% than the number B. By how many percent is the number B less than the number A?

Problem 13. Tom, Dick, and Harry produce bootleg liquor. Tom's device produces a liquid of alcoholic content a%, which fills a standard bottle in a hours, Dick's device produces a liquid of alcoholic content b%, which fills a similar bottle in b hours, and Harry produces a liquid of c%, which fills the standard bottle in c hours. To speed up the process, the friends placed the outgoing tubes of their devices into one standard bottle, which was filled in 24 hours. Find the alcoholic content of the resulting mixture.

Problem 14. Two fountains. The first fountain fills the pool in 2 hours and 30 minutes and the second, in 3 hours and 45 minutes. How long will it take both fountains, working together, to fill the pool?

Problem 15. The management of a forestry enterprise decided to cut out a part of the forest. To reassure the environmentalists, the director of the enterprise said: "We will only cut out pines; they constitute 99% of the trees in the whole forest; after cutting, they will constitute 98% of all trees". What percentage of the forest will be cut out by the forestry enterprise?

Problem 16. Mother and daughter. The mother can weed out a garden bed in 7 hours of work and, together with her daughter, in 5 hours. How long will it take the daughter to weed the garden bed alone?

Problem 17. There are 8 sheep in a flock. The first eats a portion of hay in one day, the second in two days, and so on, and the eighth sheep eats this portion of hay in eight days. Will the first two sheep or the remaining six be quicker in eating one portion of hay?

Problem 18. Leo Tolstoy's problem. A team of mowers had to mow two meadows, one of which was twice the size of the other. The whole team was mowing the larger meadow for half a working day. In the afternoon, the team split up. The first half remained on the larger meadow and finished it by the evening, while the second half mowed the smaller meadow, of which, in the evening, an unmowed part remained; it was mowed the next day by one mower who worked all day. How many mowers were there in the team?

Problem 19. (Following Isaac Newton.) The grass on a meadow grows uniformly thick and at a constant rate. Seventy cows can eat all the grass in 24 days and 30 cows, in 60 days. How many cows could graze on this meadow indefinitely long?

Problem 20. Winnie-the-Pooh and Piglet sat down at the table for a little refreshment and began to eat honey simultaneously from the same pot, without being distracted by conversations. If Winnie ate at Piglet's speed, then his meal would last 4 minutes longer and if, on the contrary, Piglet ate at Pooh's speed, his meal would take 1 minute less. How long did it take Pooh and Piglet eating together to finish off the honey from the pot?

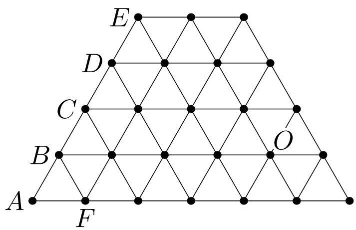
Problem 21. H.Dudeney's problem. If a hen and a half lays an egg and a half in a day and a half, how many and a half who lay better by half will lay half a score and a half in a week and a half?

Problem 22. Two pedestrians and a fly. Alice and Bob walk towards each other along a straight road towards each other at the speed of 5 km/h each. The initial distance between them is 10 km. A fly that flies at the speed of 14 km/h takes off from Alice's brow at the initial moment, reaches Bob, turns around and immediately flies back to Alice, then again to Bob, and so on. What distance will the fly cover by the time Alice and Bob meet?

Problem 23. Meeting time. At 9 a.m, the cyclists Andrew and Peter left two villages and startde to travel towards each other. Andrew can cover the distance between these villages in 6 hours and Glenn, in 4 hours. Did they meet: before noon or later? At what time exactly did they meet?

Problem 24. Two travellers simultaneously left the village A and began to walk towards village B. The first traveller walked at the speed of 5 km/h for half of the time he spent on the entire journey and then walked at the speed of 4 km/h. The second traveller walked the first half of the way at the speed of 4 km/h and the second half of the way, at the speed of 5 km/h. Which of the travellers was the first to arrive?

Problem 25. A triangular grid is made of a cord that can burn. Fire spreads along the cord at the same speed in all directions (each link burns out in exactly one minute). Which of the marked grid links (AB, BC, CD, DE, or AF) will be the last to burn if the grid is set on fire at O? In what time will this happen?



Problem 26. The road from home to school is covered by Jimmy in 20 minutes. One day, on his way to school, it occured to him that he left his pen at home. If he then continued on his way to school, he would arrive 8 minutes before the bell rings, and if he went back home to get his pen and then back to the school, he would be 10 minutes late. What part of the way to school had he covered before he found out that his pen was left at home?

Problem 27. Three cars left the city in one direction: the second 10 minutes after the first and the third 20 minutes after the second. Thirty minutes after its departure, the third car caught up with the second car and, 10 minutes later, it caught up with the first car. How long after the second car left the city did it catch up with the first car?

Problem 28. Three friends Andy, Alex, and Aaron, want to cover 60 km in three hours. They have a two seat motorcycle which runs at 50 km/h. Can they make it on time if their walking speed is 5 km/h? What exactly should they do?

Problem 29. A wolf and a hare competed in a 5.5 miles race. The race was observed by several judges so that the entire track was under their supervision. Each judge could see 1 mile of the track. At the end of the race, it turned out that the mile section of each judge was covered by the wolf in 8 minutes and by the hare in 8 minutes and 1 second. Could it happen that the hare covered the whole track faster than the wolf?

Problem 30. Crossing a desert on Mars. A traveller must cross an 80 km wide desert on Mars. It is known that, in a day, he can walk 20 km, carrying with him a supply of oxygen for 3 days. Therefore, he must prepare intermediate stations, leaving oxygen tanks there. Can he cross the desert in 6 days?

Problem 31. A dilapidated bridge. Dad, Mum, Son, and Grandma have to cross the bridge at night. They have only one lantern, without which it is impossible to cross the bridge in the dark. Unfortunately, the bridge is in so dilapidated that it can carry a maximum of two persons. Another problem is that the four persons move at different speed. Dad crosses the bridge in one minute, Mum in two minutes, Son in five minutes, and Grandma in ten minutes (two persons always walk together at the speed of the slower person). Can they all cross the bridge in 17 minutes if it is prohibited to light up the bridge from afar?

Problem 32. Three brothers, John, Peter, and Nick were returning home from fishing, and a keg of wine was waiting for them at home. John walked twice slower than Peter and thrice slower than Nick. Nick came home first and began to drink the wine, and, by the time Peter arrived, he had drunk one-seventh of the keg. When Peter came, they began to drink wine together. It is known that the brothers drink wine at the same rate. Was there any wine left for John?

Answers and Hints

Problem 1. Answer. No, it is impossible.

Note that both the first and the second tank must contain at least 26 gallons of petrol. Therefore, it is impossible to pour 50 gallons of petrol in this way.

Problem 2. Answer. 80 lb of flour in the first sack and 60 lb in the second one.

If we take one-eighth of the flour from the first sack, then seven-eighths of the original flour remains, that is, there will be 70 lb (equal amount in the sacks now!). Therefore, one-eighth of the flour from the firstsack is 10 lb, that is, there was 80 lb in the first sack and 60 lb in the second one.

Problem 3. Answer. 22 apples.

Imagine that we are watching this as a film from its end to its beginning. What do we see?

At the "beginning" (that is, at the actual end of the film!): the peasant has one apple.

1) "After" the last watchman (we see him first): the peasant has $(1+1)\times 2=4$ apples.

2) "After" the last-but-one watchman (we see him as the second one): $(1+4)\times 2=10$ apples.

3) "After" the first watchman (we see him as the last one): $(1+10)\times 2=22$.

Problem 4. Answer. 11 leagues, 29 leagues, 19 leagues, or 21 leagues (depending on the direction of travel).

In which direction did each of the musketeers ride? There is nothing about this in the statement of the problem. If they ride towards each other, then the distance between them will be 11 leagues. In the other three cases (make drawings!), the possible answers are 29 leagues; 19 leagues; 21 leagues.

Problem 5. Answer. It is more profitable to accrue income once a month.

The depositor put a certain amount of money in the bank. Then under the first type of payment, he will additionally receive 12% of the invested amount at the end of the year.

Under the second type of payment, he will receive, in the first month, 1% of the invested amount, in the second month, 1% of the slightly increased amount and in the third month, 1% of the even larger amount, and so on.

Problem 6. Answer. 60 lb.

40 lb of seawater contains $40 \cdot 0.05 = 2$ lb of salt, which amounts to 2% in the new solution; hence, there must be 50 times more solution, that is, 100 lb.

Problem 7. Answer. 60%.

Note first that $85\% + 75\% = 160\%$.

At the expense of whom was the surplus formed? At the expense of those people who speak both languages since these were counted twice. So at least 60% of the delegates speak both languages.

It is useful to discuss what happens if we remove the condition in parentheses, that is, if we assume that there may be delegates who speak neither Spanish nor English. We get a new answer: from 60% to 75%.

Problem 8. Answer. 55%.

If income of all family members were suddenly paid twice as much, the total income would increase by 100%. Of this increase, 5% would come from Mary, 15% from Mum, 25% from Dad; so, the remaining 55% would come from Grandpa.

Problem 9. Answer. At least 59.

Note that 5% of 40 is 2 and 45% of 40 is 18. Further, 47% of 40 is more than 18, but less than 19.

Problem 10. Answer. 20 balloons.

Note that, after Piglet gave the blue and the green balloon to Eeyore, the number of red balloons that Piglet had did not change and we know that it became equal to the number of the other balloons in his possession. This means that the number of balloons in Piglet's possession had decreased by 10% (if 45% is half, then 90% is all!). Thus, two balloons make up 10% of the number of balloons given to Piglet; hence the total of 20 balloons were presented to him.

Problem 11. Answer. No, it is not possible.

Since the water is in a five-litre jar and there is no third vessel into which it can be poured, it follows that all the water will be used to prepare the solution. At first, the alcohol concentration in the five-litre jar is not higher (even lower) than in the three-litre one and this non-strict inequality persists during pourings. We have equal amounts of water and alcohol; therefore, it is impossible to get a solution with concentration of more than 50% in the five-litre jar.

During the discussion, the attention of the students should be drawn to the fact that the actual volumes of the jars does not play any role.

Problem 12. Answer. By 80%.

The number A is 400% greater than the number B, hence the number A is five times greater than the number B. If we take A for 100%, then the number B will be 20% of A.

Problem 13. Answer. 72%.

Divide the bottle mentally into 100 equal "cups" and observe the quantity of pure alcohol in it. A full bottle of a% alcoholic content contains a "cups" of alcohol; therefore, Tom's device produces one "cup" of alcohol per hour. The same is true for the other two devices. During the 24 hour period, the three devices will produce 72 "cups" of alcohol and the alcoholic content of the mixture will be 72%.

Problem 14. Answer. In 1 hour 30 minutes.

In an hour, the first fountain fills 2/5 of the pool and the second one, 4/15 of the pool. Together, they will fill 2/3 of the pool in an hour, and the remaining third of the pool will be filled in another half-hour.

One can also do without fractions: in 15 hours, the first fountain will fill six pools and the second one will fill four pools. In total, together they will fill ten pools in 15 hours. Therefore, they will fill one pool in 1 hour 30 minutes.

Problem 15. Answer. 50%, or one-half.

Indeed, initially, trees other than pines (they were not cut down) constituted 1% of all trees, that is, 1/100 of all trees and, after the cutting, they will constitute 2%, that is, 1/50 of the trees.

Here, students should be reminded of Problem 5 from Lesson 2. The statements of the problems are different, but the mathematical essence is the same!

Problem 16. Answer. In 17 hours and 30 minutes.

In an hour, the daughter will do $1/5 - 1/7 = 2/35$ of the work. Therefore, she will do all the work in 17.5 hours.

Alternatively, divide all the work into 35 equal parts, then the mother does 5 parts per hour and, together with the daughter, 7 parts of the work per hour. Accordingly, the daughter can do 2 parts of the work per hour, so all the work will be done by her in 17 hours and 30 minutes.

Problem 17. Answer. The first two sheep.

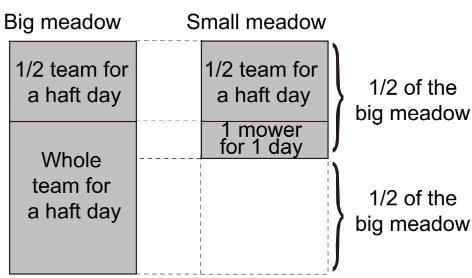
Let's compare the daily rations.

Let's check which sum is greater: $1 + 1/2$ or $1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8$.

Since $1/3 + 1/6 = 1/2$ and $1/4 + 1/5 + 1/7 + 1/8 < 1/4 \cdot 4 = 1$, one obtains the answer.

Without fractions, the solution is more difficult.

Problem 18. Answer. Eight mowers.



Let there be K mowers in the team. Then, in the first half of the day, they mowed down a field of area Ka (a is the area mowed in a working day by a single mower). During the afternoon, $0.5Ka$ was mowed down on the bigger meadow, whence the area of the bigger meadow is $1.5Ka$. Then the area of the smaller meadow is $0.75Ka$, so the area of its unmowed part is $0.25Ka$. Therefore, one mower mows $0.25Ka$ per day, and the number of mowers in the team can be found as the ratio of $2Ka$ by $0.25Ka$.

Problem 19. Answer. Three cows.

Note that 30 cows ate as much grass in 60 days as one cow can eat in 1800 days, while 70 cows ate as much grass in 24 days as one cow can eat in 1680 days. Therefore, a cow eats as much grass in 120 days as will grow in 36 days. So, in one day, the meadow will produce the amount of grass necessary for feeding $3\frac{1}{3}$ cows.

This is a good time for the teacher to say something about Sir Isaac Newton.

Problem 20. Answer. For 2 minutes 40 seconds.

If both friends ate at the speed of Pooh, then they would eat up all the honey 5 minutes faster than if they both ate at the speed of Piglet. Under the assumptions above, each eats half of the pot. It follows that Winnie consumes half of the pot of honey 5 minutes faster than Piglet. Therefore, when they ate together and Winnie had eaten half of the pot, there was as much honey left in the pot as Piglet could eat in 5 minutes. But, together, they finished the remainder in 1 minute. During that time, Piglet ate 1/5 of the remainder, and hence 4/5 of the remainder was eaten by Winnie-the-Pooh, which means that Winnie ate four times faster than Piglet. Hence the remainder was 3/4 of one-half of the pot, Winnie ate 3/5 of one-half of the pot in one minute, and so he ate the first half in 5/3 minutes.

Problem 21. Answer. Half a hen.

If one and a half hens lay one and a half eggs in a day and a half, then one hen will lay one egg in a day and a half. A hen that lays eggs better by half, that is, and a half times faster, will lay one and a half eggs in one and a half days (one egg per day). Therefore, this hen will lay ten and a half eggs (half a score means ten) in ten and a half days (a week and a half). Subtracting half a hen, we obtain the answer.

Problem 22. Answer. 14 km.

Alice and Bob will meet in one hour, so the fly will be flying for 1 hour.

Problem 23. Answer. At 11:24.

The first question of the problem can be answered without computation. Indeed, Andrew covers the distance between the villages in 6 hours; therefore, at 12:00, he will be half way between the villages. But Glenn walks faster. Therefore, at 12:00, Glenn will have passed the midpoint between the villages, and so the meeting will take place on Andrew's half of the road, that is, earlier than 12:00. When exactly? Let's calculate the time: Andrew covers 1/6 of the itinerary in an hour and Glenn, 1/4 of the itinerary; therefore, in one hour, the distance between them will diminish by 5/12 parts of the whole itinerary. So they will meet in $1 : 5/12 = 12/5$ hours, that is, at 11:24.

Problem 24. Answer. The first one arrived earlier.

The first hicker, walking at 5 km/h, covered more than half the distance, while the second one covered exactly one-half of distance.

Problem 25. Answer. The last links to burn out are AB and AF, and this will happen in 5 minutes.

The fire will reach each of the points B, C, D, E, F in 4 minutes. Therefore, the links AB and AF will be the last to burn, and this will happen in 5 minutes (the links DE, CD and BC will burn out after 4.5 minutes, because each of them will burn from both ends!).

Problem 26. Answer. 9/20 of the path.

If Jimmy had left the house 8 minutes later, then, returning for the pen, he would have been 18 minutes late for school. These 18 minutes are needed to cover the already completed path twice. Therefore, at the time he found out he had left the pen, Jimmy had been walking for 9 minutes. And since it takes him 20 minutes to get to school, he covered 9/20 of the distance in 9 minutes.

Problem 27. Answer. In 200 minutes.

1) In 15 minutes, the third car covered the same distance as the second car in 25 minutes.

2) In 20 minutes, the third car covered the same distance as the first car in 35 minutes.

3) Therefore, in 60 minutes, the third car will cover the same distance as the second car in 100 minutes and the first in 105 minutes.

4) Therefore, in 100 minutes, the second car will cover the same distance as the first car in 105 minutes, and since the first car is ahead of the second one by 10 minutes, it will take 200 minutes for the second car to catch up with the first car.

Problem 28. Answer. Yes, they can.

They can adopt the following procedure:

1) During the first hour, Andy and Alex ride on the motorcycle for 50 km, while Aaron walks 5 km. Then Andy walks for 2 hours and arrives on time.

2) During the second hour Aaron walks 5 km more, while Alex goes back 40 km and meets Aaron.

3) During the third hour, Alex and Aaron cover the remainder of the way on the motorcycle and manage to arrive on time.

Problem 29. Answer. Yes, it could.

Let's give an example of a situation in which this is possible.

Let the wolf run at the constant speed of 7.5 mph, then, it covers every mile in 8 minutes.

The hare runs more cunningly, not at uniform speed:

1) He covers the first half-mile ("fast" section) in 2 minutes 1 second.

2) He covers the second half-mile ("slow" section) in 6 minutes.

3) He covers the third half-mile ("fast" section) again in 2 minutes 1 second, and so on.

Hence the wolf covers 5.5 miles in 44 minutes, while the hare covers six "fast" sections in 12 minutes 6 seconds and five "slow" sections in 30 minutes. Thus, the hare will cover the distance in 42 minutes 6 seconds, and this is faster than the wolf.

Let us show that, in this way, the hare covers every mile in exactly 8 minutes and 1 second.

Let us colour the half-mile segments of the 5.5 mile distance in two colours (black and white) so that the "slow" sections are black and the "fast" sections are white. Then each interval of length 1 mile entirely contains either one of the black segments or one of the white segments and the remaining part of this interval is coloured differently. This means that in running any mile, the hare runs rapidly for a half-mile (which takes 2 minutes and 1 second) and the other half-mile, he runs slowly (which takes 6 minutes). In total, the hare covers every mile in 8 minutes 1 second.

The problem is difficult, but it can be discussed with 13-year olds.

Problem 30. Answer. Yes, he can.

1) In the first two days, he can set up a station 20 km from the starting point, and leave a one-day oxygen supply there; and then return to the starting point.

2) In the next four days, he will succeed in crossing the desert, taking with him a supply of oxygen for three days, because he will replenish it at the intermediate station.

Problem 31. Answer. Yes, they can.

The following scheme for crossing the bridge is possible:

1) dad and mum, two minutes;
2) dad back with the lantern, one minute;
3) grandson with grandma, ten minutes;
4) mum back with the lantern, two minutes;
5) dad and mum, two minutes.

Total: 17 minutes.

Problem 32. Answer. No, he didn't get any.

John walked with half the speed of Peter and one-third of the speed of Nick. Therefore, John spent twice as much time on the road as Peter and three times as much time as Nicholas. We can say that John, Peter, and Nicholas spend $6t$, $3t$, and $2t$ hours on the road, respectively. Thus Nickolas was drinking ale alone for t hours and, together with Peter, for $3t$ hours. Nicholas alone had drunk one-seventh of the keg and, together with Peter, six-sevenths of the keg, that is, by the arrival of John, the keg was empty.

Appendix. Handout Materials

Lesson 1

Problem 1. Gathering mushrooms. John picked 36 more mushrooms than his sister Mary. On the way home, Mary asked him: "Give me some mushrooms, so that I'll have as many mushrooms as you." How many mushrooms should John give her?

Problem 2. Geese and pigs in the barnyard. Peter counted the number of their heads: there were 30; then he counted the number of their legs: there were 84. How many geese and how many pigs were there in the barnyard?

Problem 3. Donkeys were grazing in a clearing. Several boys came up to them. "Let each one of us mount a donkey", suggested the eldest boy. But then two boys were left without donkeys. "Let's try to mount in twos", the eldest boy suggested again. Then two boys mounted each donkey, but this time one donkey was left without a rider. How many donkeys and how many boys were there in the clearing?

Problem 4. How old is John? When John was asked his age, he thought about it and said: "I am three times younger than my dad, but three times older than Tommy". Then little Tommy appeared and said "Daddy is 40 years older than me". How old is John?

Problem 5. An ancient problem. To buy an ice cream, Peter was short by seven pence and Mary, by one penny. Then they joined together all the money they had. But they still did not have enough to buy even one portion. How much did a portion of ice cream cost?

Problem 6. The teacher posed a difficult problem during a lesson. As a result, the number of boys in the class who solved this problem was equal to the number of girls who did not solve it. Who is in the majority in the class: those who solved the problem or the girls?

Lesson 2

Problem 1. Peter bought two books. The first book was cheaper by 75% than the second one. By how many percent is the second book more expensive than the first one?

Problem 2. A customer's dream. The price of potatoes dropped by 20%. By how many percent more potatoes can one buy with the same amount of money?

Problem 3. Where is it cheaper? In two nearby shops, the price of milk was the same. Then, in one shop, its price was cut by 40%, while, in the other, it was first cut by 20%, and later by 25%. In which of the shops is the milk cheaper now?

Problem 4. Garden plot. The length of a rectangular plot of land was increased by 50%, but its width was reduced by 10%. How has the area of the plot changed?

Problem 5. Dried mushrooms. The humidity of fresh mushrooms is 99% and that of dried ones is 98%. How does the mass of mushrooms change after drying?

Problem 6. Two positive numbers. One positive number was increased by 1% and the other one was increased by 4%. Can the sum of these numbers increase by 3% as a result?

Lesson 3

Problem 1. Carlson at the Kid's birthday. The Kid eats a jar of jam in 6 minutes and Carlson, twice as fast. How long will it take them to eat this jam together?

Problem 2. An ancient problem. A man drinks up a cask of ale in 14 days and, together with his wife, he will finish the same cask in 10 days. How many days will it take his wife alone to drink up the same cask?

Problem 3. Cats and mice. Five cats caught five mice in five minutes. How many cats can catch ten mice in ten minutes?

Problem 4. Beavers and a dam. Ten beavers calculated that they can build a dam in 8 days. After they had worked for two days, it turned out that, in view of an impending flood, they had to finish the work in 2 days. How many beavers do they need to call for help?

Problem 5. Firewood. In one day, two workers can either saw up three logs into chunks or chop up six such chunks into firewood for the chimney. How many such logs should they saw into chunks in a similar way in order to have time to chop up all of the resulting chunks on the same day?

Problem 6. A classical topic. Three workers have dug a hole. They worked in turns; each worked as long as it would take the other two to dig half the hole working together. Working like this, they have dug out a certain hole. How many times faster would they have finished the job if they had worked together?

Lesson 4

Problem 1. Dragonfly and two ants. Two ants, Basil and Cyril, went on a visit to their friend, the Dragonfly. Basil crawled all the way, while Cyril rode half the way on a Caterpillar, whose speed was half the speed of crawling, and rode the rest of the way on Grasshopper, which was ten times faster than crawling. Which of the ants came first if they started at the same time and covered the same distance?

Problem 2. Caterpillar on a pole. A caterpillar crawls along a pole, climbing 5 feet upwards during the day and descending 4 feet during the night. The height of the pole is 10 feet. How long will it take the caterpillar to reach the top of the pole?

Problem 3. Circular route. Two buses run on a circular route, with a 21 minute interval between them. What will the interval between buses be if there are three buses running on this route?

Problem 4. The road to work. An engineer usually arrives by train to the train station at 8 a.m. At 8.00 a.m. sharp, a car from the factory pulls up to the train station and takes the engineer to the factory. Once, the engineer arrived at the train station at 7 a.m and decided to walk to meet the car. When he met the car, he got in and arrived at the factory 20 minutes earlier than usual. At what time did the engineer meet the car?

Problem 5. Encounters on the way. Two ferries depart simultaneously from the opposite banks of the river and cross the river at constant speed perpendicularly to the banks. The ferries meet each other at the distance of 720 m from the nearest bank. When they reach the shore, they immediately go back and then they meet at 400 m from the other shore. What is the width of the river?

Problem 6. Train on the bridge. The train passes a bridge 450 m long in 45 seconds and runs past a signal pole in 15 seconds. Calculate the length of the train and its speed.

Lesson 5

Problem 1. A bicycle tire burst when the cyclist had travelled two-thirds of the way. It took him twice as long to walk the rest of the way. How many times faster did the cyclist ride than he walked?

Problem 2. George and Helen. George left the house 5 minutes after Helen, walking twice as fast as she did. How long will it take George to catch up with Helen?

Problem 3. One hundred metres. Three runners, Andrew, Boris, and Alex compete in a 100 metre event. When Andrew reached the finish line, Boris was 10 metres behind him. When Boris reached the finish line, Alex was 10 metres behind Boris. How many metres was Alex behind Andrew at the moment when Andrew reached the finish line?

Problem 4. Average speed, what is it? A person walked for a certain time at the speed of 4 km/h, and then, for twice as much time, at the speed of 7 km/h. What was his average speed of motion?

Problem 5. If a cyclist rides at 10 km/h, he will be one hour late. If he rides at 15 km/h, then he will arrive one hour early. At what speed should he or she ride so as to arrive on time?

Problem 6. Two trains move towards each other on parallel tracks at the same speed of 60 km/h. A passenger in the second train noticed that the first train ran past him in six seconds. What is the length of the first train?

Lesson 6

Problem 1. Two swimmers simultaneously jumped off a raft and swam in opposite directions: one downstream and the other upstream. After 5 minutes, they simultaneously turned and swam back. Which of the swimmers will reach the raft first?

Problem 2. A powerboat covers 90 km downstream and 70 km upstream in the same period of time. How far will a raft travel in that same period of time?

Problem 3. Down the Volga river. A river boat takes 5 days to go from Nizhny Novgorod to Astrakhan (down the Volga river), and 7 days to come back. How long will it take a raft to float down from Nizhny Novgorod to Astrakhan?

Problem 4. The floating hat. While canoeing down the river, Peter lost his hat under the bridge, but continued to paddle in the same direction. After 15 minutes, he noticed the loss, returned, and found the hat floating 1 km from the bridge. What is the speed of the current of the river?

Problem 5. Counting escalator steps. Two people walk down, without skipping any steps, on a down going escalator. One descends faster than the other. Which of them will touch more steps and why?

Problem 6. A clever trick. With its tank full of fuel, a boat can cover 72 km upstream or 120 km downstream. What is the greatest distance that the boat can cover, leaving enough fuel for the return journey?

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Table of Contents